Optimization:
finding the biggest/smallest/highest/lowest, etc.

Lots of non-standard problems!

Logistic growth rate

Small populations AND large populations grow slowly

N: density of the population, G: growth rate
Logistic growth law:

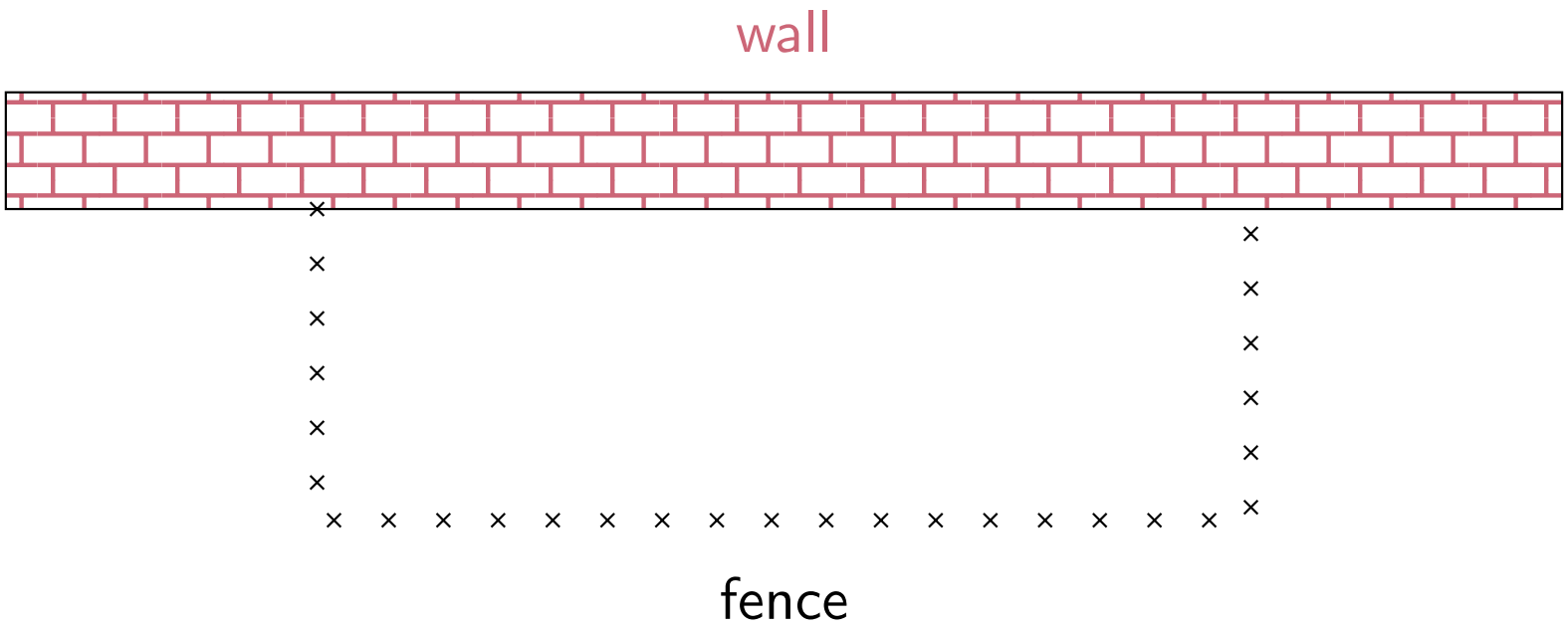
$$G(N) = rN\left(\frac{K-N}{K}\right)$$

r, K: constants

Example 1:

- For which values of N is $G(N) = 0$?
- What does $G(N) < 0$ mean?
- Give an interpretation of the constant K in our model.
- What population density achieves maximum growth rate?

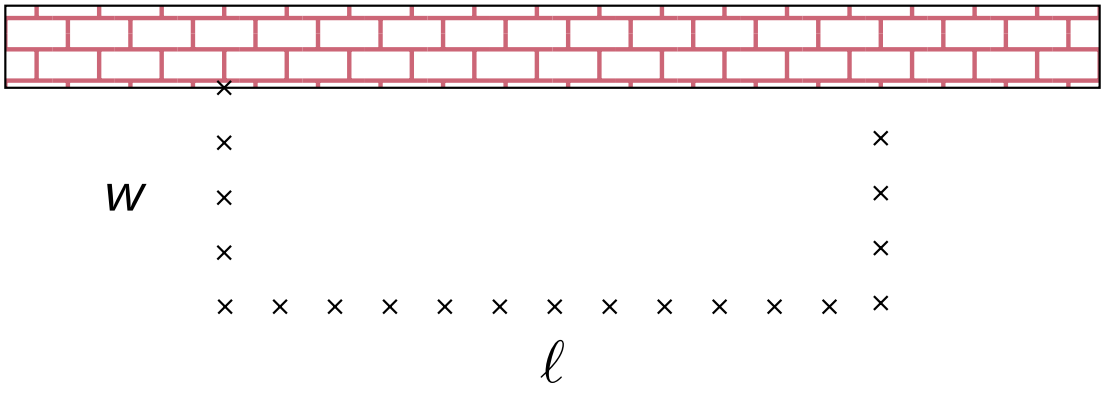
Fencing a pen



Example 2:

Given 60 metres of fencing material, what is the largest rectangular area you can enclose?

