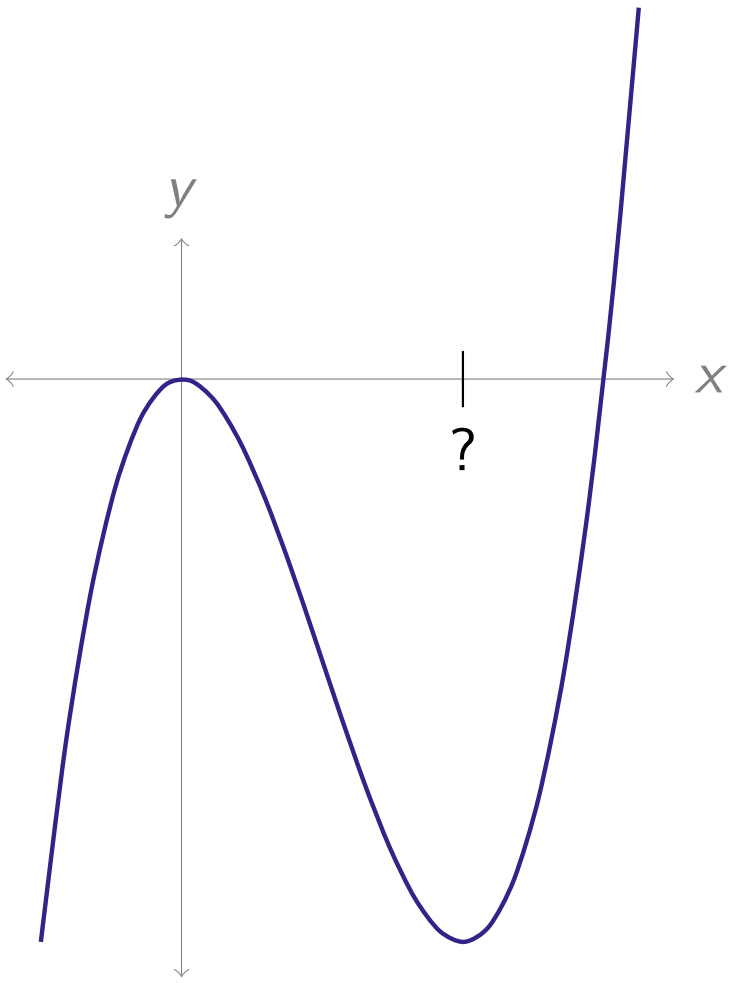
Overview

Graphing More Accurately

- First and second derivatives
- Critical points
- Extrema

Up with hope, down with dope, Increasing functions have positive slope

Using our previous method, we can sketch $f(x) = x^3 - 3x^2$:

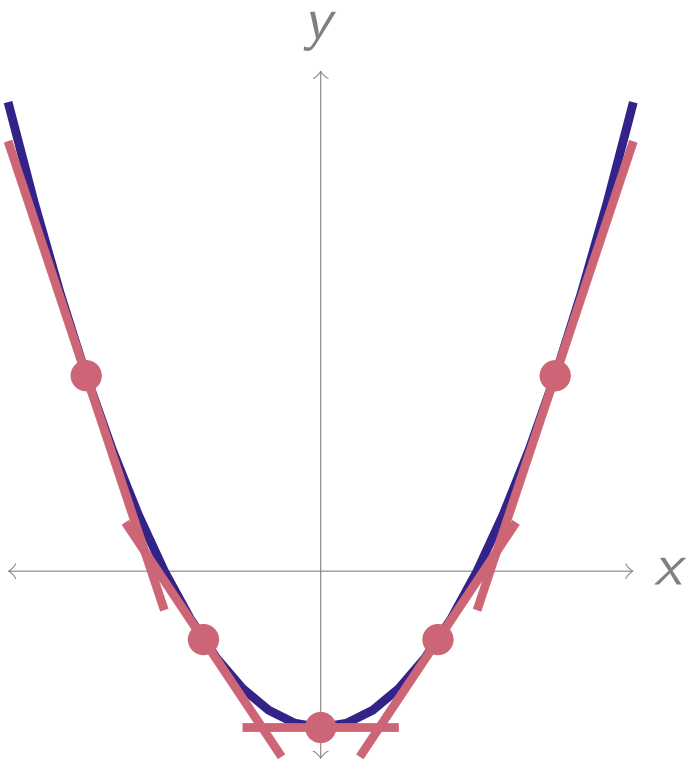


Sketch:

$$f(x) = \frac{x^2}{x+1}$$

- Blow-up discontinuity: $x = -1$
- $f'(x) = \frac{x(x+2)}{(x+1)^2}$
- $f''(x) = \frac{2}{(x+1)^3}$

