Overview

Graphing More Accurately

- First and second derivatives

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- Critical points
- Extrema

Up with hope, down with dope,
Increasing functions have positive slope

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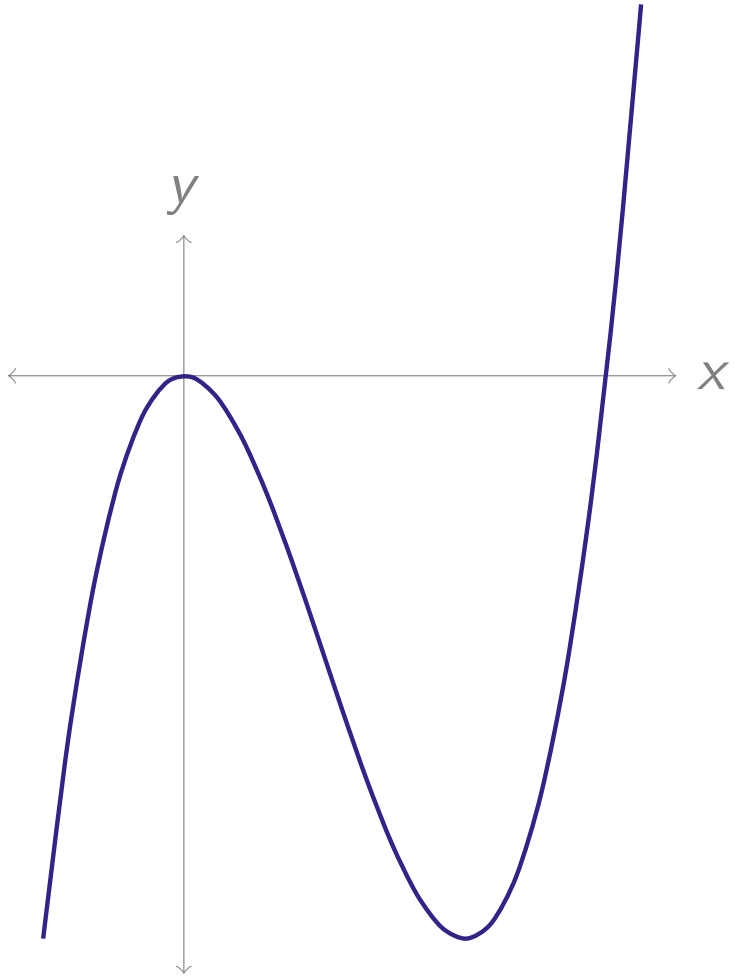

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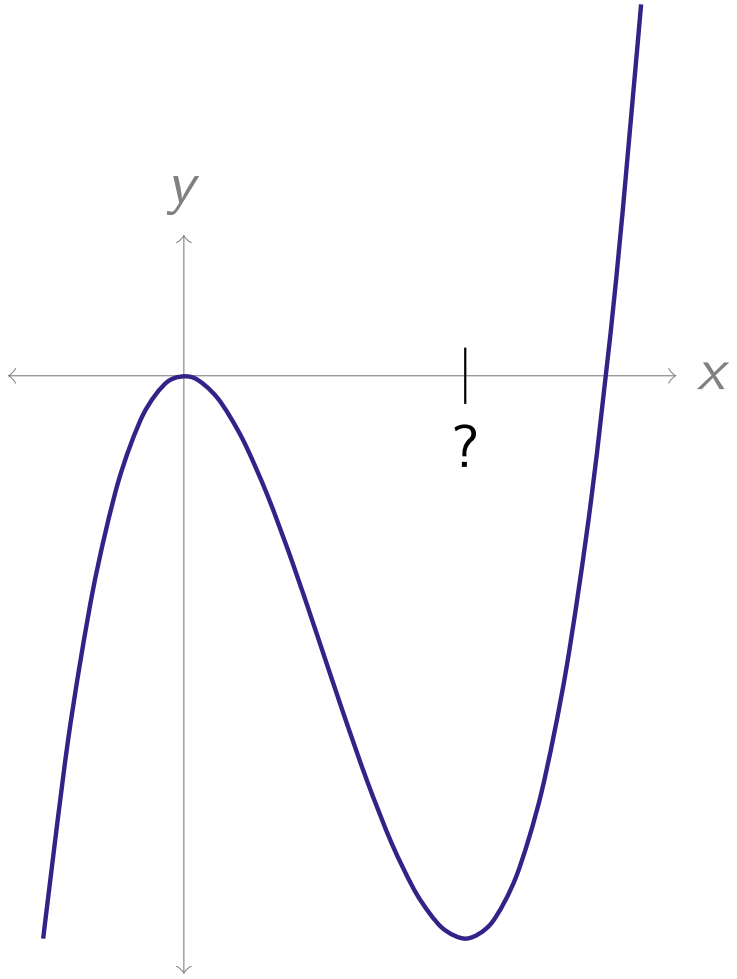

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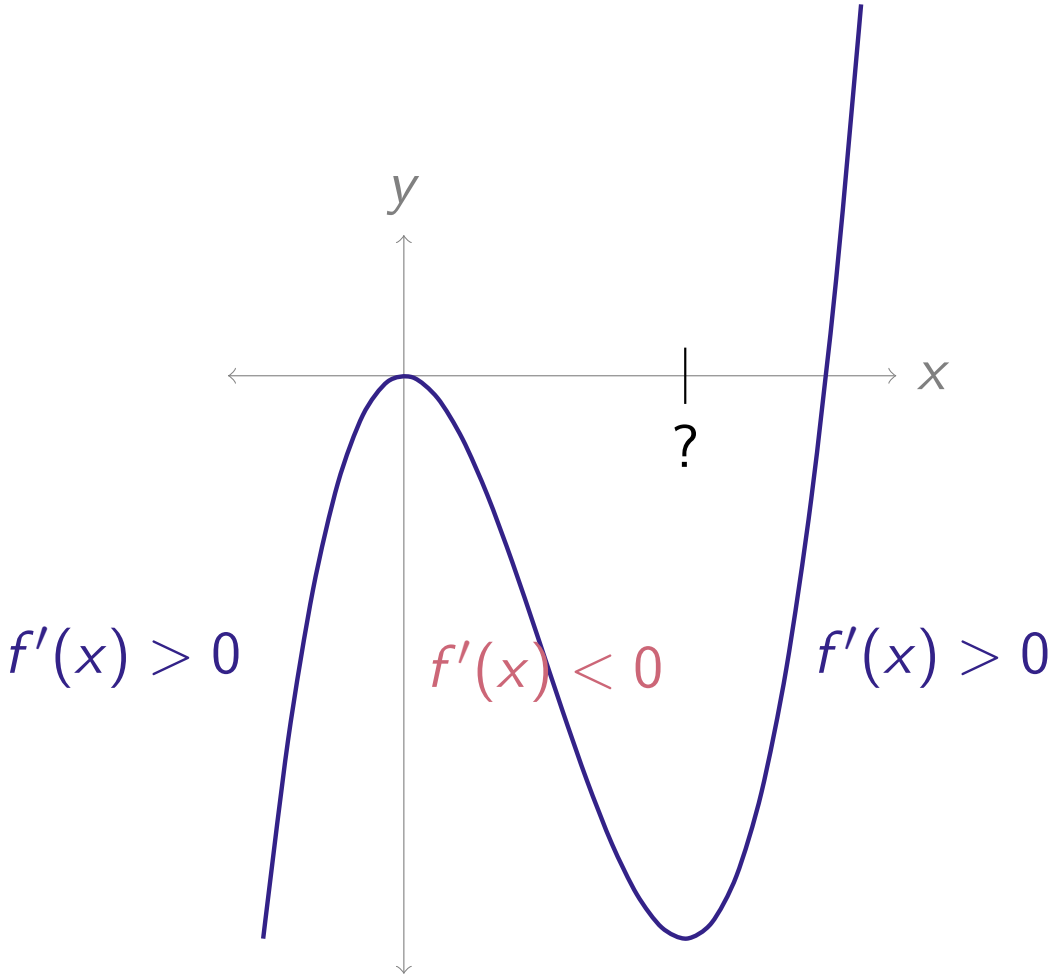