Fencing a pen

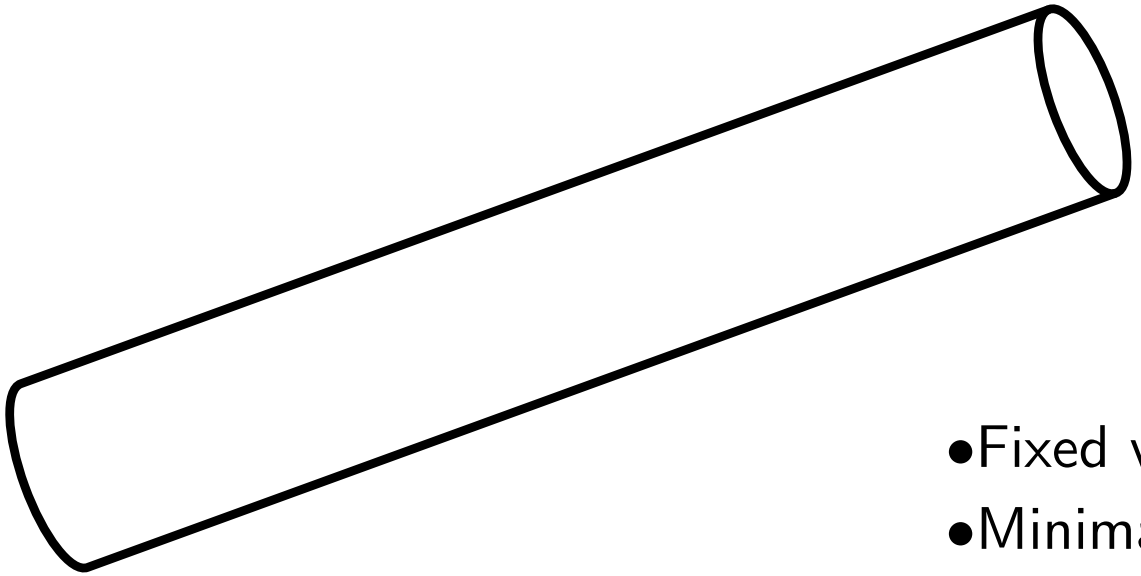


Solution 2:

- Area is $\ell \times w$, where ℓ is the length of the side parallel to the wall, and w is the length of the side perpendicular to the wall.
- The amount of fencing we have is 60 metres, so $\ell + 2w = 60$.
- Then $\ell = 60 - 2w$
- So, our area (in terms of only one variable w) is $A = (60 - 2w)w$
- This is a parabola pointing down. Its maximum occurs when $w = 15$.
- Then the area of the pen is $(60 - 15)(15) = 675$ sq m.

Cell surface area

Some cells are roughly shaped like cylinders.



- Fixed volume, V
- Minimal surface area

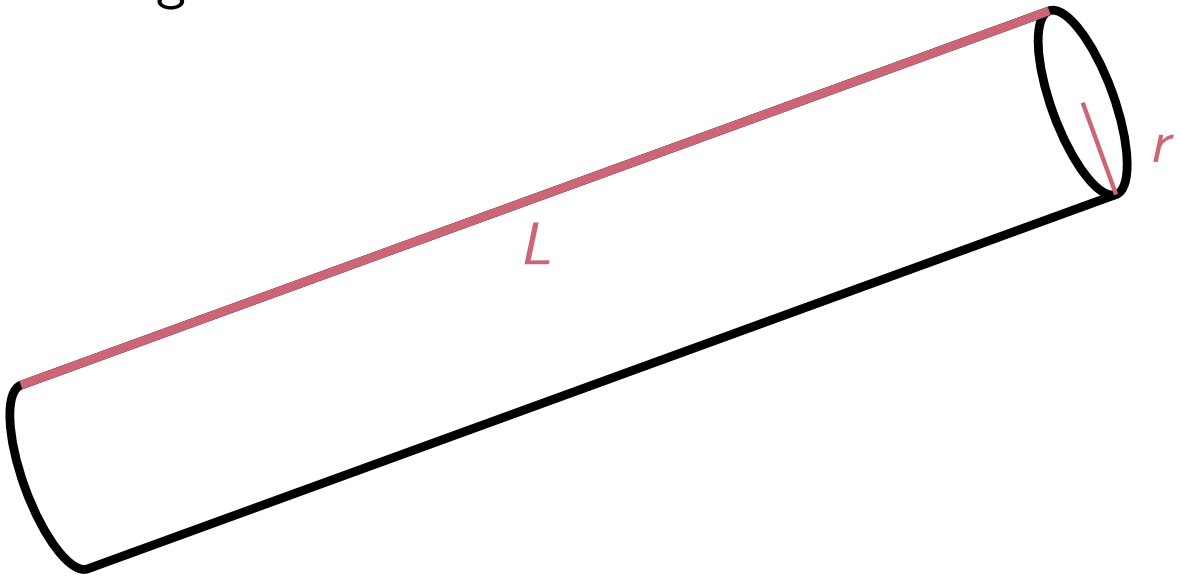
Q: What are the dimensions of the cell?

Cell surface area: analyse the problem

V: volume (fixed constant)

r: radius

L: length



Question: what are L and r (in terms of V) giving minimal surface area?

Cell surface area: plan a strategy

Find L, r that make the surface area of the cylinder minimal.

