economic interpretation of partial derivative:

$P(K, L)$

production (# items)

capital investment ($)

Labour (worker-hours)

$P_K$ : $\lim_{h \to 0} \frac{P(K+h, L) - P(K, L)}{h}$ } $\Delta P$ } $\Delta K$

partial derivative with respect to K

$\approx \frac{P(K+1, L) - P(K, L)}{1} = P(K+1, L) - P(K, L)$

Change in productivity per dollar invested

$P_L$ Change in productivity per hour worked

$P_L : \frac{\Delta P}{\Delta L}$ If $\Delta L = 1$, $P_L = \Delta P$

(ex) $G(W, C)$

cost of garment \$

cost of wool \$/lb

cost of cotton \$/lb

$G_W = 2$

$G_C = 5$

Question: If wool increases by \$3 / lb, then G increases by \$6 ?

$$2 = G_W \approx \frac{\Delta G}{\Delta w} = \frac{\Delta G}{3} \Rightarrow \Delta G = 6$$

# Higher-Order Derivatives

We always have to specify our "variable"

(ex) $f(x,y) = x\sin y$

1ST Partial Derivs: $f_x = \sin y$ $\qquad$ $f_y = x\cos y$

2ND " " : $\boxed{f_{xy} = \cos y}$ $\qquad f_{xx} = 0$ $\qquad f_{yy} = -\sin y \cdot x$ $\qquad \boxed{f_{yx} = \cos y}$

think: $(f_x)_y$

"mixed partials" (both variables)

## Clairaut's Thm

Assume that $f$ is defined on an open set $D$ of $\mathbb{R}^2$ $(x,y)$, and that $f_{xy}$ and $f_{yx}$ are continuous throughout $D$. Then:

$$f_{xy} = f_{yx}$$

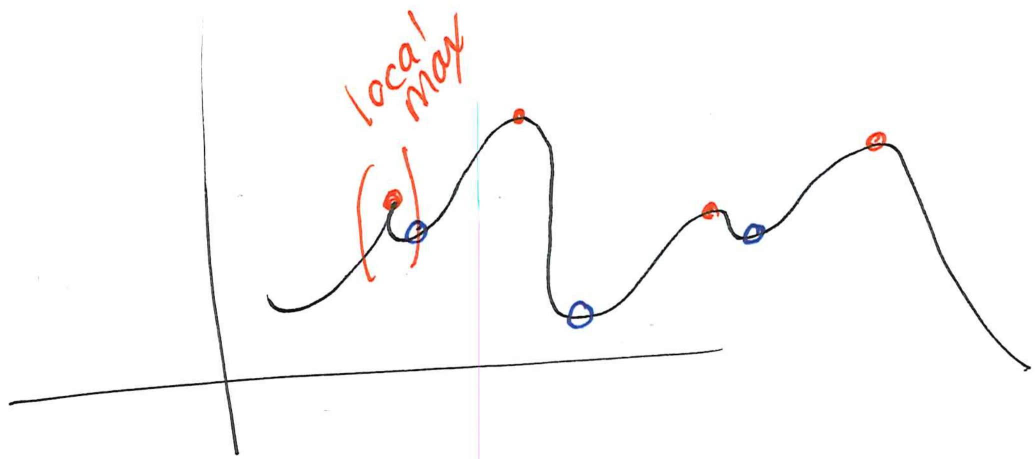
ex) Is it possible to have a function $f(x,y)$, defined everywhere, with $f_x = 3x$ and $f_y = 3x$ ?

Then: $f_{xy} = 0$ and $f_{yx} = 3$

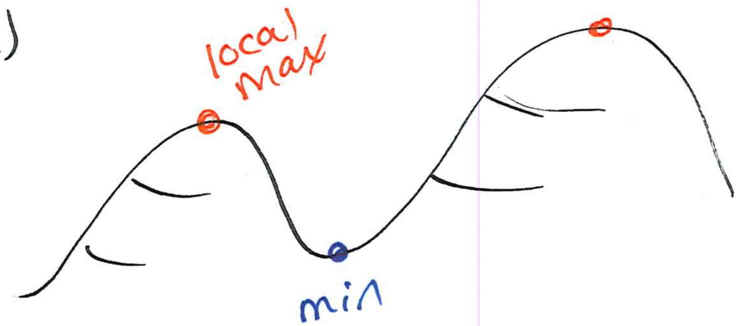
NOT POSSIBLE!