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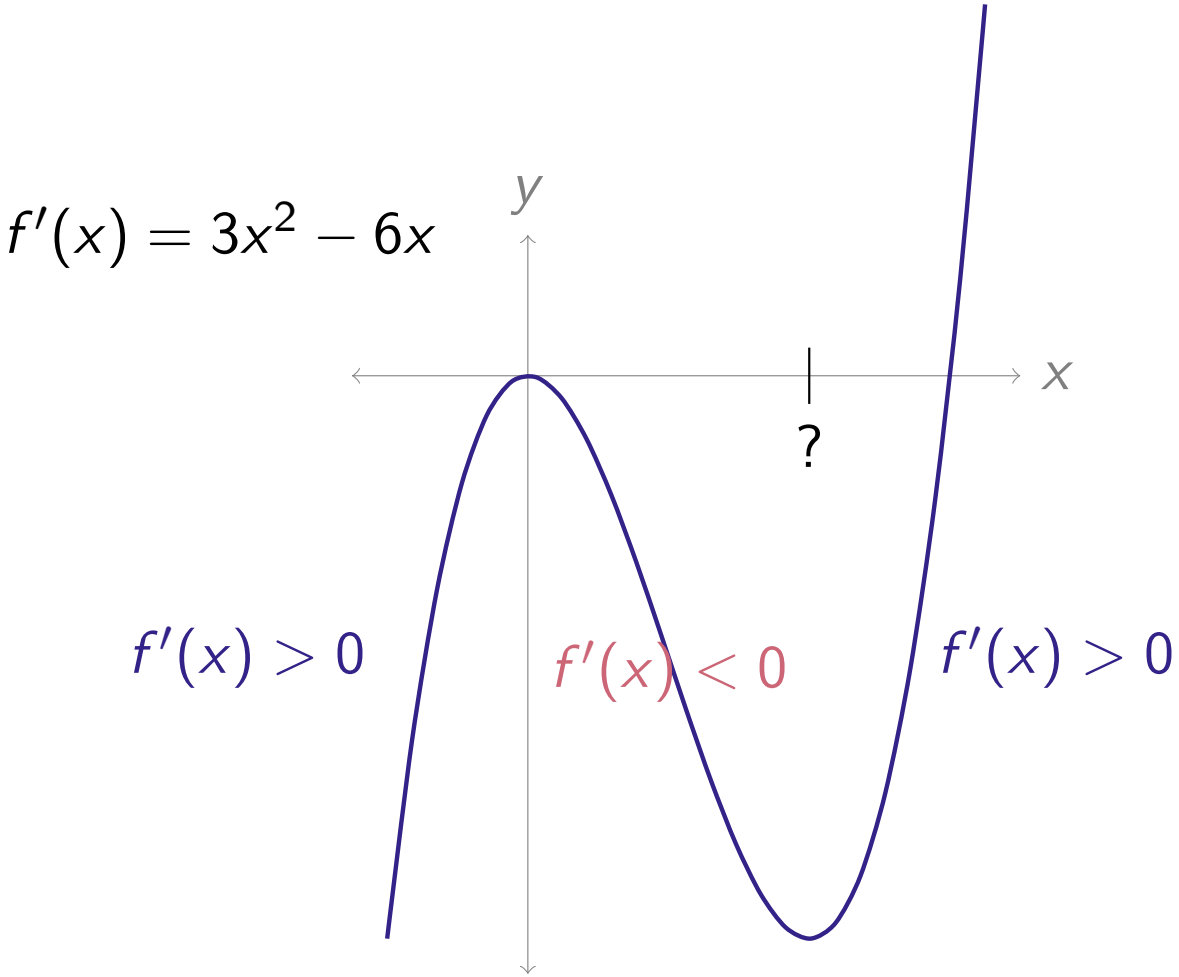

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$$f'(x) = 3x^2 - 6x$$
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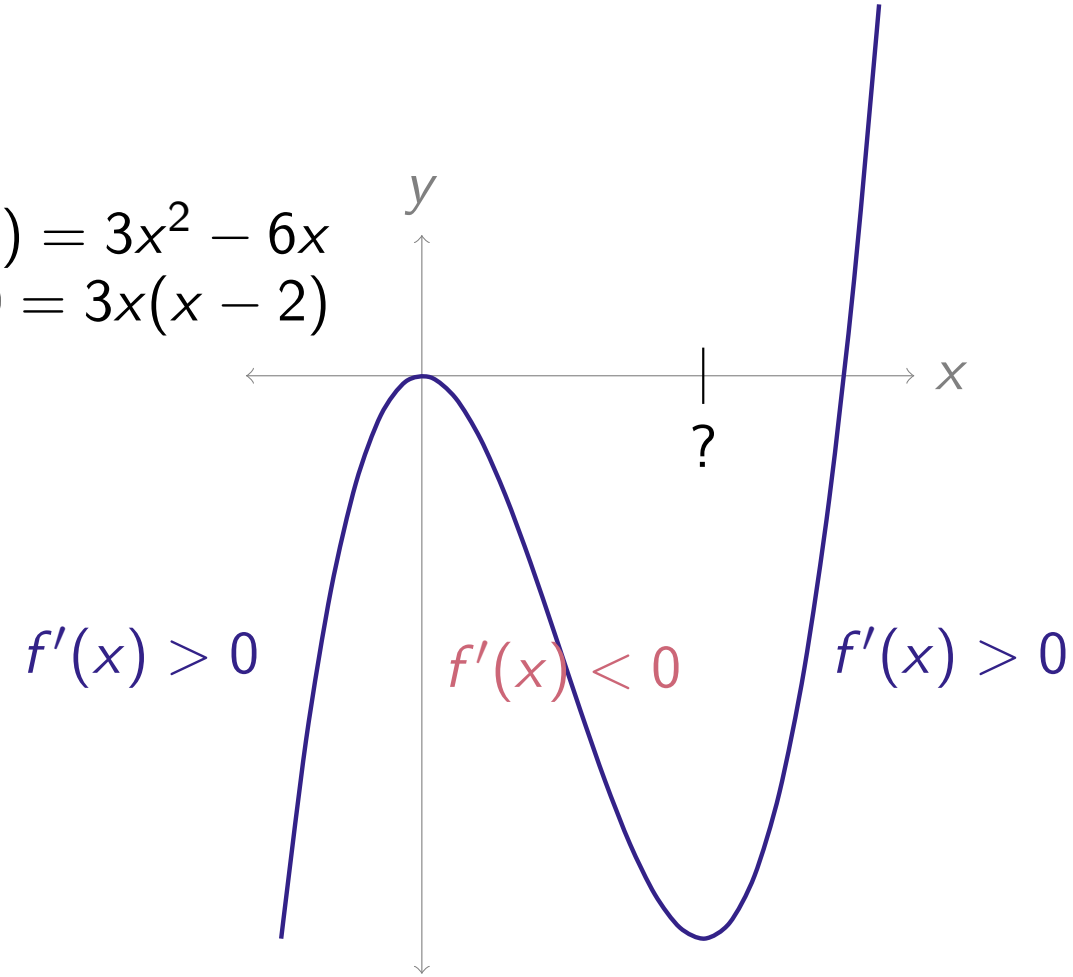