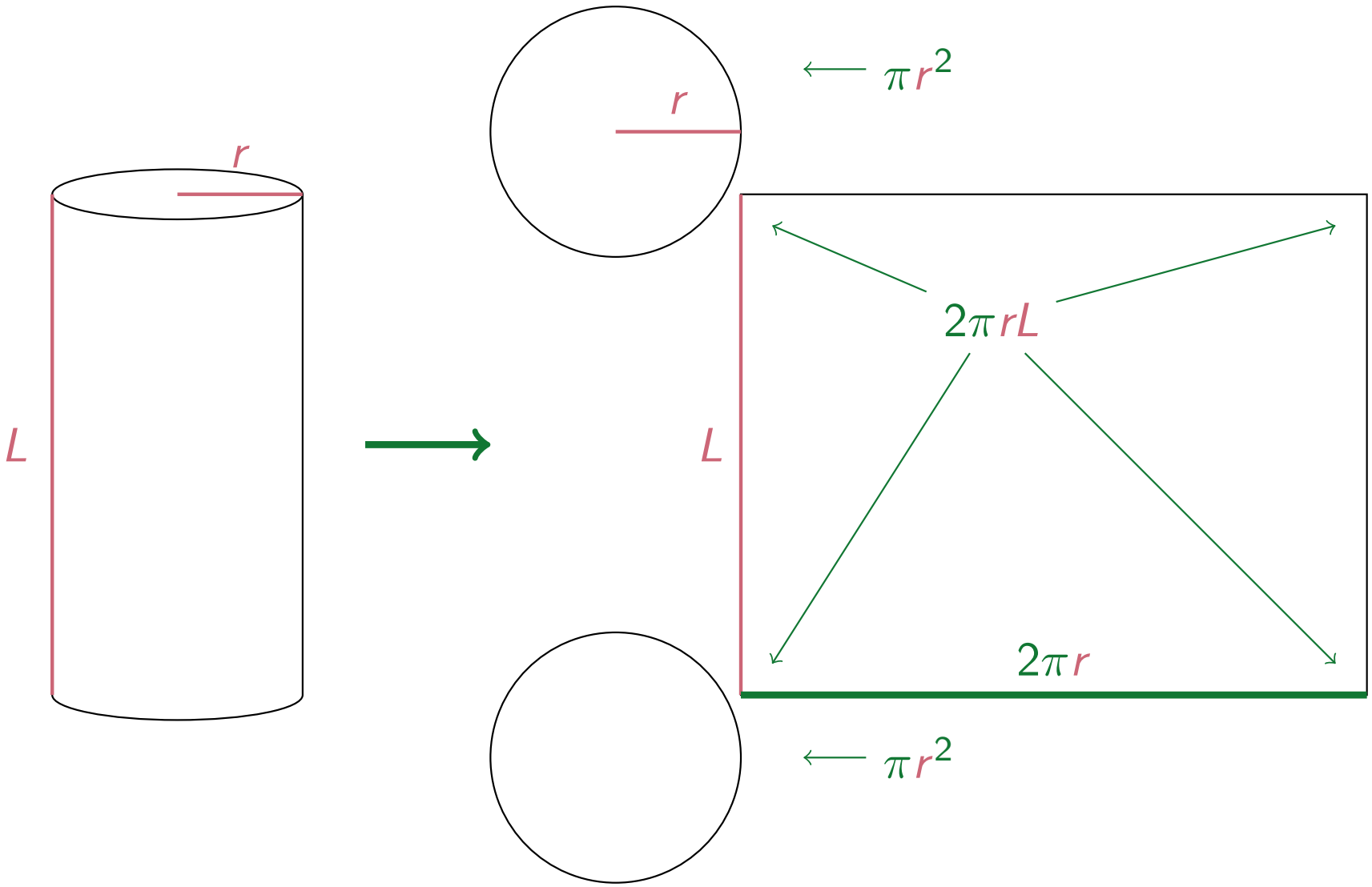
$\uparrow$ Find global minimum (using critical points, etc.)

Find an equation for the surface area of the cylinder

Question: what are L and r (in terms of V) giving minimal surface area?

Cell surface area: implement strategy

Task: find an equation for the surface area of the cylinder.



Cell surface area: plan a strategy

Find L, r that make the surface area of the cylinder minimal.

Find global minimum
(using critical points, etc.)

Find an equation for the surface area of the cylinder

✓ $SA = 2\pi r^2 + 2\pi rL$

Problems: **two** variables

Haven't used V

Question: what are L and r (in terms of V) giving minimal surface area?

Cell surface area: plan a strategy

Find L, r that make the surface area of the cylinder minimal.

↑ Find global minimum (using critical points, etc.)

Simplify surface area equation into **one** variable

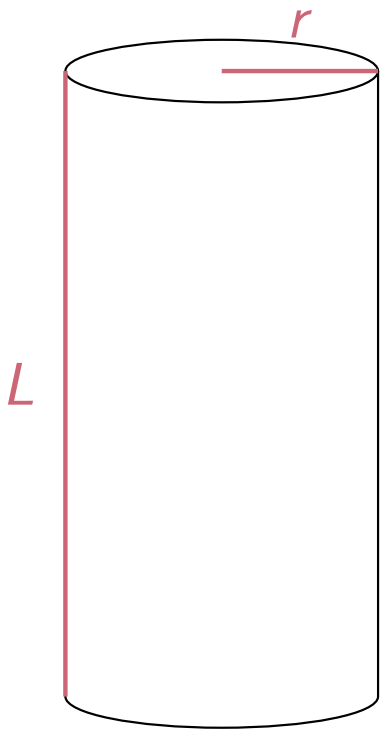
↑ Write L as a function of r

Find an equation for the surface area of the cylinder: $SA = 2\pi r^2 + 2\pi rL$

Question: what are L and r (in terms of V) giving minimal surface area?

Cell surface area: implement strategy

Task: write L as a function of r.



- Given: the volume is a constant, V.
- The volume of a cylinder is: (Area of base)$\times$ (height)
- So,

$$V = \pi r^2 L$$

$$\boxed{L = \frac{V}{\pi r^2}}$$

Cell surface area: plan a strategy

Find L, r that make the surface area of the cylinder minimal.

↑ Find global minimum (using critical points, etc.)

Simplify surface area equation into **one** variable

$$SA = 2\pi r^2 + 2\pi r\left(\frac{V}{\pi r^2}\right)$$

$$SA = 2\pi r^2 + 2Vr^{-1}$$

↑ Write L as a function of r

✓ $L = \frac{V}{\pi r^2}$

Find an equation for the surface area of the cylinder: $SA = 2\pi r^2 + 2\pi rL$

Question: what are L and r (in terms of V) giving minimal surface area?

Cell surface area: implement strategy

Task: find minimum value of the function $SA(r) = 2\pi r^2 + 2Vr^{-1}$, $r > 0$.

$$\begin{aligned} SA'(r) &= 4\pi r - 2Vr^{-2} \\ 0 &= 4\pi r - 2Vr^{-2} \\ 2Vr^{-2} &= 4\pi r \\ 2V &= 4\pi r^3 \\ \frac{V}{2\pi} &= r^3 \\ r &= \sqrt[3]{\frac{V}{2\pi}} \end{aligned}$$

The only critical point (with positive r) is the one above. We should check that it's a minimum. We choose to use the second derivative test.

$$SA''(r) = 4\pi + 4Vr^{-3}$$

Since this is positive for all positive r, our critical point is indeed a minimum.

