Overview

Graphing More Accurately
- First and second derivatives
- Critical points
- Extrema
Up with hope, down with dope,
Increasing functions have positive slope

Using our previous method, we can sketch $f(x) = x^3 - 3x^2$: 
Sketch:

\[ f(x) = \frac{x^2}{x + 1} \]

- **Blow-up discontinuity:** \( x = -1 \)
- \( f'(x) = \frac{x(x+2)}{(x+1)^2} \)
- \( f''(x) = \frac{2}{(x+1)^3} \)
Concavity

Slopes are increasing

\[ f''(x) > 0 \]

“concave up”

Slopes are decreasing

\[ f''(x) < 0 \]

“concave down”
Mnemonic

+ +  

- -
Concavity

Where is the function concave up? Concave down?
Sketch:

\[ f(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 - 3x^2 \]

- Consider behaviour near origin and far away from origin
- First derivative: increasing and decreasing; find \( f(x) \) at special points
- Second derivative: concavity
Using info from first derivative:
Using info from second derivative:
Sketch graphs with the following properties, or explain that none exist.

<table>
<thead>
<tr>
<th></th>
<th>concave up</th>
<th>concave down</th>
</tr>
</thead>
<tbody>
<tr>
<td>increasing</td>
<td><img src="image1" alt="Graph" /></td>
<td><img src="image2" alt="Graph" /></td>
</tr>
<tr>
<td>decreasing</td>
<td><img src="image3" alt="Graph" /></td>
<td><img src="image4" alt="Graph" /></td>
</tr>
</tbody>
</table>
Sign Changes in a Factored Function

\[ f(x) = (x - 1) \ (x - 2)^2 \ (x - 3) \]
Example 1:

\[ f(x) = (x - 3)(x - 1)^2x(x + 2)^3(x + 5)^4 \]

Where is \( f(x) \) positive? Where is it negative?
Special Points:

- Roots / zeroes
- Critical point (i.e. $f'(x) = 0$ or $f'(x)$ DNE)
- Maxima, minima
- CP that is not an extremum
Each of the following functions have a critical point at $x = 0$. Match the derivatives with their graphs.

(a) $f'(x)$ negative when $x < 0$
   $f'(x)$ negative when $x > 0$

(b) $f'(x)$ positive when $x < 0$
   $f'(x)$ positive when $x > 0$

(c) $f'(x)$ negative when $x < 0$
   $f'(x)$ positive when $x > 0$

(d) $f'(x)$ positive when $x < 0$
   $f'(x)$ negative when $x > 0$
Match the second derivatives with their graphs.

(a) \( f''(x) \) negative

(c) \( f''(x) \) negative when \( x < 0 \)
\( f''(x) \) positive when \( x > 0 \)

(b) \( f''(x) \) positive

(d) \( f''(x) \) positive when \( x < 0 \)
\( f''(x) \) negative when \( x > 0 \)
Suppose \( x = a \) is a critical point of the continuous function \( f(x) \). That is, \( f'(a) = 0 \) or \( f'(x) \) does not exist at \( a \).

\( a \) \( f'(a) \) changes from neg to pos

\( b \) \( f'(a) \) changes from pos to neg

\( c \) \( f''(a) = 0 \)

\( d \) \( f''(a) \) changes from neg to pos

\( e \) \( f''(a) \) changes from pos to neg

\( f \) \( f''(a) \) positive and \( f'(a) = 0 \)

\( g \) \( f''(a) \) negative and \( f'(a) = 0 \)

I. inflection point

II. local max

III. local min

IV. not a local extrema

V. could be local max, local min, or neither
Tests

The information from the previous questions forms a method for finding local extrema.

- First, find all critical points (where \( f'(x) = 0 \) or \( f'(x) \) doesn’t exist). Local extrema can ONLY occur at critical points.
- Now, we have to classify those critical points: they could be local maxes, local mins, or neither. Two options:
  - Using the second derivative:
    - If \( f''(x) > 0 \), the CP is a local max.
    - If \( f''(x) < 0 \), the CP is a local min.
    - If \( f''(x) = 0 \),
  - Using the first derivative:
    - If \( f'(x) \) does not change signs at the CP, then
    - If \( f'(x) \) changes from increasing to decreasing, the CP is a local max.
    - If \( f'(x) \) changes from decreasing to increasing, the CP is a local min.
Chapter 6: Sketching the graph of a function using calculus tools

6.2 Special points on the graph of a function

\[ f(x) = \frac{1}{4}x^4 - 2x^3 + \frac{9}{2}x^2 + 1 \]

1. Find all critical points of \( f(x) \)
2. Are they local maxima, local minima, or neither?

Challenge: sketch the function.
Chapter 6: Sketching the graph of a function using calculus tools

6.2 Special points on the graph of a function

Sketch.

\[ f(x) = \frac{1}{4}x^4 - 2x^3 + \frac{9}{2}x^2 + 1 \]

local min at \((0, 1)\)
also \textbf{global} min

inflection points at \((3, \frac{31}{4})\)
and \((1, \frac{11}{4})\)
Global extrema over a closed interval

Where do global extrema occur?
A cell of age \( t \) hours has a probability\(^1\) of dividing given by the function

\[
P(t) = \frac{at}{t^3 + 16},
\]

where \( a \) is the constant \( \frac{9\sqrt{3}}{2\pi} \).

From time \( t = 0 \) to time \( t = 10 \), when is the cell likeliest to divide? When is it least likely to divide?

---

\(^1\) Actually, we’re giving a probability distribution function... it’s a little more complicated than what we said. Still, a higher value of \( P(t) \) means a cell age that is more reproduce-y, so we just want to find the max and min of \( P(t) \).
Sketch

\[ f(x) = \frac{(x - 1)^2}{x^3} \]

Include the following, if they exist:

- Roots (zeroes)
- Discontinuities
- Asymptotes
- Local and global extrema (maxes and mins)
- Inflection points and concavity
Sketch

\[ f(x) = \frac{(x-1)^2}{x^3} \]

\[ f'(x) = \frac{-(x-1)(x-3)}{x^4} \]

\[ f''(x) = \frac{2(x^2-6x+6)}{x^5} \]
Sketch

\[ f(x) = \frac{(x - 1)^2}{x^3} \]
Sketch:

\[ f(x) = \frac{10x}{x^3 + 16} \]

- \[ f'(x) = \frac{20(8-x^3)}{(x^3+16)^2} \]
- \[ f''(x) = \frac{60x^2(x^3-32)}{(x^3+16)^3} \]