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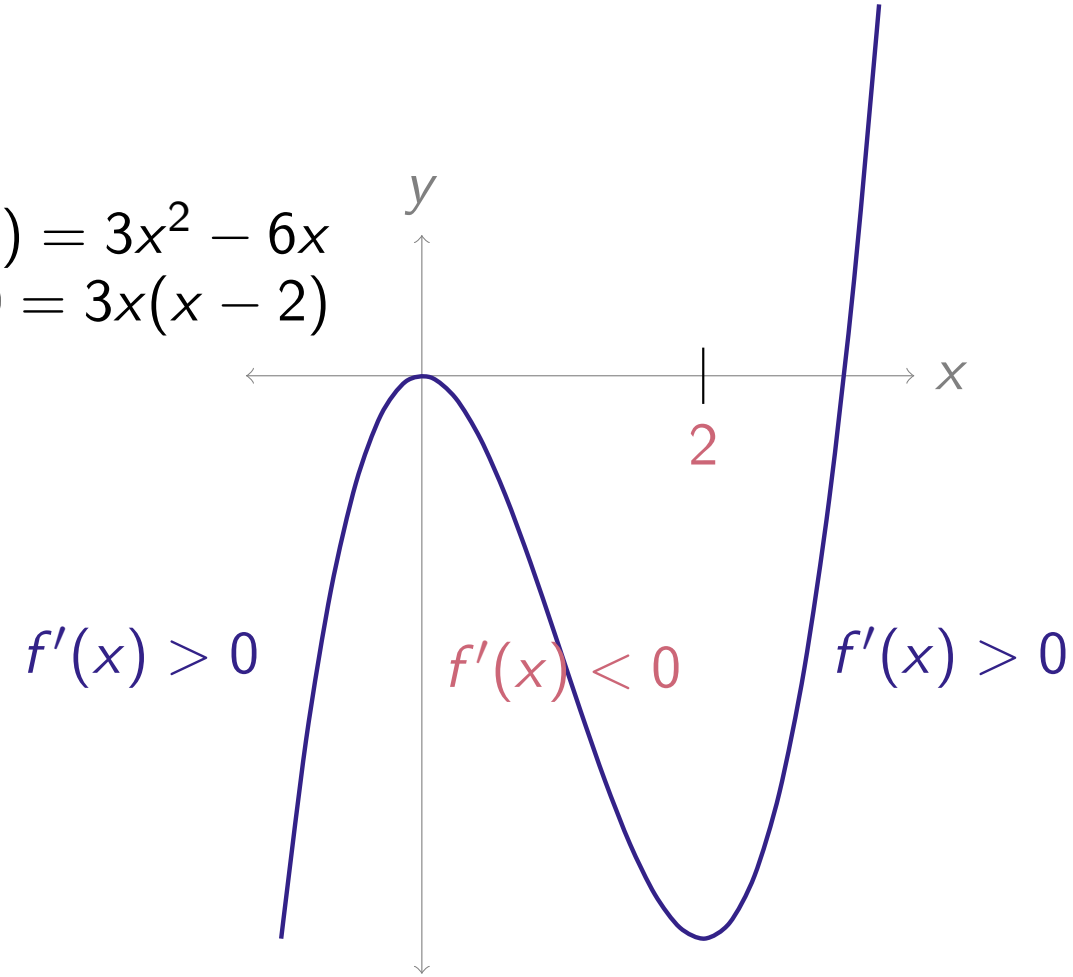

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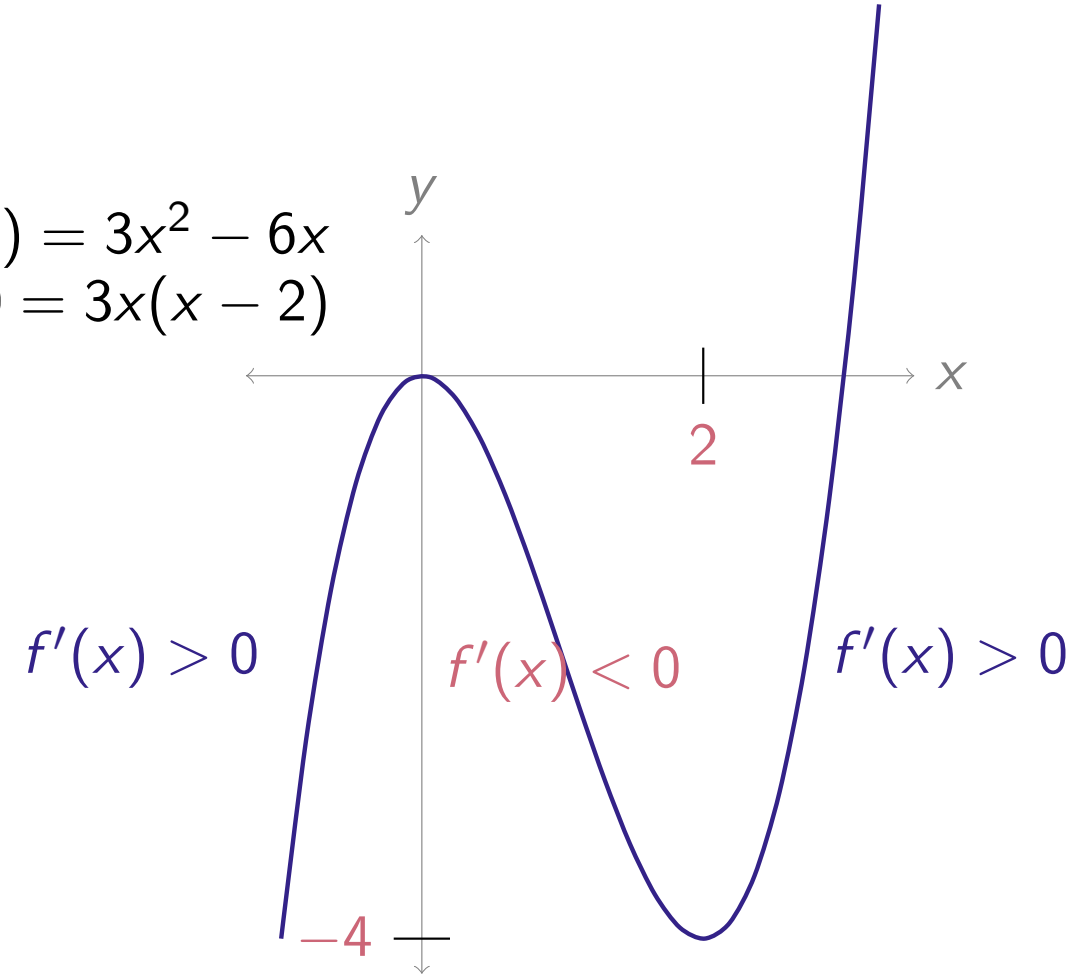

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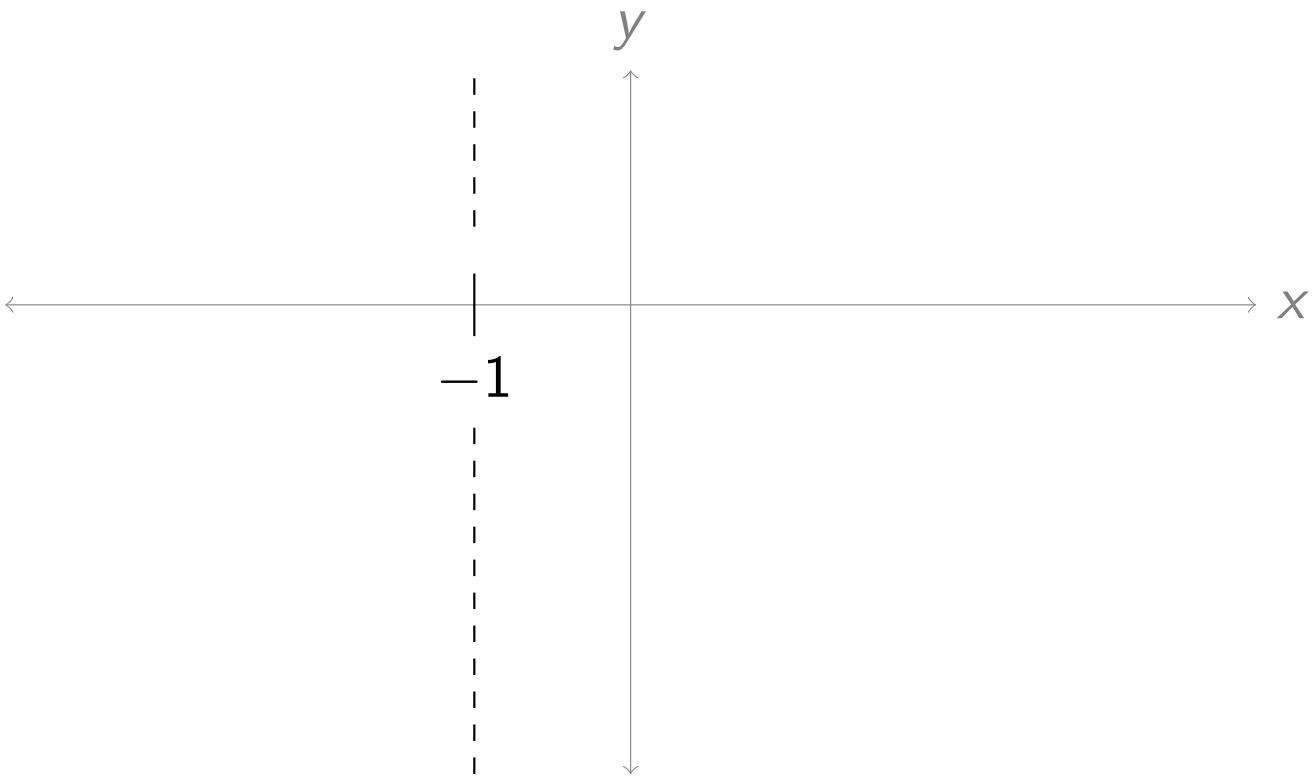