Cell surface area: plan a strategy

Find L, r that make the surface area of the cylinder minimal.

$r = \sqrt[3]{\frac{V}{2\pi}}$

$L = ?\sqrt[3]{\frac{4V}{\pi}}$

↑ Find global minimum (using critical points, etc.)

Simplify surface area equation into **one** variable

↑ Write L as a function of r

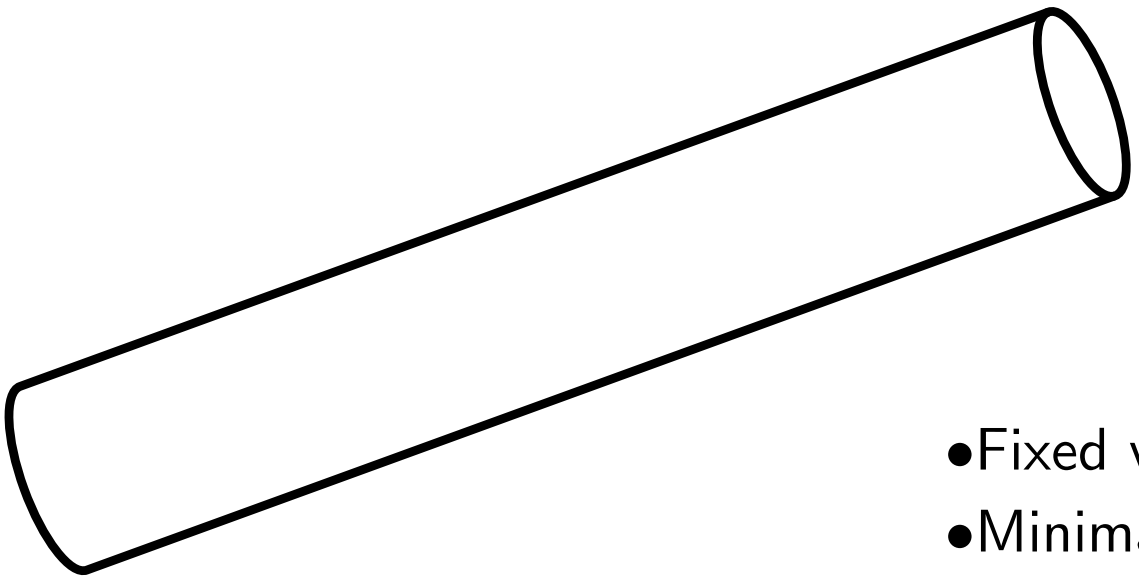
✓ $L = \frac{V}{\pi r^2}$

Find an equation for the surface area of the cylinder: $SA = 2\pi r^2 + 2\pi rL$

Question: what are L and r (in terms of V) giving minimal surface area?

Cell surface area

Some cells are roughly shaped like cylinders.

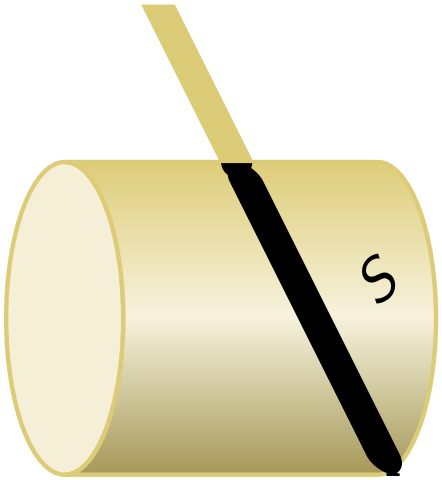


- Fixed volume, V
- Minimal surface area

Q: What are the dimensions of the cell?