# Ch. 12.8 Local Extrema
## (Max/Min Problems)

Recall:




$\mathbb{R}^3$ (xyz)




local max:

can't walk uphill

local min:

can't walk downhill

Thm) If $f$ has a local max or min at $(a,b)$, then if $f_x$ and $f_y$ exist at $(a,b)$, then $f_x(a,b) = f_y(a,b) = 0$.

Def: If $f_x(a,b) = f_y(a,b) = 0$
OR if one DNE,
then $(a,b)$ is a critical point.

ex) If $f_x = \frac{1}{x}$ and $f_y = 0$

then $(0,0)$ is a CP

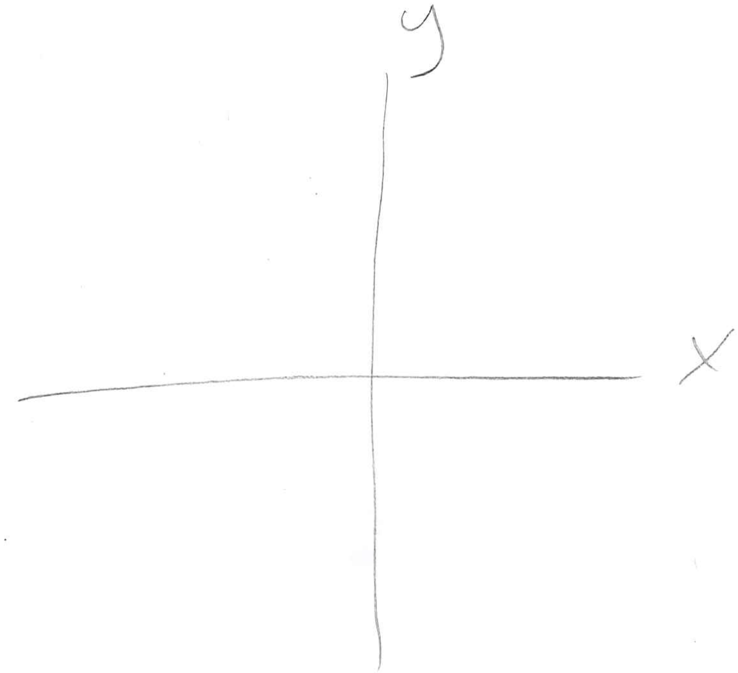
(ex) $z = 16 - 4x^2 - y^2$

$\overset{\shortparallel}{f(x,y)}$

$$\begin{cases} f_x = -8x \\ f_y = -2y \end{cases}$$

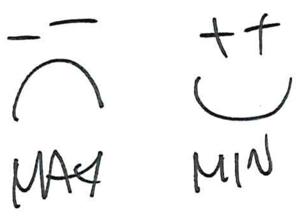
CP: $$\begin{cases} 0 = -8x \\ 0 = -2y \end{cases}$$

$$\boxed{CP: (0,0)}$$



Saddle pt: a point $(a,b)$
that is a critical pt,
but <u>not</u> a max or min

i.e. for any small neighborhood around $(a,b)$:
there exists $(x,y)$ such that
$f(x,y) > f(a,b)$ ← $(a,b)$ not max




and there exists $(x,y)$ such that
$f(x,y) < f(a,b)$ ← $(a,b)$ not min

## Second Derivative Test

Suppose the second partial derivatives of $f$ are continuous near $(a,b)$, and

$$f_x(a,b) = f_y(a,b) = 0$$

Define $D = f_{xx}(x,y) \cdot f_{yy}(x,y) - [f_{xy}(x,y)]^2$

① If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, then $f$ has local MAX at $(a,b)$