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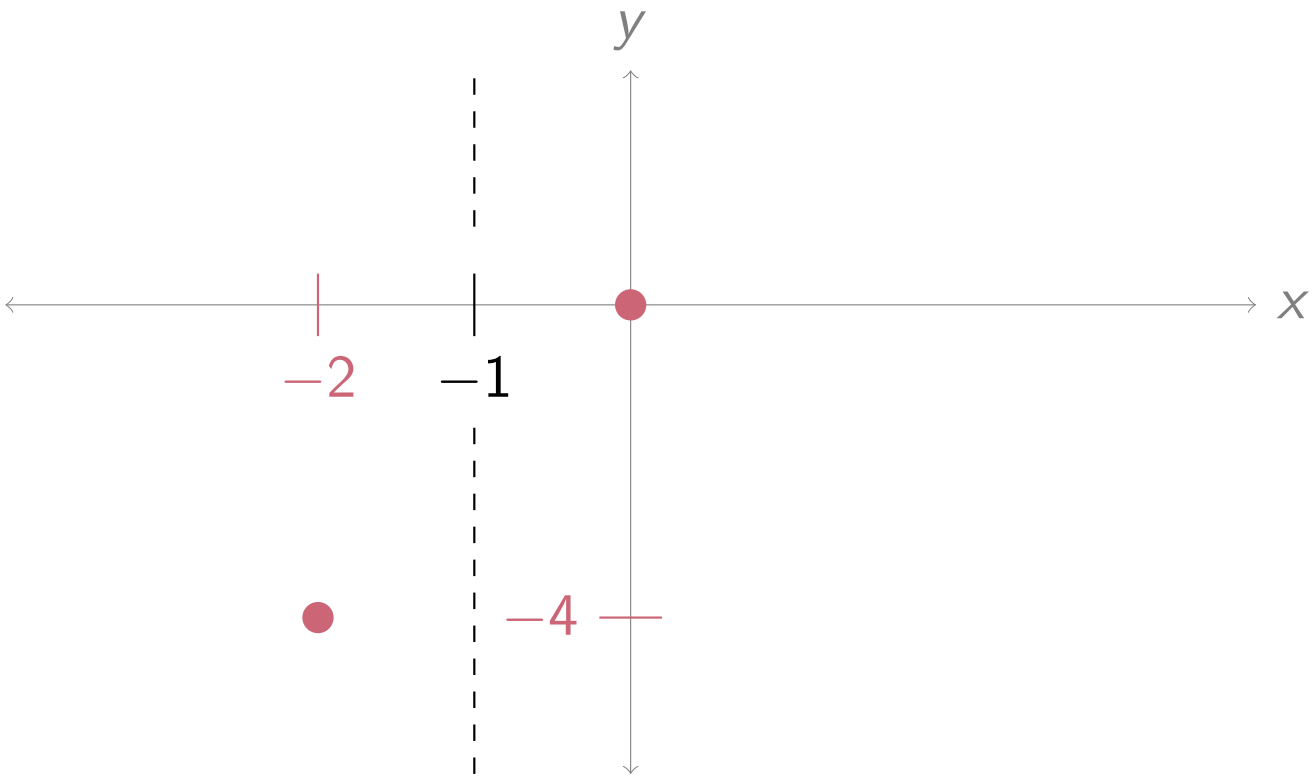

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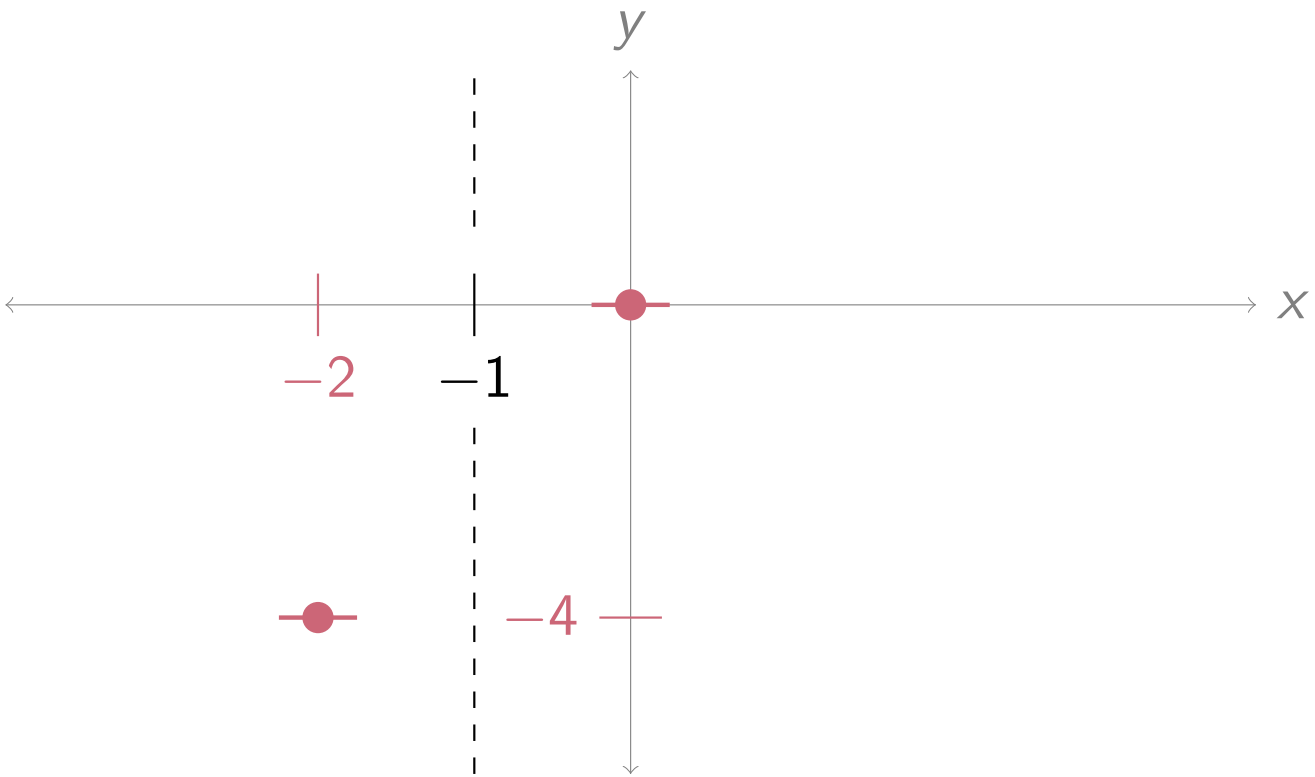

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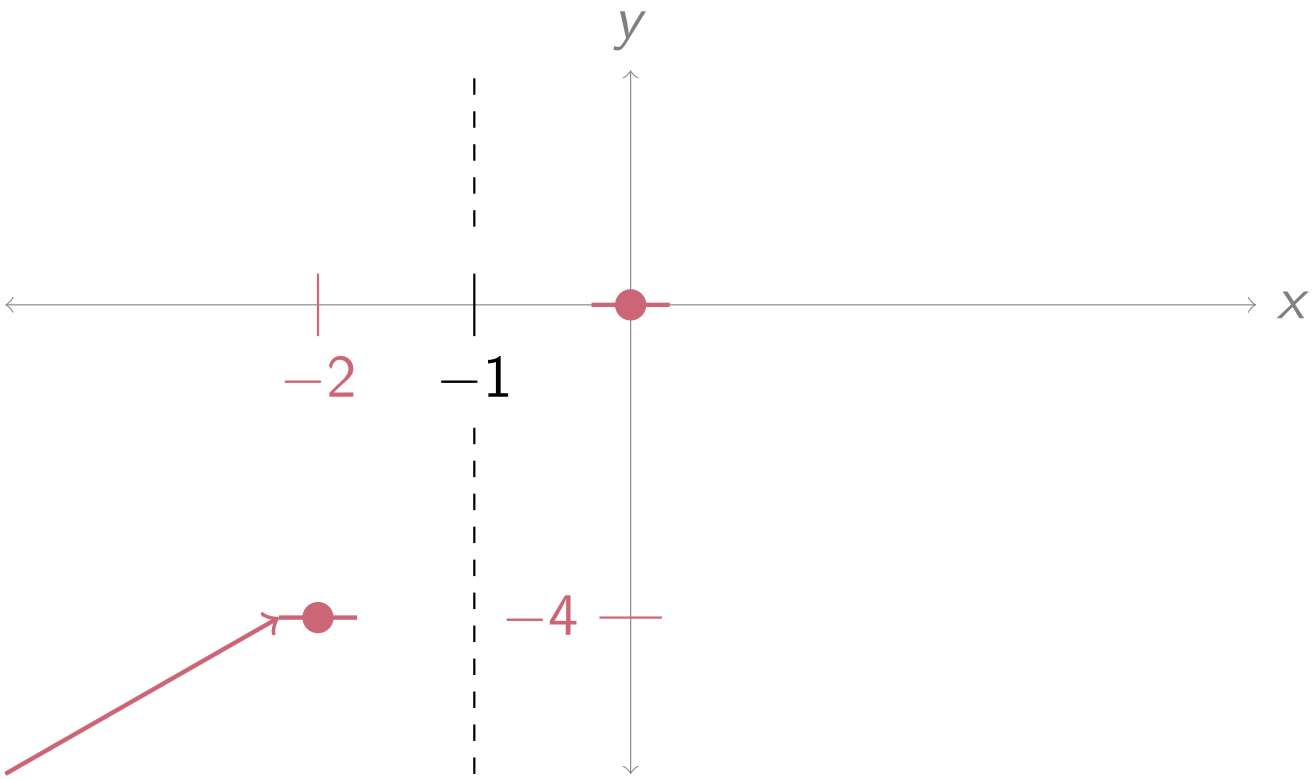