Radius: $\sqrt[3]{\frac{V}{2\pi}}$ Length: $\sqrt[3]{\frac{4V}{\pi}}$

Kepler's Wedding



wine cask
length S determines price

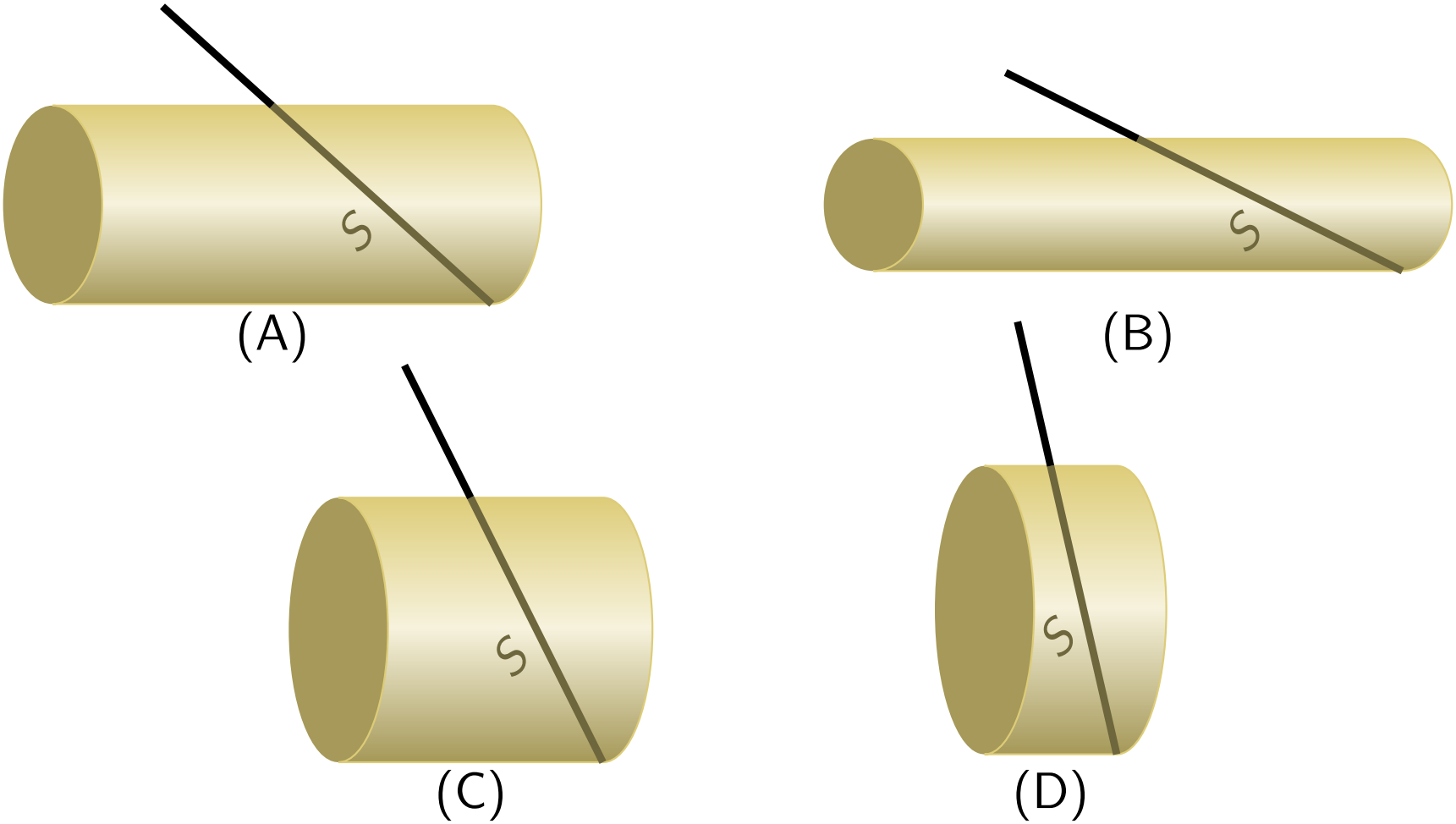
wine seller

Kepler

- comfortable with math
- wants cheap wine

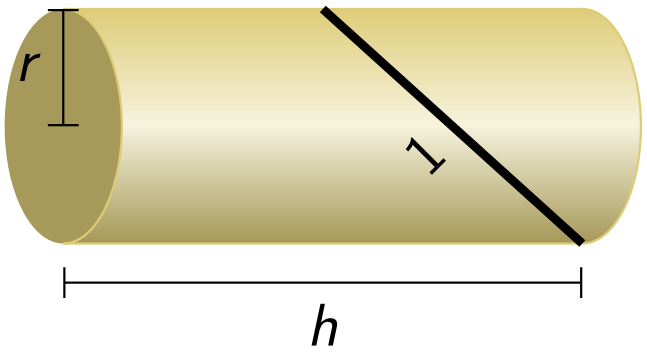
Kepler's Wedding: Choose the Biggest Cask

Suppose S is the same in all these casks. (Therefore, they all have the same price.) Which has the biggest volume of wine?



Kepler's Wedding

Suppose $S = 1$ metre. What are the height and radius of a (cylindrical) cask with maximum volume?



Kepler's Wedding

Optimal cask:

Kepler's Wedding, Continued

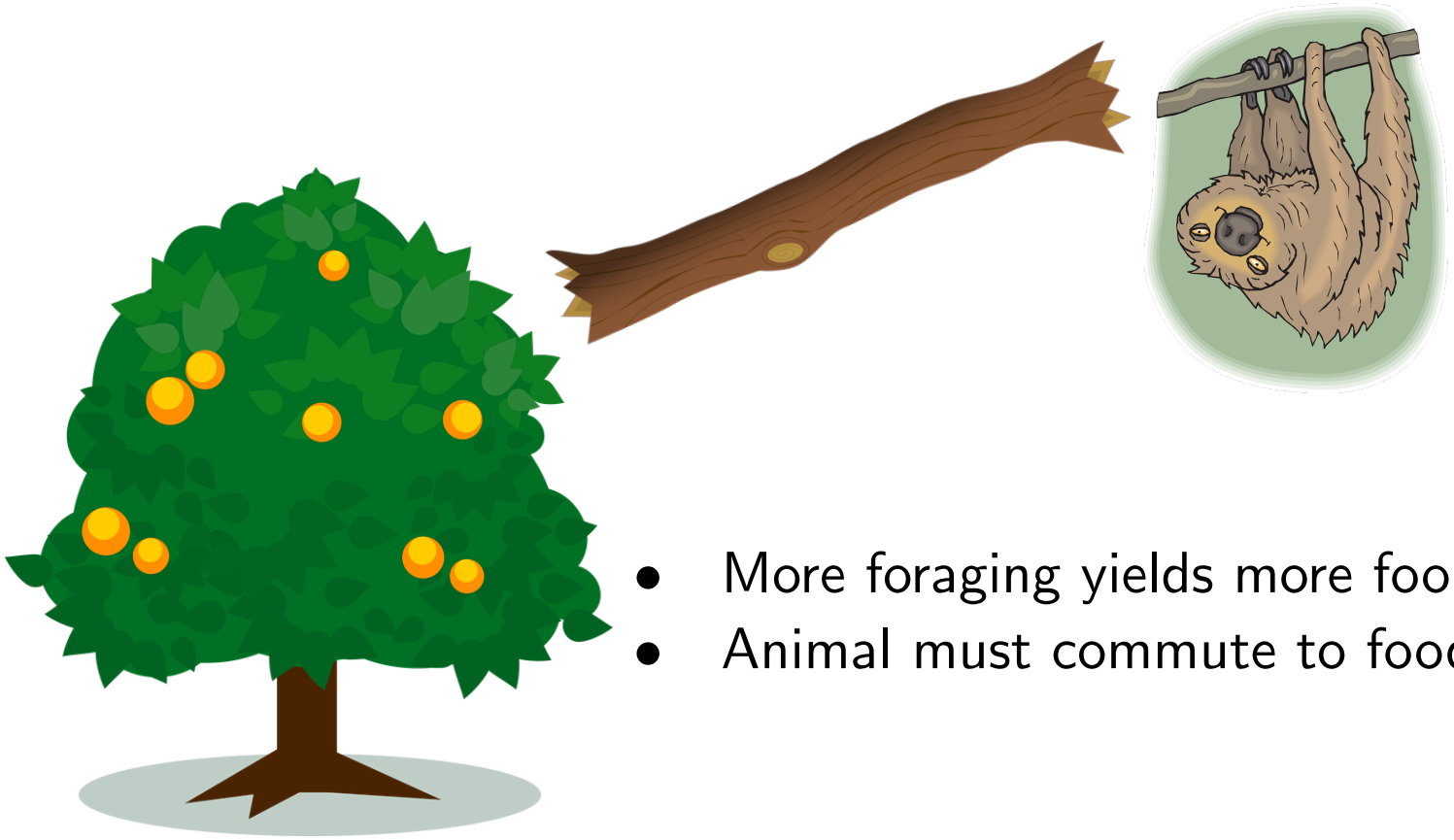
When $S = 1$, we found:

- $V = \frac{\pi}{16}\left(4h - h^3\right)$
- The only critical point was at $h = \frac{2}{\sqrt{3}} \approx 0.67$.

Example 3:

Suppose the casks all have heights that are between 0.5 metres and 1 metre. What is height of the cask with the **least** amount of wine?

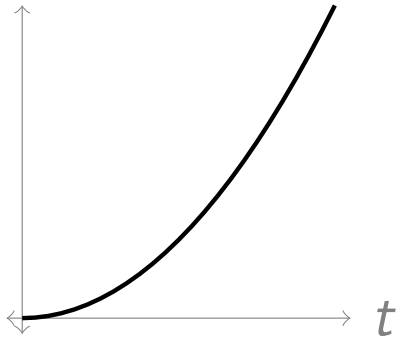
Optimal Foraging



- More foraging yields more food
- Animal must commute to food patch

Optimal Foraging

Each nut has a hard shell, and it takes time to crack one at a time before I can eat the nutty goodness inside. At first, the bees attacked, and I got very little honey. Over time, they tired, and eating honey got easier and easier. Collecting food was going great, until the birds showed up and started stealing away my stash.



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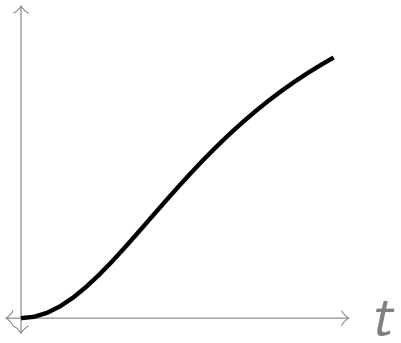

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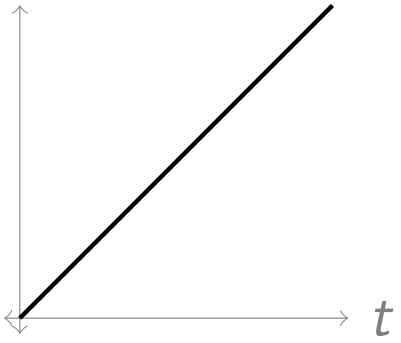

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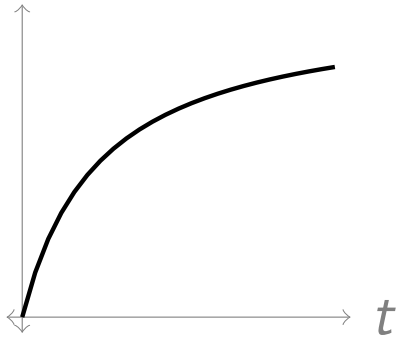
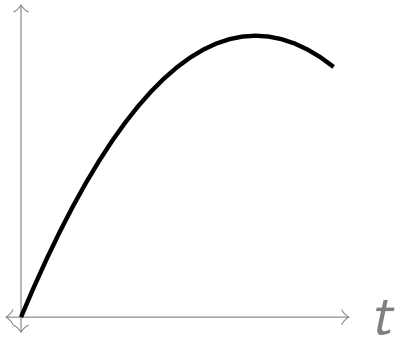
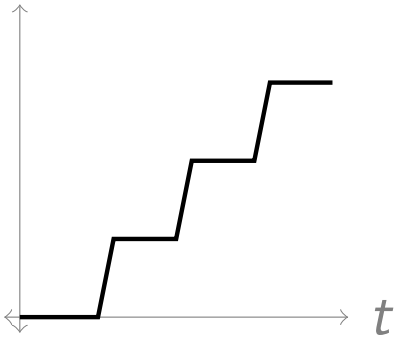
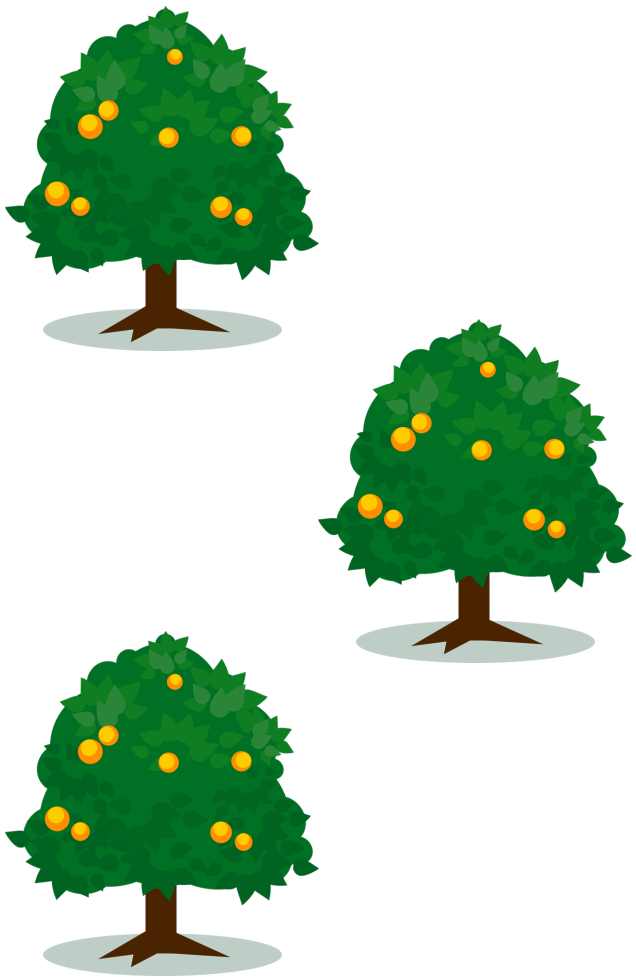






