Optimization: finding the biggest/smallest/highest/lowest, etc.

Lots of non-standard problems!
Chapter 7: Optimization

7.1 Simple biological optimization problems

Logistic growth rate

Small populations AND large populations grow slowly

\( N \): density of the population, \( G \): growth rate

**Logistic growth law:**

\[
G(N) = rN \left( \frac{K - N}{K} \right)
\]

\( r, K \): constants

**Example 1:**

- For which values of \( N \) is \( G(N) = 0? \)
- What does \( G(N) < 0 \) mean?
- Give an interpretation of the constant \( K \) in our model.
- What population density achieves maximum growth rate?
Fencing a pen

Example 2:

Given 60 metres of fencing material, what is the largest rectangular area you can enclose?
**Fencing a pen**

**Solution 2:**

- Area is $\ell \times w$, where $\ell$ is the length of the side parallel to the wall, and $w$ is the length of the side perpendicular to the wall.
- The amount of fencing we have is 60 metres, so $\ell + 2w = 60$.
- Then $\ell = 60 - 2w$
- So, our area (in terms of only one variable $w$) is $A = (60 - 2w)w$
- This is a parabola pointing down. Its maximum occurs when $w = 15$.
- Then the area of the pen is $(60 - 15)(15) = 675$ sq m.
Cell surface area

Some cells are roughly shaped like cylinders.

- Fixed volume, $V$
- Minimal surface area

Q: What are the dimensions of the cell?
Cell surface area: analyse the problem

\( V \): volume (fixed constant)
\( r \): radius
\( L \): length

Question: what are \( L \) and \( r \) (in terms of \( V \)) giving minimal surface area?
Cell surface area: plan a strategy

Find $L, r$ that make the surface area of the cylinder minimal.

Find an equation for the surface area of the cylinder

Find global minimum (using critical points, etc.)

Question: what are $L$ and $r$ (in terms of $V$) giving minimal surface area?
Cell surface area: implement strategy

Task: find an equation for the surface area of the cylinder.

\[ \text{Surface Area} = 2\pi r^2 + 2\pi rL \]
Cell surface area: plan a strategy

Find \( L, r \) that make the surface area of the cylinder minimal.

Find an equation for the surface area of the cylinder

\[ SA = 2\pi r^2 + 2\pi rL \]

Problem: two variables

Question: what are \( L \) and \( r \) (in terms of \( V \)) giving minimal surface area?
Cell surface area: plan a strategy

Find $L$, $r$ that make the surface area of the cylinder minimal.

Find an equation for the surface area of the cylinder: $SA = 2\pi r^2 + 2\pi rL$

Simplify surface area equation into one variable

Write $L$ as a function of $r$

Find global minimum (using critical points, etc.)

Question: what are $L$ and $r$ (in terms of $V$) giving minimal surface area?
Cell surface area: implement strategy

Task: write $L$ as a function of $r$.

- Given: the volume is a constant, $V$.
- The volume of a cylinder is: (Area of base) $\times$ (height)
- So,

$$V = \pi r^2 L$$

$$L = \frac{V}{\pi r^2}$$
Cell surface area: plan a strategy

Find $L, r$ that make the surface area of the cylinder minimal.

\[ \text{Find global minimum (using critical points, etc.)} \]

\[ \text{Simplify surface area equation into one variable} \]

\[ SA = 2\pi r^2 + 2\pi r \left( \frac{V}{\pi r^2} \right) \]
\[ SA = 2\pi r^2 + 2V r^{-1} \]

\[ \text{Write } L \text{ as a function of } r \]
\[ \checkmark \quad L = \frac{V}{\pi r^2} \]

Find an equation for the surface area of the cylinder: \[ SA = 2\pi r^2 + 2\pi rL \]

Question: what are $L$ and $r$ (in terms of $V$) giving minimal surface area?
Cell surface area: implement strategy

Task: find minimum value of the function \( SA(r) = 2\pi r^2 + 2V r^{-1}, r > 0. \)

\[
SA'(r) = 4\pi r - 2V r^{-2} \\
0 = 4\pi r - 2V r^{-2} \\
2V r^{-2} = 4\pi r \\
2V = 4\pi r^3 \\
\frac{V}{2\pi} = r^3 \\
r = \sqrt[3]{\frac{V}{2\pi}}
\]