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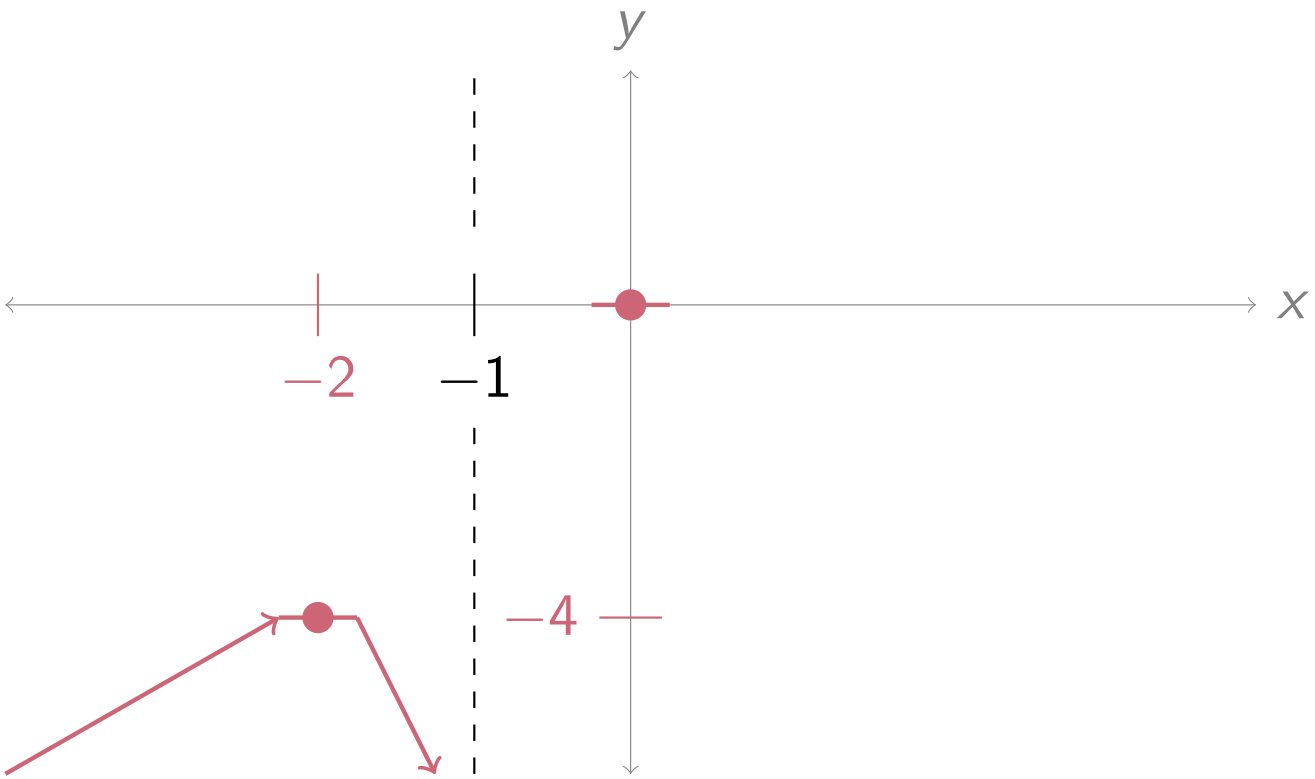

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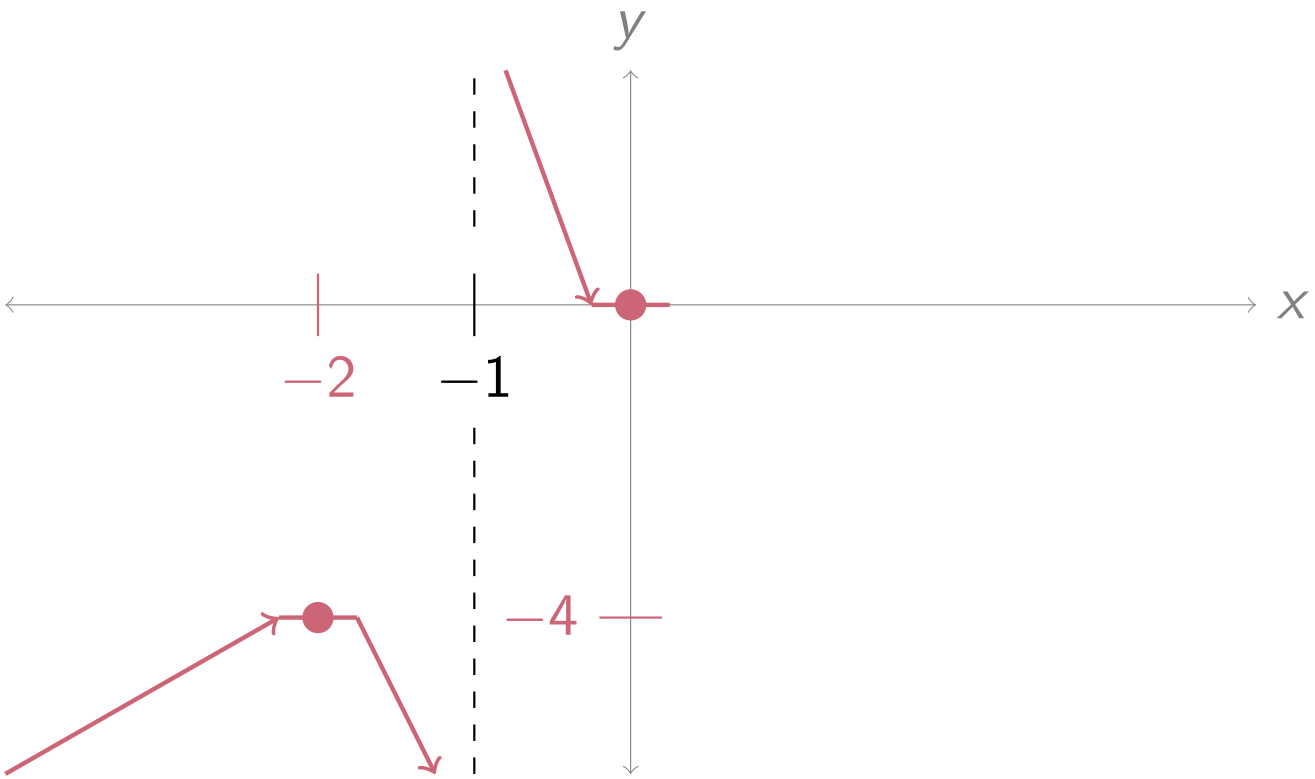