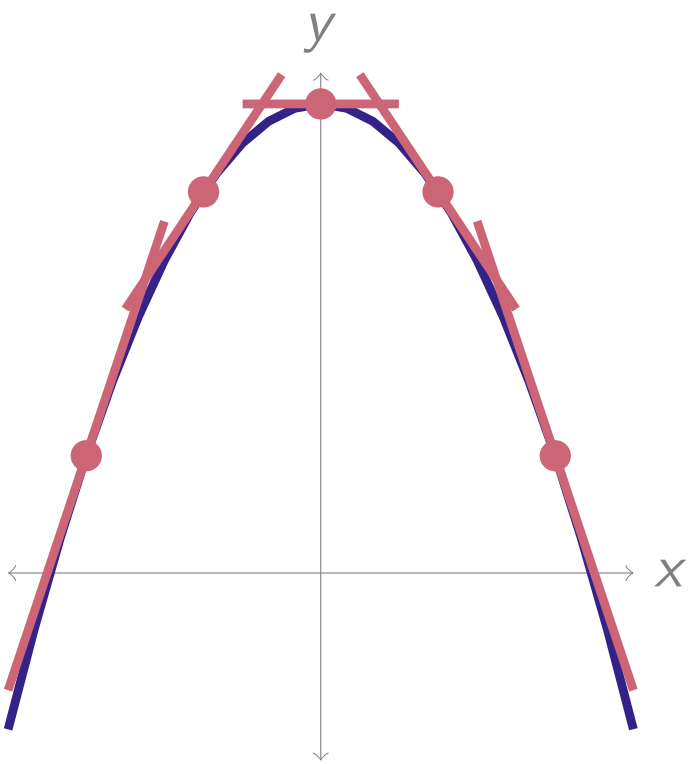
Concavity



Slopes are increasing

$f''(x) > 0$

"concave up"



Slopes are decreasing

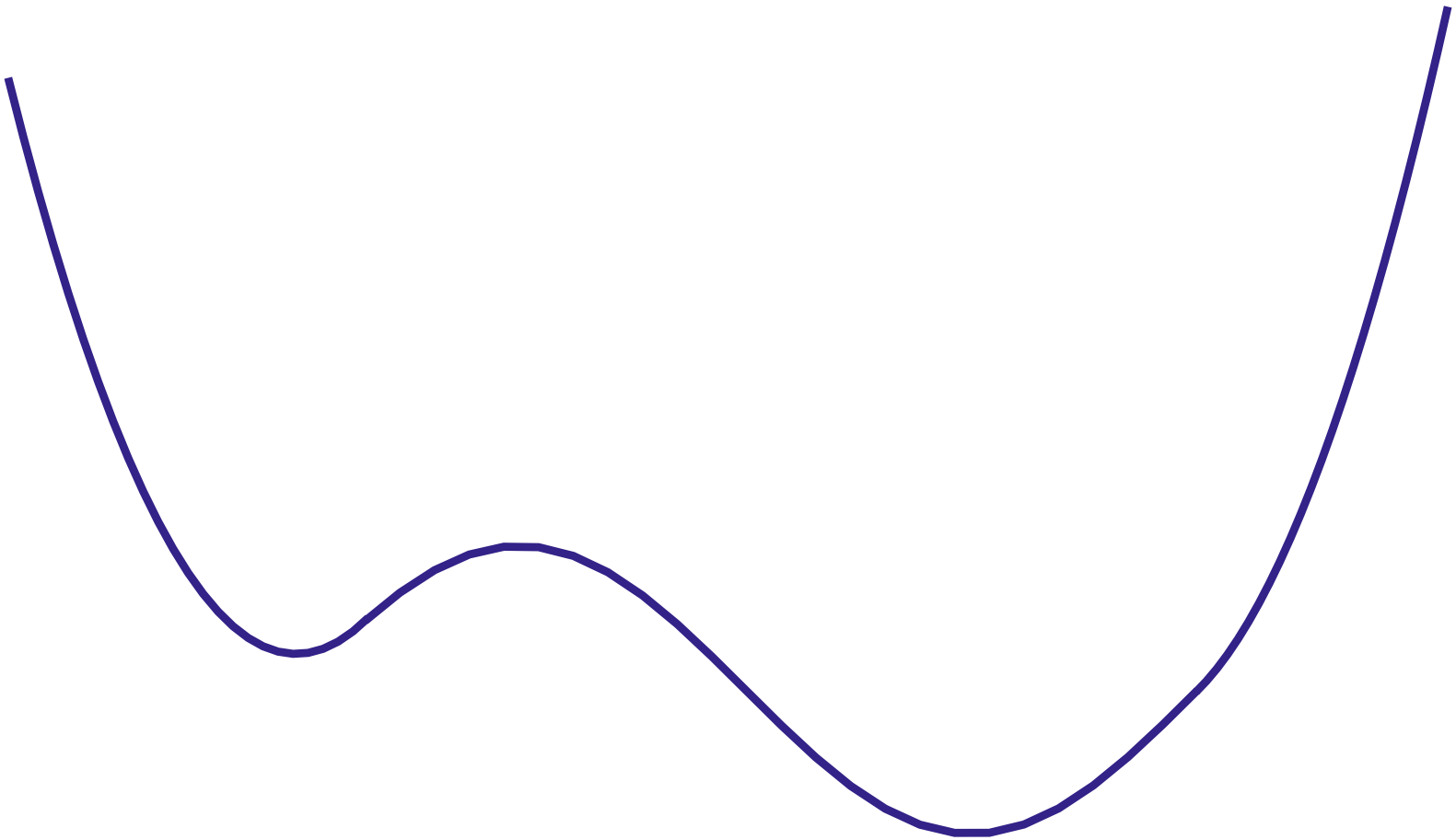
$f''(x) < 0$

"concave down"

Mnemonic

Concavity

Where is the function concave up? Concave down?

