Chapter 6: Sketching the graph of a function using calculus tools

Overview

Graphing More Accurately
- First and second derivatives
- Critical points
- Extrema

6.1 Overall shape of the graph of a function

Up with hope, down with dope,
Increasing functions have positive slope

Using our previous method, we can sketch $f(x) = x^3 - 3x^2$:

$$f(x) = x^3 - 3x^2$$

![Graph of $f(x) = x^3 - 3x^2$]

Notes

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Blow-up discontinuity: $x = -1$

- $f'(x) = \frac{x(x+2)}{(x+1)^2}$
- $f''(x) = \frac{2}{(x+1)^3}$

Sketch:

$$f(x) = \frac{x^2}{x + 1}$$

Notes
Concavity

Slopes are increasing
\[ f''(x) > 0 \quad \text{“concave up”} \]

Slopes are decreasing
\[ f''(x) < 0 \quad \text{“concave down”} \]

Mnemonic

Where is the function concave up? Concave down?

Notes
Sketch:
\[ f(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 - 3x^2 \]

- Consider behaviour near origin and far away from origin
- First derivative: increasing and decreasing; find \( f(x) \) at special points
- Second derivative: concavity

Using info from first derivative:

Using info from second derivative:
Sketch graphs with the following properties, or explain that none exist.

<table>
<thead>
<tr>
<th></th>
<th>concave up</th>
<th>concave down</th>
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</thead>
<tbody>
<tr>
<td>increasing</td>
<td>x</td>
<td>y</td>
</tr>
<tr>
<td>decreasing</td>
<td>y</td>
<td>x</td>
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**Sign Changes in a Factored Function**

\[ f(x) = (x - 1)(x - 2)^2(x - 3) \]

**Example 1:**

\[ f(x) = (x - 3)(x - 1)^2x(x + 2)^3(x + 5)^4 \]

Where is \( f(x) \) positive? Where is it negative?
Special Points:
- Roots / zeroes
- Critical point (i.e. $f'(x) = 0$ or $f'(x)$ DNE)
- Maxima, minima
- CP that is not an extremum

Each of the following functions have a critical point at $x = 0$.
Match the derivatives with their graphs.

(a) $f'(x)$ negative when $x < 0$
(b) $f'(x)$ positive when $x > 0$
(c) $f'(x)$ negative when $x < 0$
(d) $f'(x)$ positive when $x > 0$

Match the second derivatives with their graphs.

(a) $f''(x)$ negative
(b) $f''(x)$ positive
(c) $f''(x)$ negative when $x < 0$
(d) $f''(x)$ positive when $x < 0$
Suppose $x = a$ is a critical point of the continuous function $f(x)$. That is, $f'(a) = 0$ or $f'(x)$ does not exist at $a$.

- $f'(a)$ changes from neg to pos
- $f'(a)$ changes from pos to neg
- $f''(a) = 0$
- $f''(a)$ changes from neg to pos
- $f''(a)$ changes from pos to neg
- $f''(a)$ positive and $f'(a) = 0$
- $f''(a)$ negative and $f'(a) = 0$

**I. inflection point**

**II. local max**

**III. local min**

**IV. not a local extrema**

**V. could be local max, local min, or neither**

**Tests**

The information from the previous questions forms a method for finding local extrema.

- First, find all critical points (where $f'(x) = 0$ or $f'(x)$ doesn’t exist). Local extrema can ONLY occur at critical points.
- Now, we have to classify those critical points: they could be local maxes, local mins, or neither. **Two options:**
  - Using the second derivative:
    - If $f''(x) > 0$, the CP is a local max
    - If $f''(x) < 0$, the CP is a local min
    - If $f''(x) = 0$, not a local extremum
  - Using the first derivative:
    - If $f'(x)$ does not change signs at the CP, then
    - If $f'(x)$ changes from increasing to decreasing, the CP is a local max
    - If $f'(x)$ changes from decreasing to increasing, the CP is a local min

\[ f(x) = \frac{1}{4}x^4 - 2x^3 + \frac{9}{2}x^2 + 1 \]

- Find all critical points of $f(x)$
- Are they local maxima, local minima, or neither?
  **Challenge:** sketch the function.
Chapter 6: Sketching the graph of a function using calculus tools 6.2 Special points on the graph of a function

\[
f(x) = \frac{1}{4}x^4 - 2x^3 + \frac{9}{2}x^2 + 1
\]

Sketch.

local min at \((0, 1)\) also **global** min
inflection points at \((3, \frac{31}{4})\) and \((1, \frac{11}{4})\)

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Global extrema over a closed interval

Where do global extrema occur?

Cell Division

A cell of age \(t\) hours has a probability\(^1\) of dividing given by the function

\[
P(t) = \frac{at}{t^3 + 16},
\]

where \(a\) is the constant \(\frac{9\sqrt{3}}{2\pi}\).

From time \(t = 0\) to time \(t = 10\), when is the cell likeliest to divide? When is it least likely to divide?

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\(^1\)Actually, we’re giving a probability distribution function... it’s a little more complicated than what we said. Still, a higher value of \(P(t)\) means a cell age that is more reproduce-y, so we just want to find the max and min of \(P(t)\).
Sketch

\[ f(x) = \frac{(x - 1)^2}{x^3} \]

Include the following, if they exist:

- Roots (zeroes)
- Discontinuities
- Asymptotes
- Local and global extrema (maxes and mins)
- Inflection points and concavity

Notes

\[ f(x) = \frac{(x - 1)^2}{x^3} \]

\[ f'(x) = \frac{-(x - 1)(x - 3)}{x^4} \]

\[ f''(x) = \frac{2(x^2 - 6x + 6)}{x^5} \]
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6.3 Sketching the graph of a function

Sketch:

\[ f(x) = \frac{10x}{x^3 + 16} \]

- \( f'(x) = \frac{20(x^3 + 16) - 10x(3x^2)}{(x^3 + 16)^2} \)
- \( f''(x) = \frac{60x^2(x^3 - 32)}{(x^3 + 16)^3} \)