Optimal Foraging with Multiple, Equivalent Patches



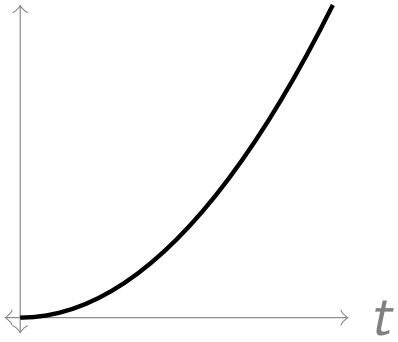
- Strategy 1: always go to the patch with the most amount of food

- Strategy 2: stay in a patch until you eat everything

Optimal Foraging

The energy gained from t minutes at a patch is given by $\boxed{f(t) = \dfrac{Et}{k+t}}$

where E and k are positive constants.



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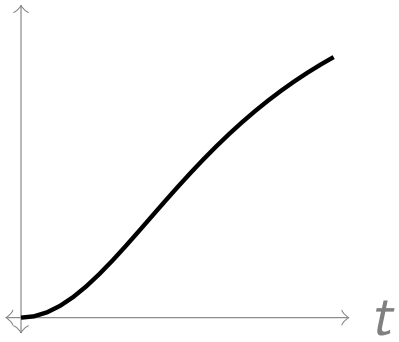

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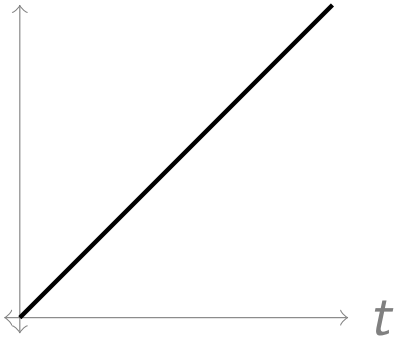

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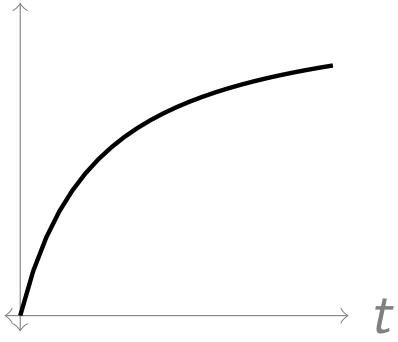
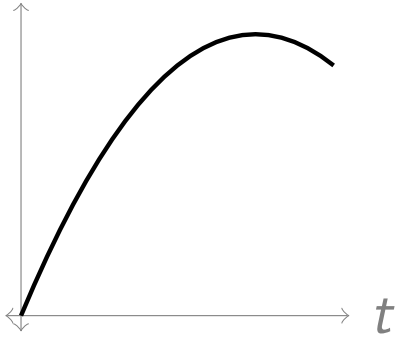
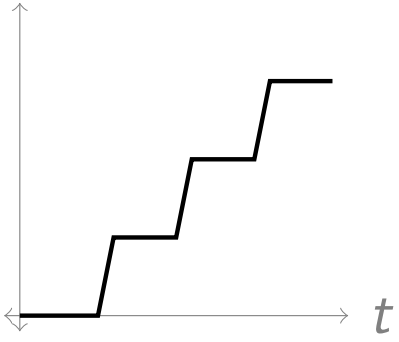


D



E

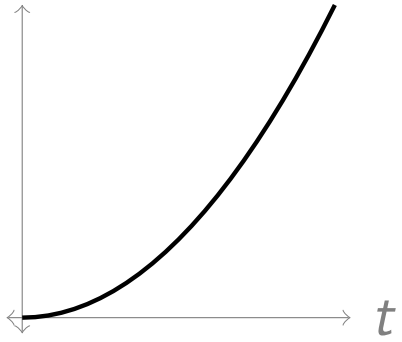


F

Optimal Foraging

The energy gained from t minutes at a patch is given by $\boxed{f(t) = \frac{Et}{k+t}}$.

Interpret E and k.



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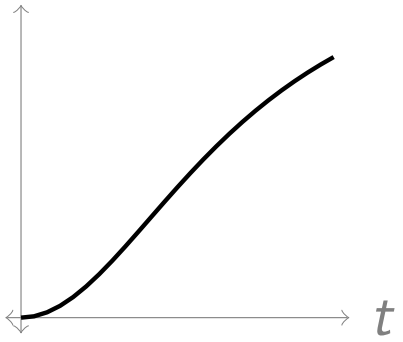

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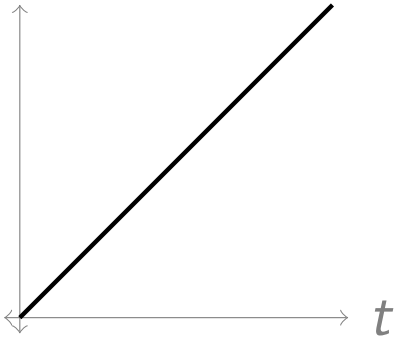