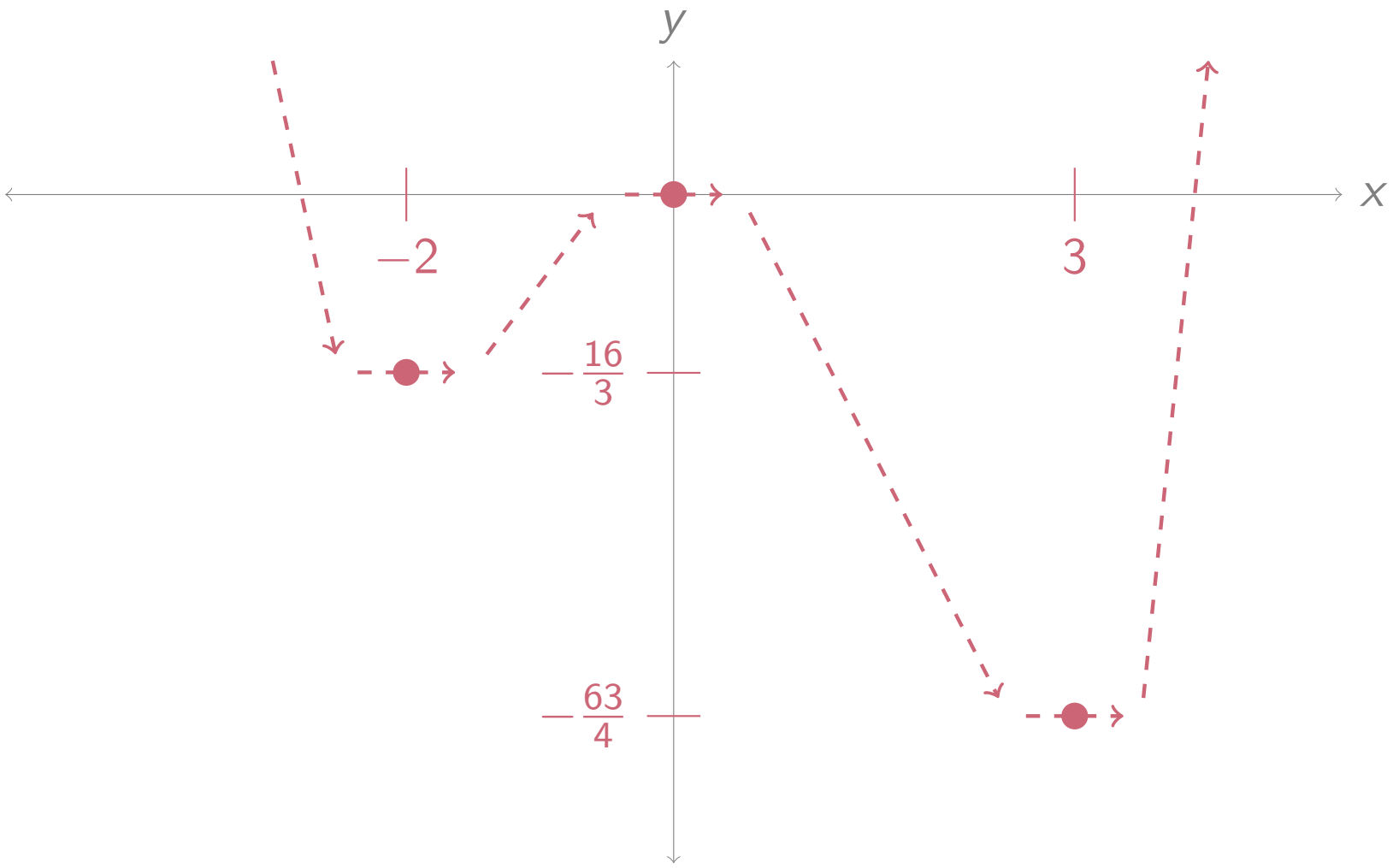


Sketch:

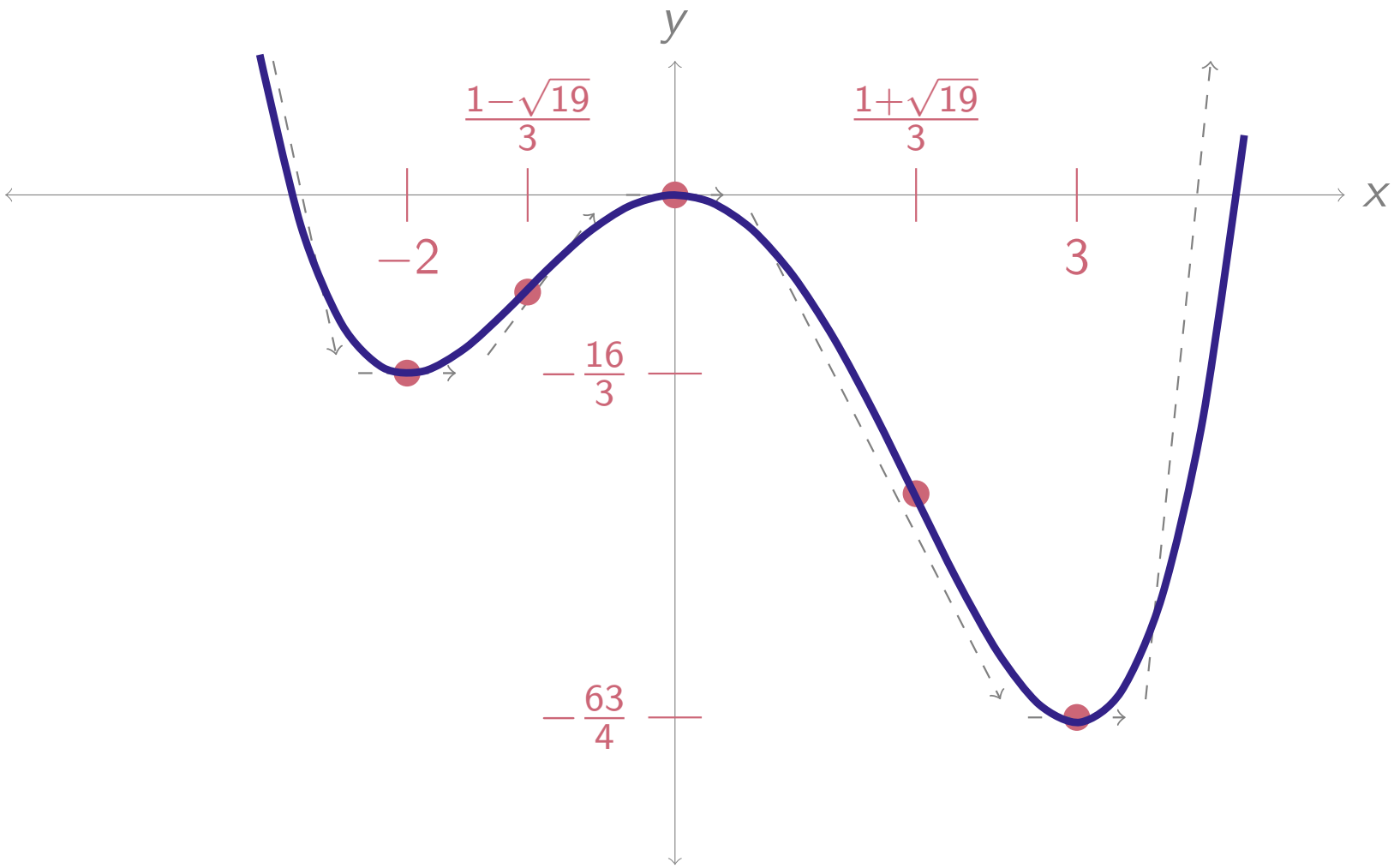
$$f(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 - 3x^2$$