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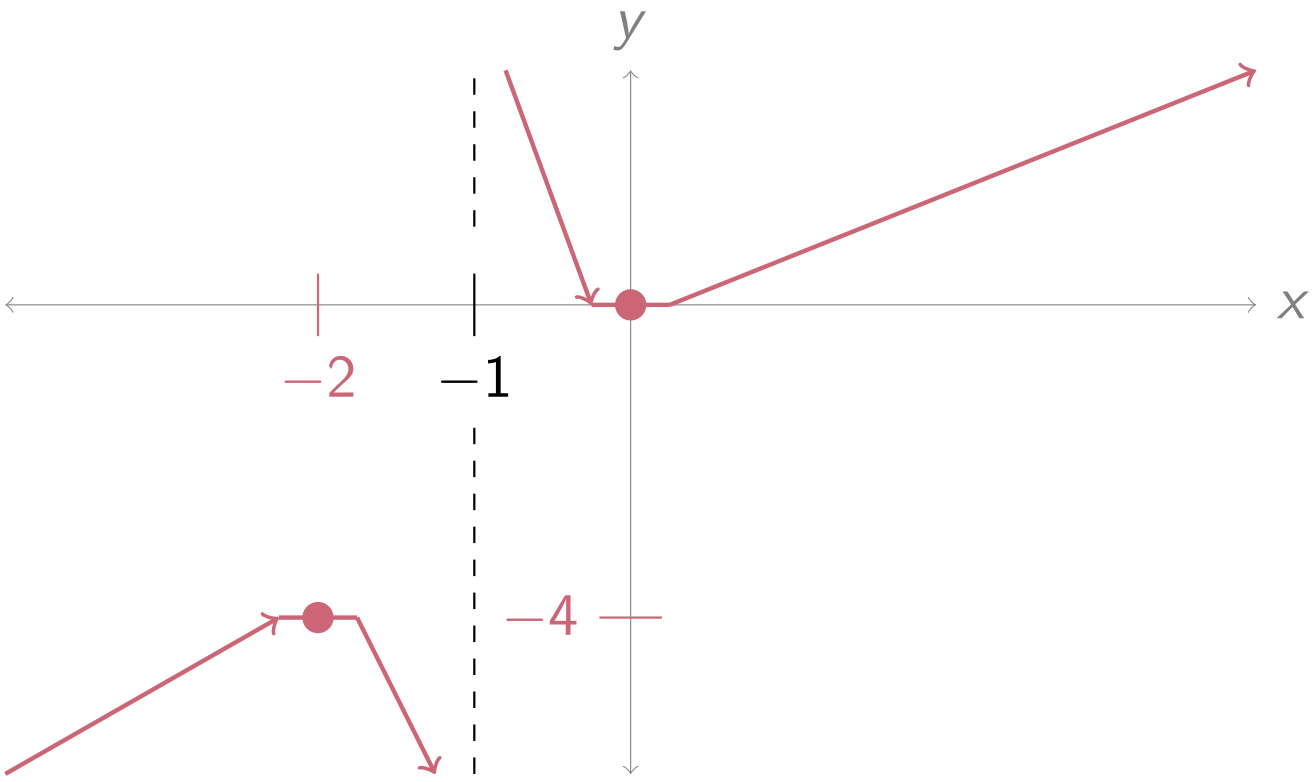

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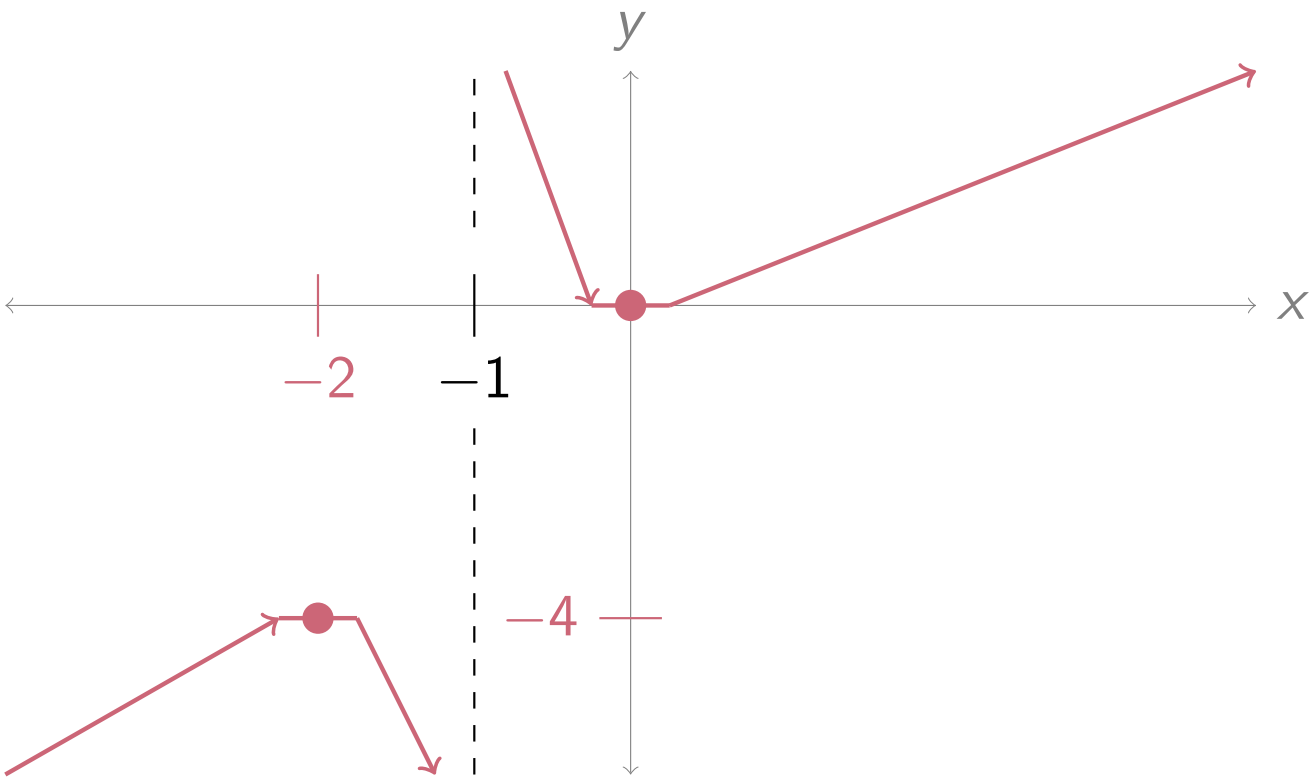

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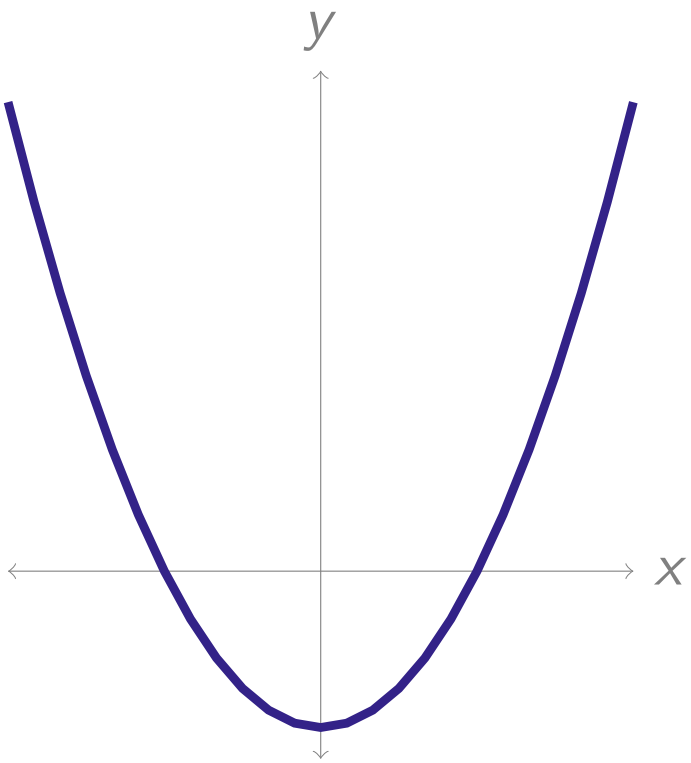