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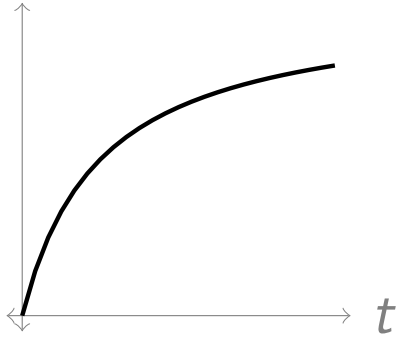

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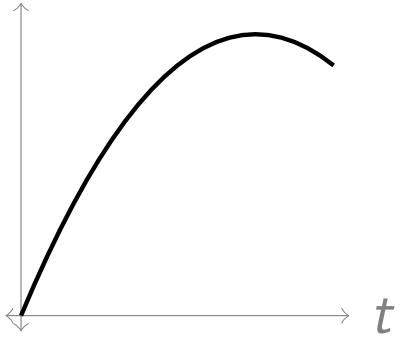
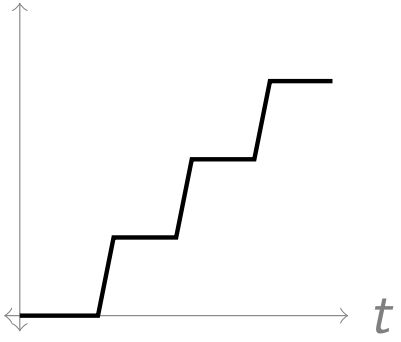


E



F

What to Optimize

Idea 1: optimize total energy consumed.
Drawback:

Idea 2:

$$R(t) = \frac{\text{energy consumed}}{\text{time spent}}$$

Optimize **efficiency** of energy consumption

Energy gained after t hours in a patch: $\dfrac{Et}{k+t}$

Time needed to travel to a patch: τ.

$$R(t) = \frac{\text{energy consumed}}{\text{time spent}}$$
$$= \frac{\frac{Et}{k+t}}{t+\tau} = \frac{Et}{(k+t)(\tau+t)}$$

Example 4: Find t that makes $R(t)$ maximum.

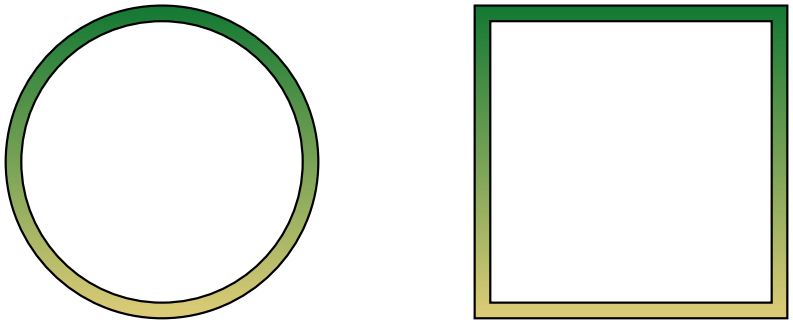
This is the optimal residence time.

Let $R(t) = \frac{Et}{(k+t)(\tau+t)}$.
We can make a sketch. We see a single global max.
Finding the critical point:

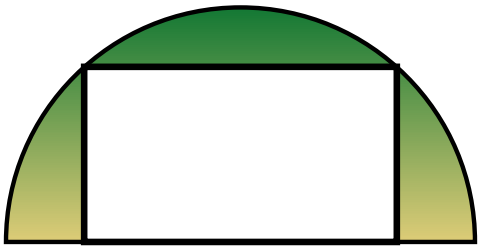
$$\begin{aligned}
R'(t) &= \frac{(k+t)(\tau+t)(E) - Et((k+t)+(\tau+t))}{(k+t)^2(\tau+t)^2} \\
&= \frac{E(k\tau - t^2)}{(k+t)^2(\tau+t)^2} \\
0 &= \frac{E(k\tau - t^2)}{(k+t)^2(\tau+t)^2} \\
t^2 &= k\tau \\
t &= \sqrt{k\tau}
\end{aligned}$$

We ignore the root for $t < 0$.

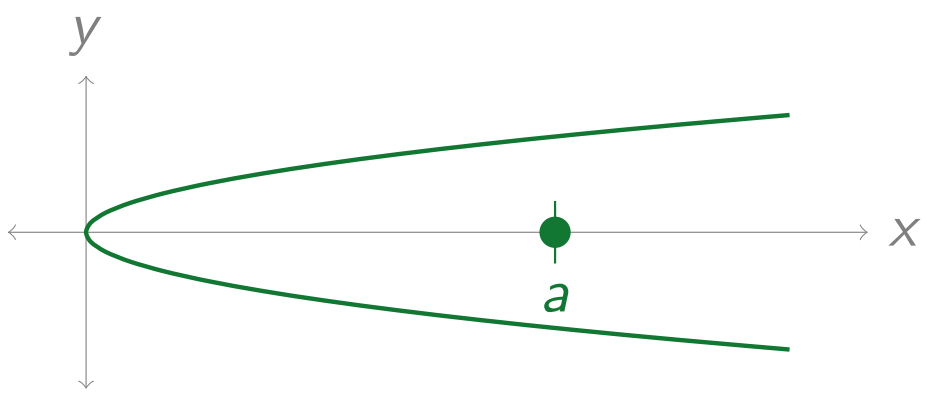
You have L metres of rope, and you want to use it to form a circle and a square. How would you enclose the most area? The least?



A rectangle is inscribed in a semicircle of radius R so that one side of the rectangle lies along a diameter of the semicircle. Find the largest and smallest possible perimeter of such a rectangle.



Find the minimum distance from the point $(a, 0)$ to the parabola $y^2 = 8x$.



Find a point A on the positive x-axis and a point B on the positive y-axis such that (i) the triangle AOB contains the first quadrant portion of the parabola $y = 1 - x^2$ and (ii) the area of the triangle AOB is as small as possible