- Consider behaviour near origin and far away from origin
- First derivative: increasing and decreasing; find $f(x)$ at special points
- Second derivative: concavity

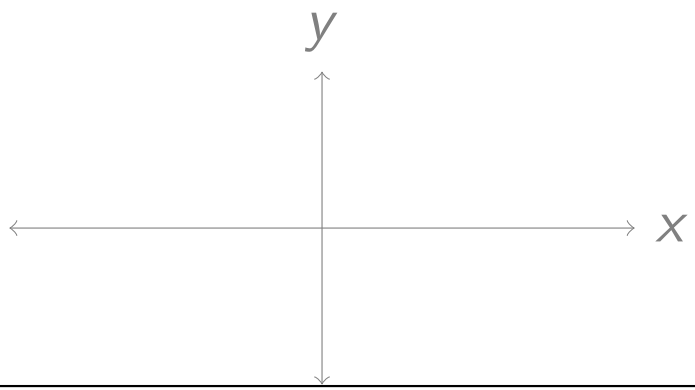
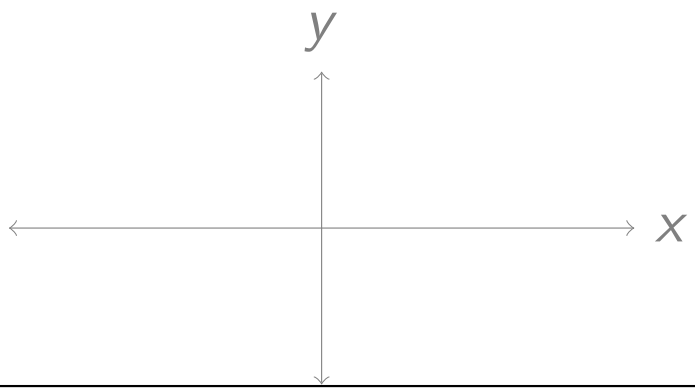
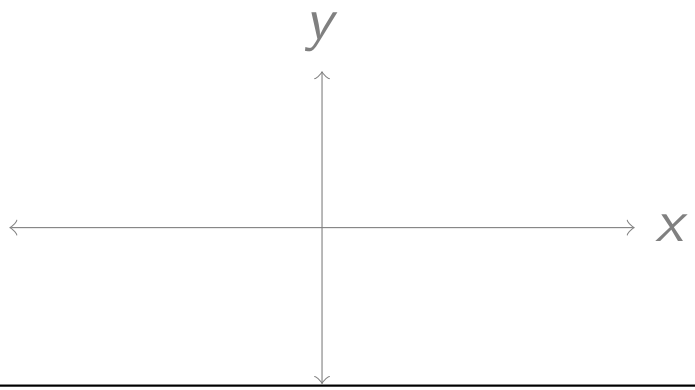
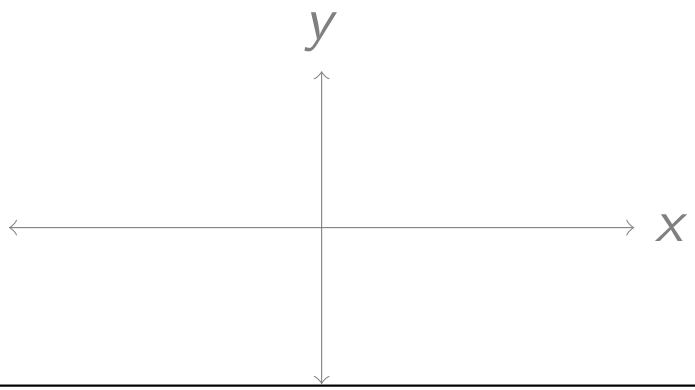
Using info from first derivative:



Using info from second derivative:

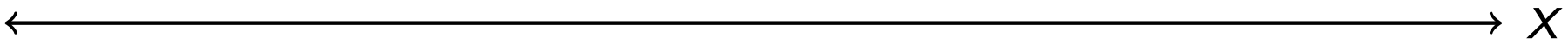


Sketch graphs with the following properties, or explain that none exist.

	concave up	concave down
increasing		
decreasing		

Sign Changes in a Factored Function

$$f(x) = (x-1)\ (x-2)^2\ (x-3)$$

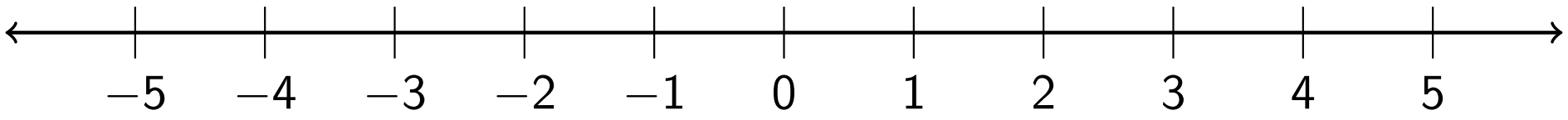


Sign Changes in a Factored Form

Example 1:

$$f(x) = (x-3)(x-1)^2 x(x+2)^3(x+5)^4$$

Where is $f(x)$ positive? Where is it negative?



Special Points:

- Roots / zeroes
- critical point (i.e. $f'(x) = 0$ or $f'(x)$ DNE)
- maxima, minima
- CP that is not an extremum

Each of the following functions have a critical point at $x = 0$.
Match the derivatives with their graphs.

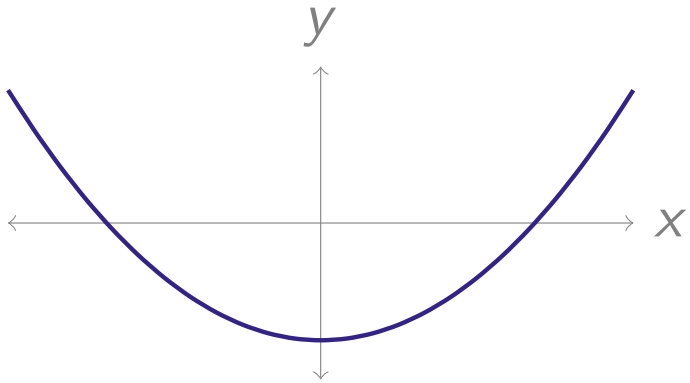
(a) $f'(x)$ negative when $x < 0$
$f'(x)$ negative when $x > 0$

(b) $f'(x)$ positive when $x < 0$
$f'(x)$ positive when $x > 0$

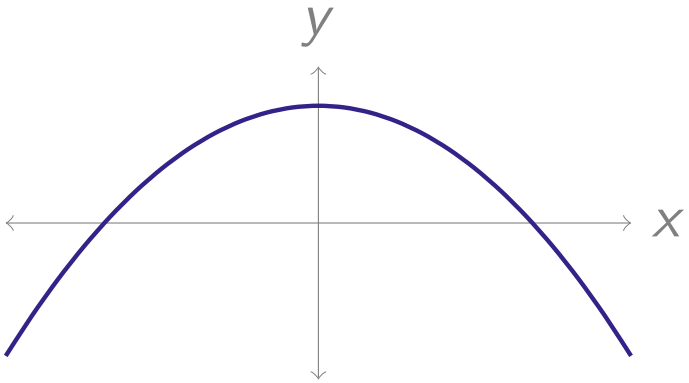
(c) $f'(x)$ negative when $x < 0$
$f'(x)$ positive when $x > 0$

(d) $f'(x)$ positive when $x < 0$
$f'(x)$ negative when $x > 0$

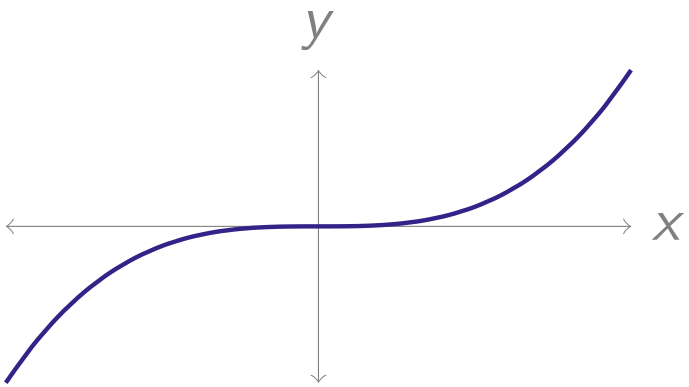
I



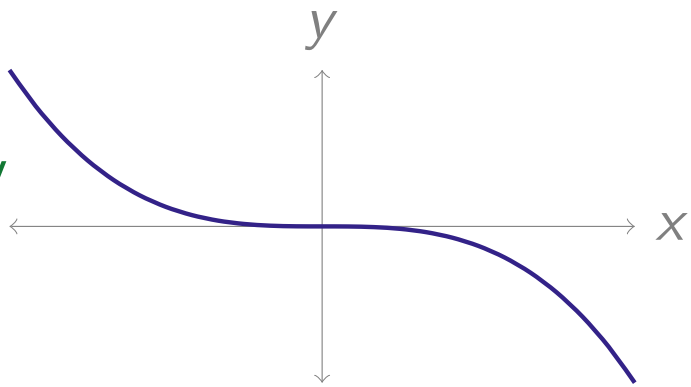
II



III



IV



Match the second derivatives with their graphs.