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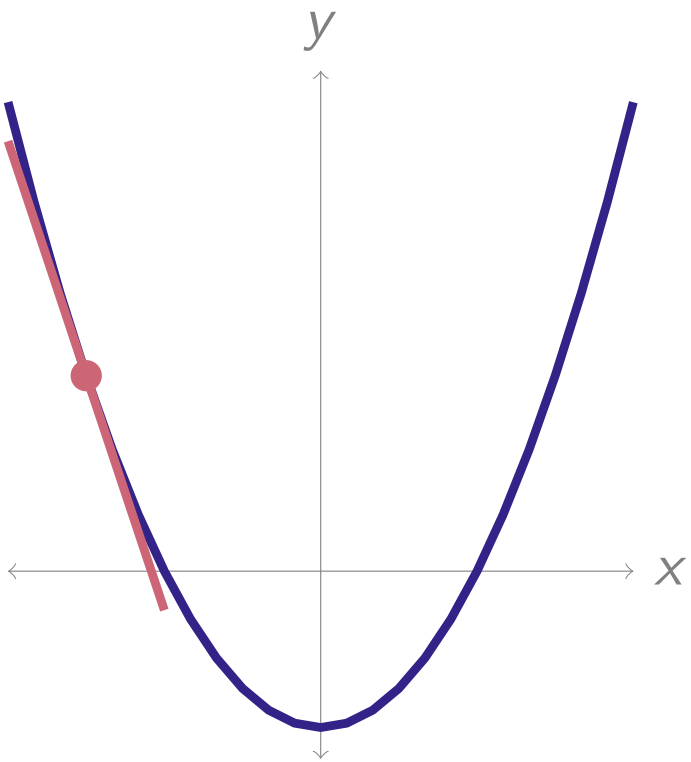

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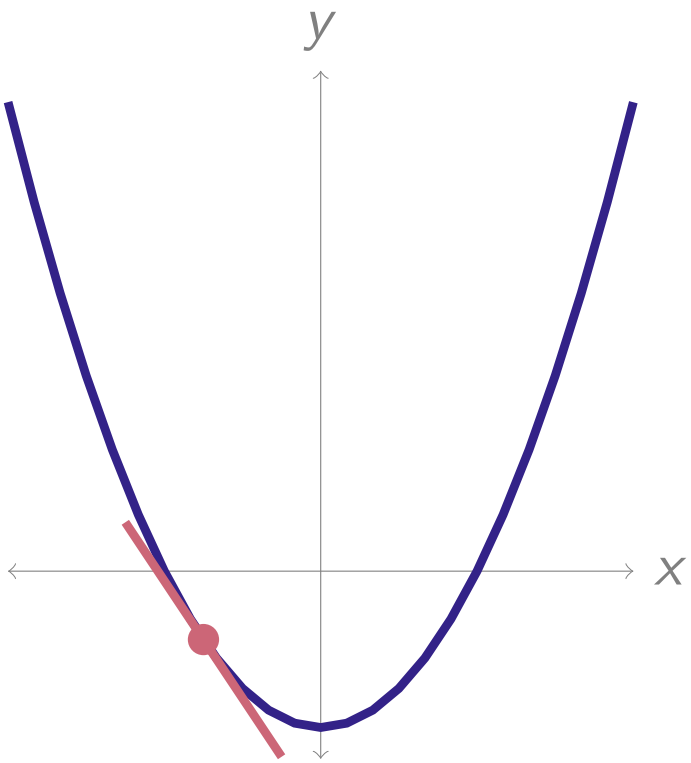

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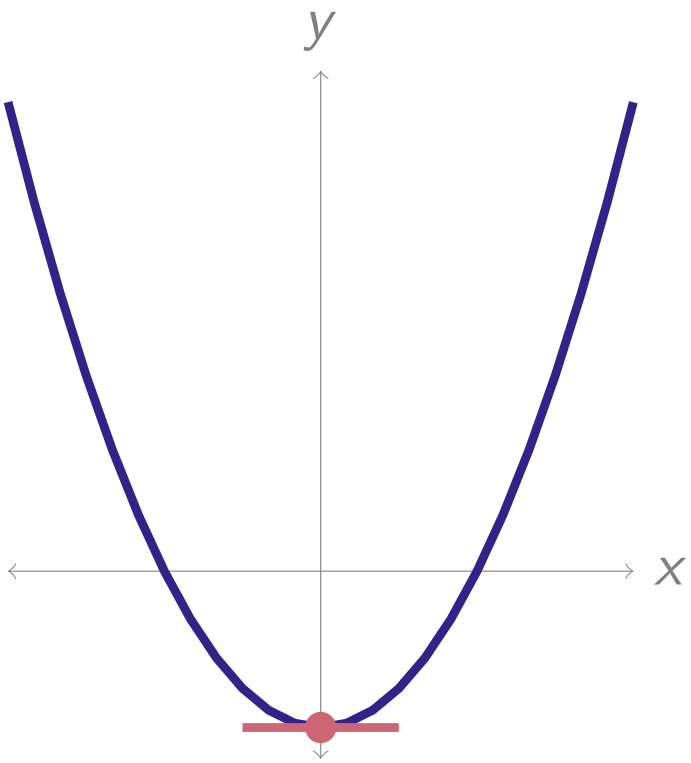

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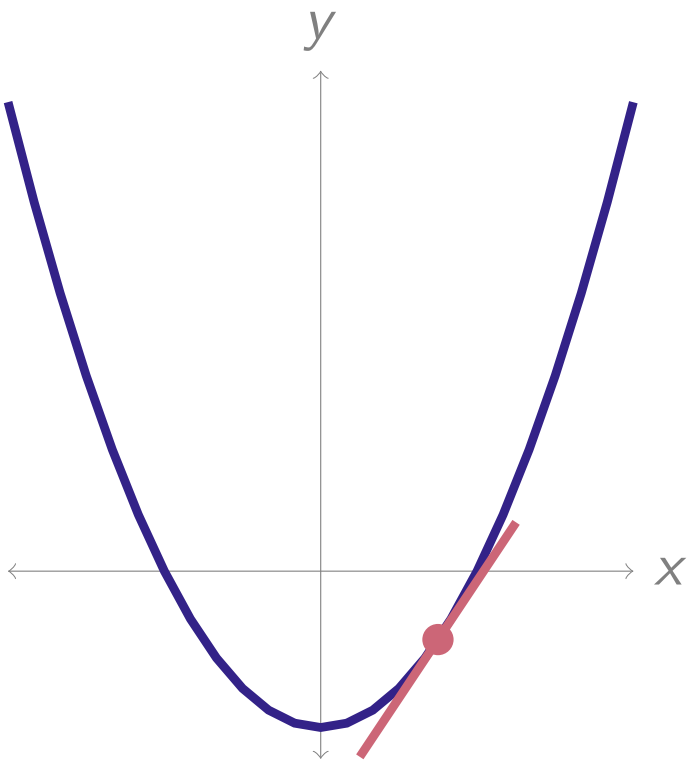

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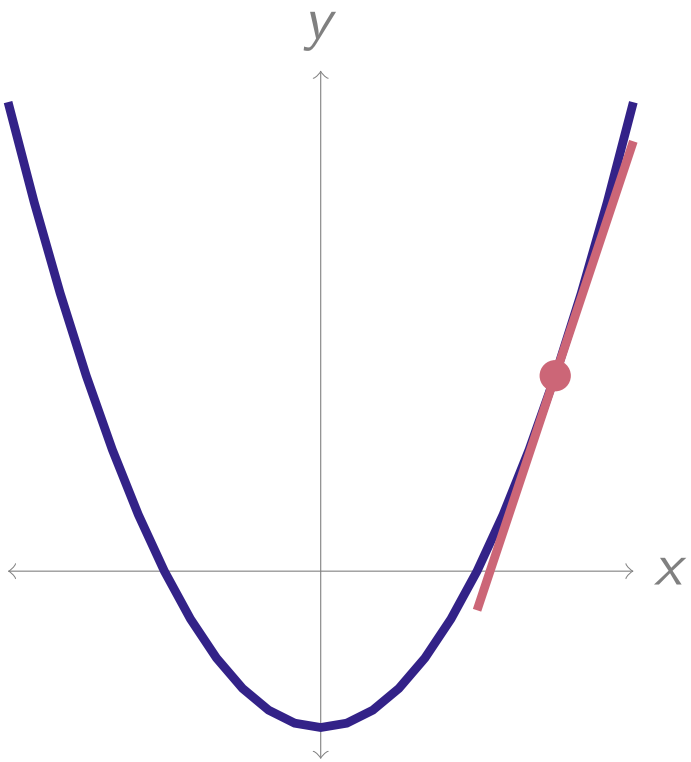