② If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, then $f$ has local MIN at $(a,b)$

③ If $D(a,b) < 0$, then $(a,b)$ is a saddle point

④ If $D(a,b) = 0$, test inconclusive

(ex) $f(x,y) = x^2 - 2x - y^2 - 4y - 4$

Find & classify all CPs.

$$\begin{cases} f_x = 2x - 2 \\ f_y = -2y - 4 \end{cases}$$

$$\begin{cases} 0 = 2x - 2 \\ 0 = -2y - 4 \end{cases}$$

$$\begin{cases} x = 1 \\ y = -2 \end{cases}$$

$f_{xx} = 2$

$f_{xy} = 0$

$f_{yy} = -2$

Only CP: $(1, -2)$

Use 2nd Deriv Test

$$D = f_{xx} f_{yy} - f_{xy}^2$$
$$= (2)(-2) - 0$$
$$= -4 < 0$$

So: $(1,-2)$ is a saddle point

ex $f(x,y) = x^2 - 6x + \frac{1}{4}y^4 - 8y$

Find + classify CPs

$\begin{cases} f_x = 2x-6 \\ f_y = y^3-8 \end{cases}$ $\begin{cases} 0 = 2x-6 \\ 0 = y^3-8 \end{cases}$ $x=3$, $y=2$ $\boxed{CP: (3,2)}$

$D: f_{xx} f_{yy} - f_{xy}^2$

$f_{xx} = 2$

$f_{xy} = 0$

$f_{yy} = 3y^2$

$D(3,2) =$
$(2)(3 \cdot 2^2) - 0$
$= 2 \cdot 12 = 24 > 0$

$\boxed{D(a,b) > 0}$

$\boxed{f_{xx} > 0}$

$\boxed{(3,2) \text{ local min}}$

$++$

ex $f(x,y) = x^2 + xy^2 - 2x + 1$

Find ~~& classify~~ all CPs

$$\begin{cases} f_x = 2x + y^2 - 2 \\ f_y = 2xy \end{cases}$$

$$\begin{cases} 0 = 2x + y^2 - 2 \\ 0 = 2xy \end{cases}$$

$$\begin{cases} 0 = y^2 - 2 \\ x = 0 \end{cases} \quad \text{or} \quad \begin{matrix} 0 = 2x - 2 \\ y = 0 \end{matrix}$$

$$\begin{cases} y = \pm\sqrt{2} \\ x = 0 \end{cases} \quad \text{or} \quad \begin{matrix} x = 1 \\ y = 0 \end{matrix}$$

$$\boxed{\begin{aligned} CP:\ &(0, \sqrt{2}) \\ &(0, -\sqrt{2}) \\ &(1, 0) \end{aligned}}$$

$f_{xx} = 2$

$f_{xy} = 2y$

$f_{yy} = 2x$

$D(x,y) = 2(2x) - (2y)^2$
$= 4x - 4y^2$

$\boxed{(0, \sqrt{2})}$ $D(0, \sqrt{2}) = 0 - 4(2) < 0$ SADDLE PT

$\boxed{(0, -\sqrt{2})}$ $D(0, -\sqrt{2}) = 0 - 4(2) < 0$ SADDLE PT

$\boxed{(1, 0)}$ $D(1,0) = 4 - 0 > 0$ LOCAL MIN
$f_{xx} = 2 > 0$

# Using Inspection to Classify CPs

ex $f(x,y) = \cos(xy)$

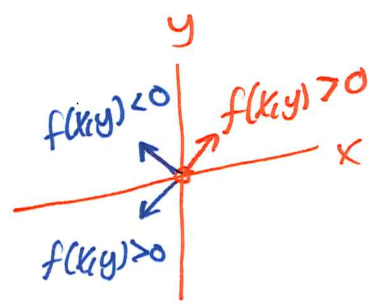
CP: $(0,0)$

$f(0,0) = 1$ MAX

(biggest cosine ever gets)

ex $f(x,y) = x\sin y$

CP: $(0,0)$




(0,0) is a saddle pt