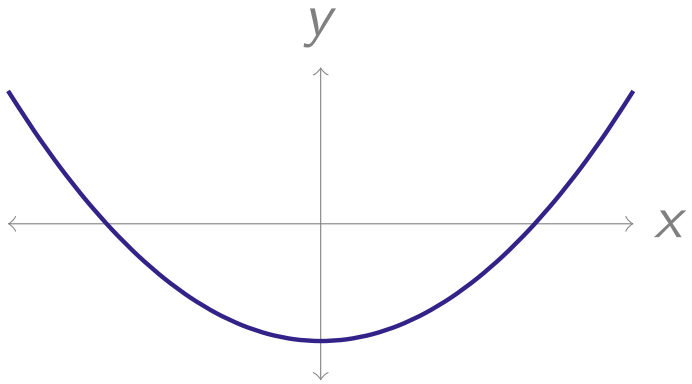
(a) $f''(x)$ negative

(b) $f''(x)$ positive

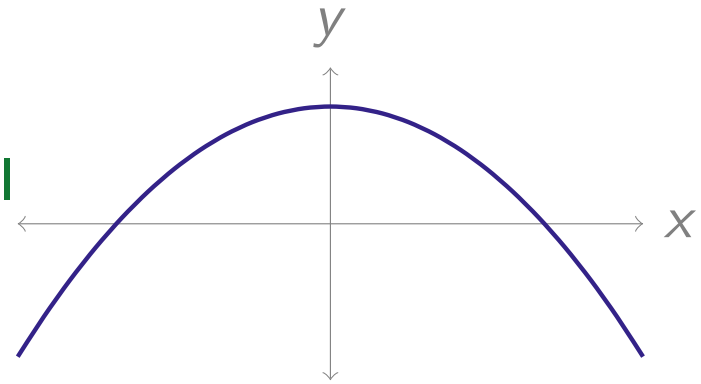
(c) $f''(x)$ negative when $x < 0$
$f''(x)$ positive when $x > 0$

(d) $f''(x)$ positive when $x < 0$
$f''(x)$ negative when $x > 0$

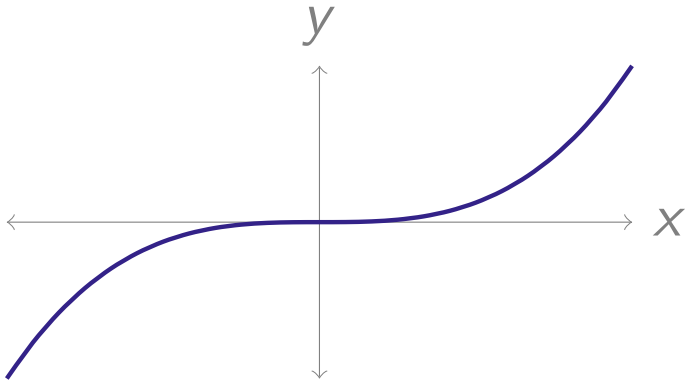
I



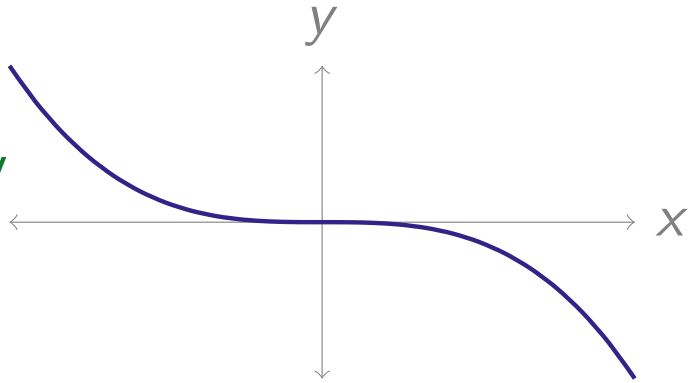
II



III



IV



Suppose $x = a$ is a critical point of the continuous function $f(x)$. That is, $f'(a) = 0$ or $f'(x)$ does not exist at a.

(a) $f'(a)$ changes from neg to pos

(b) $f'(a)$ changes from pos to neg

(c) $f''(a) = 0$

(d) $f''(a)$ changes from neg to pos

(e) $f''(a)$ changes from pos to neg

(f) $f''(a)$ positive and $f'(a) = 0$

(g) $f''(a)$ negative and $f'(a) = 0$

I. inflection point

II. local max

III. local min

IV. not a local extrema

V. could be local max, local min, or neither

Tests

The information from the previous questions forms a method for finding local extrema.