The only critical point (with positive \( r \)) is the one above. We should check that it’s a minimum. We choose to use the second derivative test.

\[
SA''(r) = 4\pi + 4V r^{-3}
\]

Since this is positive for all positive \( r \), our critical point is indeed a minimum.
Cell surface area: plan a strategy

Find \( L, r \) that make the surface area of the cylinder minimal.

\[
\begin{align*}
    r &= \sqrt[3]{\frac{V}{2\pi}} \\
    L &= \sqrt[3]{\frac{4V}{\pi}}
\end{align*}
\]

\( L \) and \( r \) (in terms of \( V \)) giving minimal surface area?

Find an equation for the surface area of the cylinder: \( SA = 2\pi r^2 + 2\pi rL \)

Question: what are \( L \) and \( r \) (in terms of \( V \)) giving minimal surface area?
Cell surface area

Some cells are roughly shaped like cylinders.

- Fixed volume, $V$
- Minimal surface area

Q: What are the dimensions of the cell?

Radius: $\sqrt[3]{\frac{V}{2\pi}}$  
Length: $\sqrt[3]{\frac{4V}{\pi}}$
Kepler’s Wedding

- Kepler is comfortable with math.
- Kepler wants cheap wine.

- The wine cask's length $S$ determines the price.

- The wine seller:
Kepler’s Wedding: Choose the Biggest Cask

Suppose $S$ is the same in all these casks. (Therefore, they all have the same price.) Which has the biggest volume of wine?
Kepler’s Wedding

Suppose $S = 1$ metre. What are the height and radius of a (cylindrical) cask with maximum volume?
Kepler’s Wedding

Optimal cask:
When $S = 1$, we found:

- $V = \frac{\pi}{16} (4h - h^3)$
- The only critical point was at $h = \frac{2}{\sqrt{3}} \approx 0.67$.

Example 3:

Suppose the casks all have heights that are between 0.5 metres and 1 metre. What is height of the cask with the least amount of wine?
Optimal Foraging

- More foraging yields more food
- Animal must commute to food patch
Optimal Foraging

Each nut has a hard shell, and it takes time to crack one at a time before I can eat the nutty goodness inside. At first, the bees attacked, and I got very little honey. Over time, they tired, and eating honey got easier and easier. Collecting food was going great, until the birds showed up and started stealing away my stash.
Optimal Foraging with Multiple, Equivalent Patches

- **Strategy 1:** always go to the patch with the most amount of food
- **Strategy 2:** stay in a patch until you eat everything
Optimal Foraging

The energy gained from $t$ minutes at a patch is given by

$$f(t) = \frac{Et}{k + t}$$

where $E$ and $k$ are positive constants.
Optimal Foraging

The energy gained from $t$ minutes at a patch is given by $f(t) = \frac{Et}{k + t}$.

Interpret $E$ and $k$.

A  

B  

C  

D  

E  

F
What to Optimize

Idea 1: optimize total energy consumed.
Drawback:

Idea 2:

$$R(t) = \frac{\text{energy consumed}}{\text{time spent}}$$
Optimize **efficiency** of energy consumption

Energy gained after $t$ hours in a patch: \( \frac{Et}{k + t} \)

Time needed to travel to a patch: $\tau$.

\[
R(t) = \frac{\text{energy consumed}}{\text{time spent}} = \frac{Et}{k + t} = \frac{Et}{t + \tau} = \frac{Et}{(k + t)(\tau + t)}
\]

**Example 4:** Find $t$ that makes $R(t)$ maximum.

This is the optimal residence time.
Let \( R(t) = \frac{Et}{(k+t)(\tau+t)} \).

We can make a sketch. We see a single global max.

Finding the critical point:

\[
R'(t) = \frac{(k + t)(\tau + t)(E) - Et((k + t) + (\tau + t))}{(k + t)^2(\tau + t)^2}
\]

\[
= \frac{E(k\tau - t^2)}{(k + t)^2(\tau + t)^2}
\]

\[
0 = \frac{E(k\tau - t^2)}{(k + t)^2(\tau + t)^2}
\]

\[
t^2 = k\tau
\]

\[
t = \sqrt{k\tau}
\]

We ignore the root for \( t < 0 \).