- First, find all critical points (where $f'(x) = 0$ or $f'(x)$ doesn't exist). Local extrema can ONLY occur at critical points.
- Now, we have to classify those critical points: they could be local maxes, local mins, or neither. Two options:
- Using the second derivative:
  - If $f''(x) > 0$, the CP is a local
  - If $f''(x) < 0$, the CP is a local
  - If $f''(x) = 0$,
- Using the first derivative:
  - If $f'(x)$ does not change signs at the CP, then
  - If $f'(x)$ changes from increasing to decreasing, the CP is a local
  - If $f'(x)$ changes from decreasing to increasing, the CP is a local

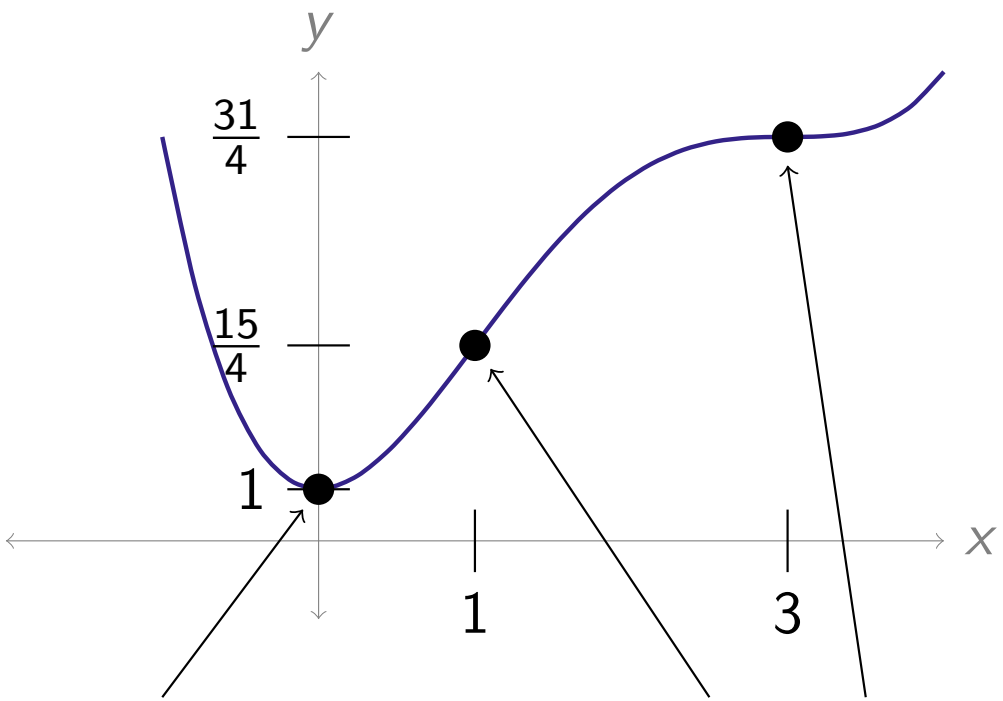
$$f(x) = \frac{1}{4}x^4 - 2x^3 + \frac{9}{2}x^2 + 1$$

1. Find all critical points of $f(x)$
2. Are they local maxima, local minima, or neither?

Challenge: sketch the function.

$$f(x) = \frac{1}{4}x^4 - 2x^3 + \frac{9}{2}x^2 + 1$$

Sketch.

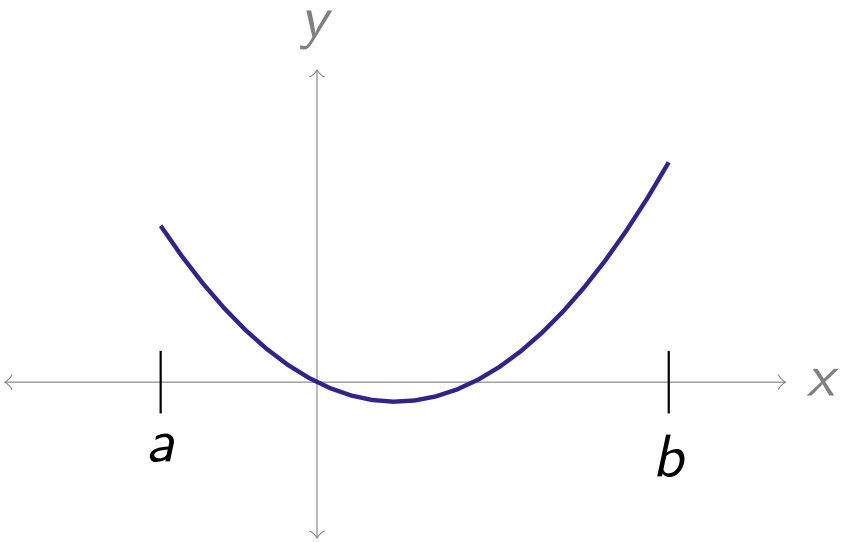


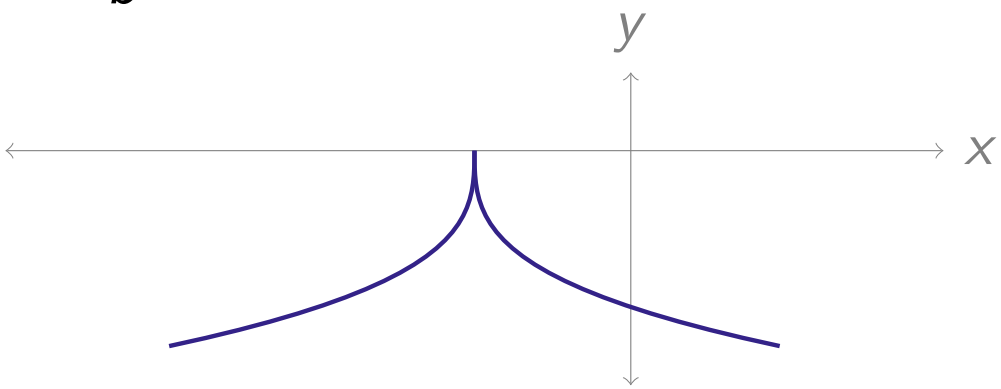
local min at $(0, 1)$
also **global** min

inflection points at $(3, \frac{31}{4})$
and $(1, \frac{11}{4})$

Global extrema over a closed interval

Where do global extrema occur?





Cell Division

A cell of age t hours has a probability[1] of dividing given by the function

$$P(t) = \frac{at}{t^3 + 16},$$

where a is the constant $\frac{9\sqrt[6]{3}}{2\pi}$.

Frpm time $t = 0$ to time $t = 10$, when is the cell likeliest to divide? When is it least likely to divide?

[1] Actually, we're giving a probability distribution function... it's a little more complicated than what we said. Still, a higher value of $P(t)$ means a cell age that is more reproduce-y, so we just want to find the max and min of $P(t)$.

Sketch

$$f(x) = \frac{(x-1)^2}{x^3}$$

Include the following, if they exist:

- Roots (zeroes)
- Discontinuities
- Asymptotes
- Local and global extrema (maxes and mins)
- Inflection points and concavity

Sketch

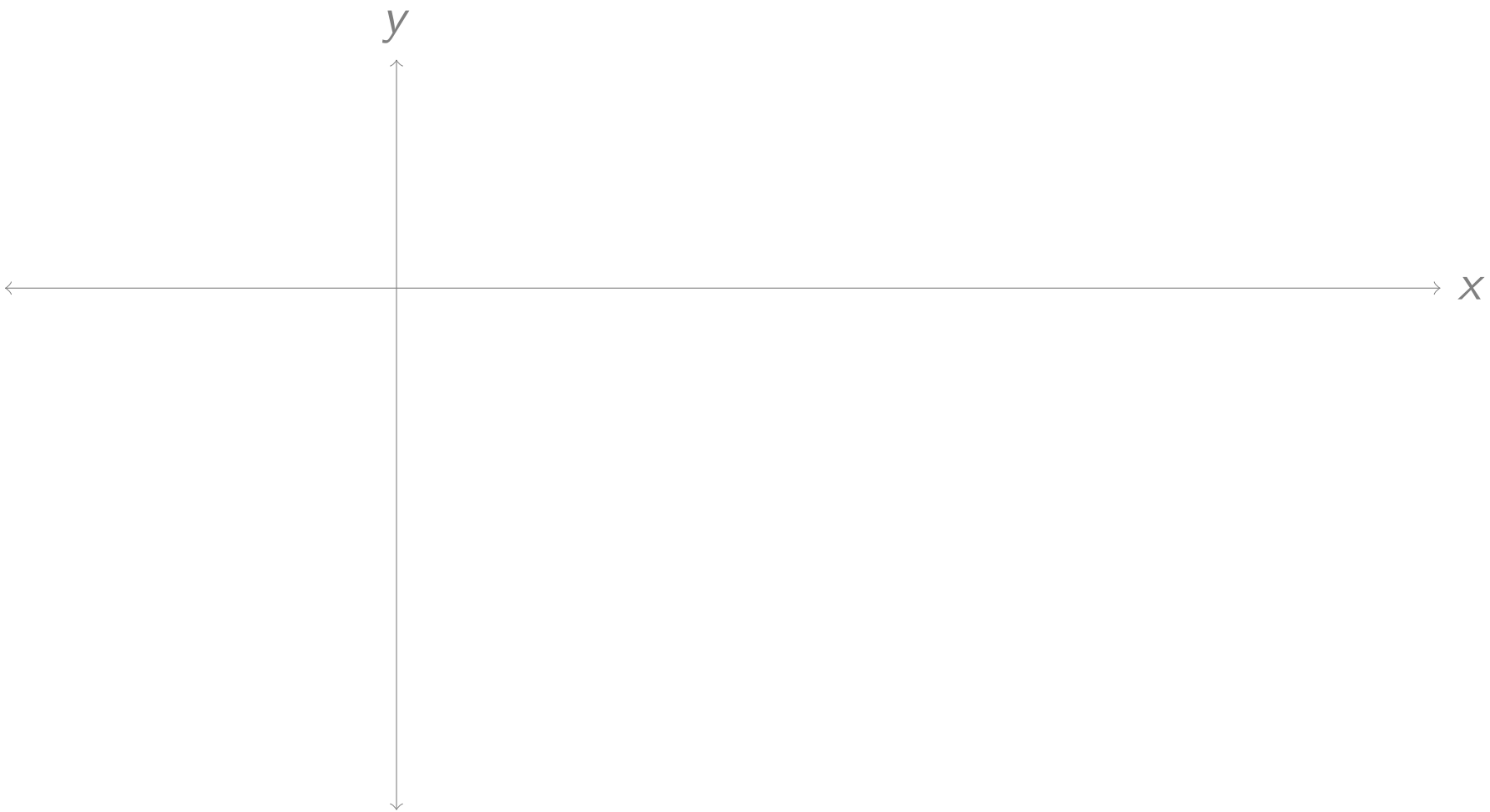
$$f(x) = \frac{(x-1)^2}{x^3}$$

$$f'(x) = \frac{-(x-1)(x-3)}{x^4}$$

$$f''(x) = \frac{2(x^2-6x+6)}{x^5}$$

Sketch

$$f(x) = \frac{(x-1)^2}{x^3}$$



Sketch:

$$f(x) = \frac{10x}{x^3 + 16}$$

- $f'(x) = \frac{20(8-x^3)}{(x^3+16)^2}$
- $f''(x) = \frac{60x^2(x^3-32)}{(x^3+16)^3}$