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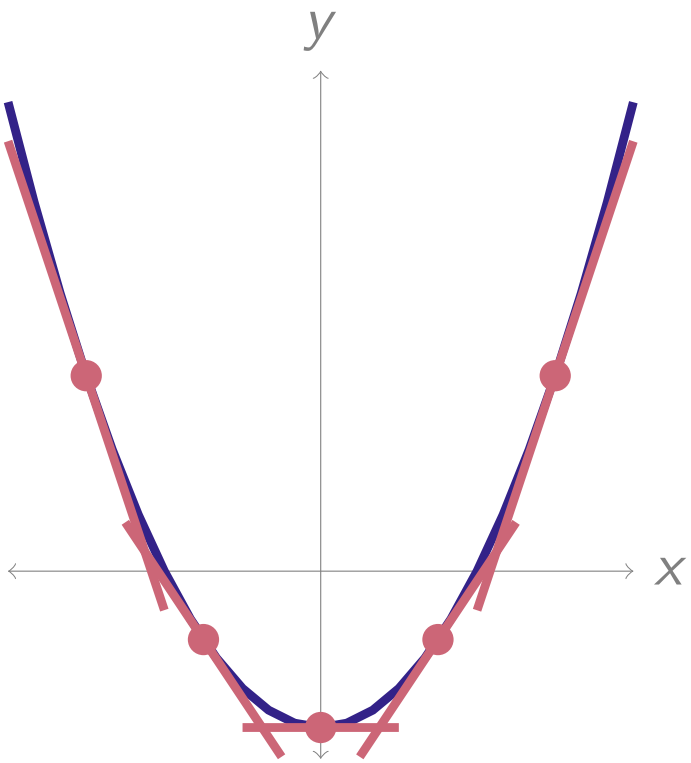

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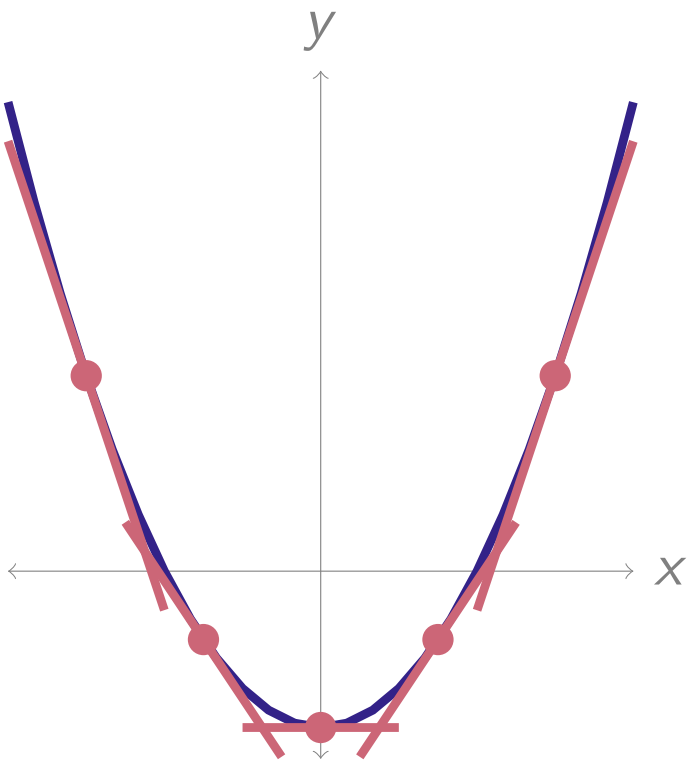

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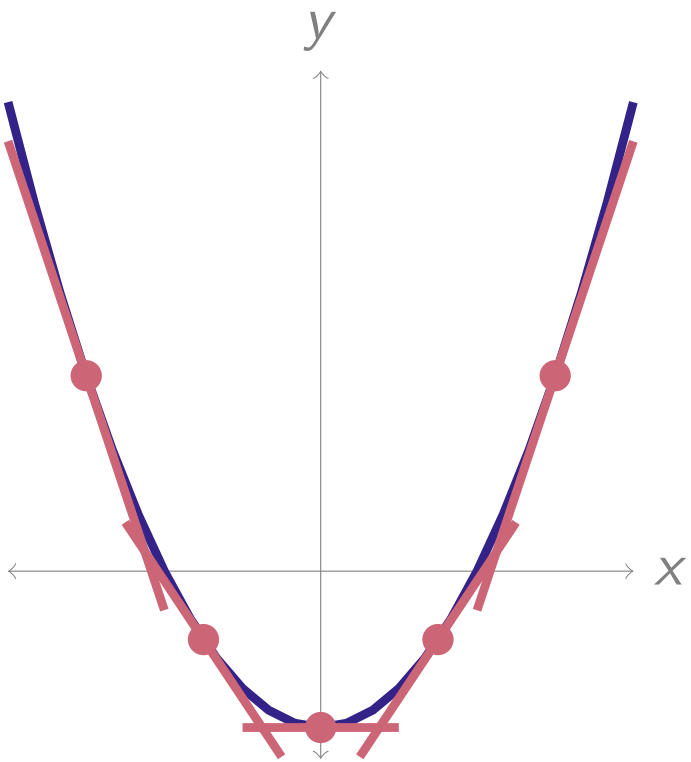

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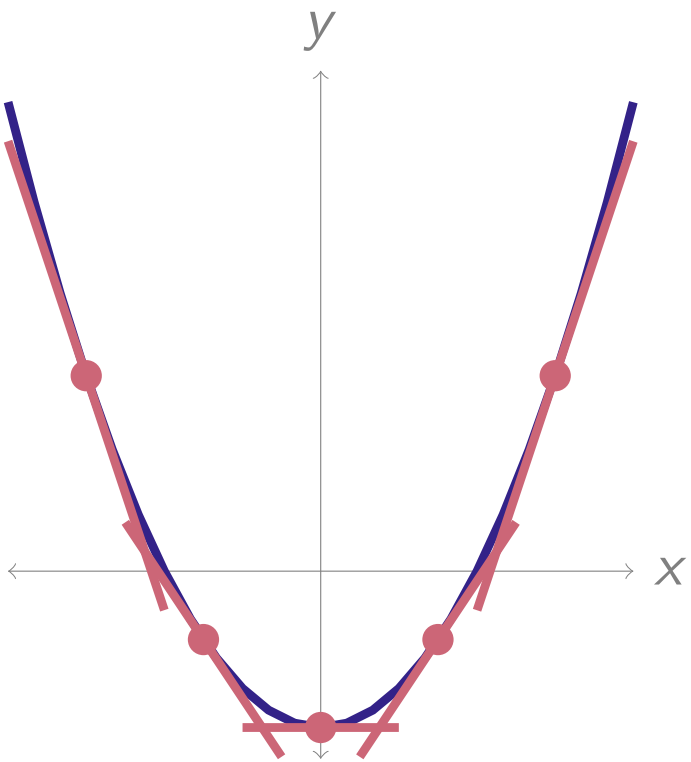

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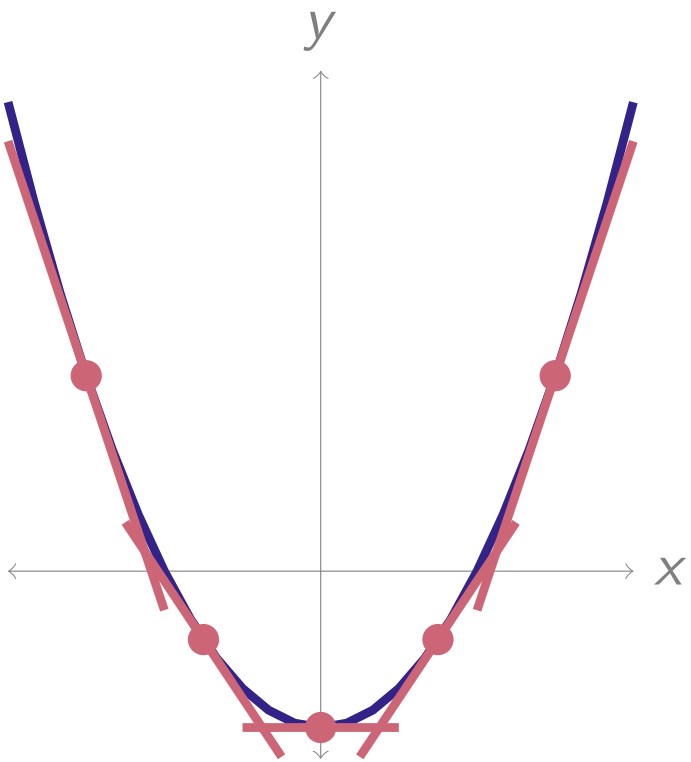

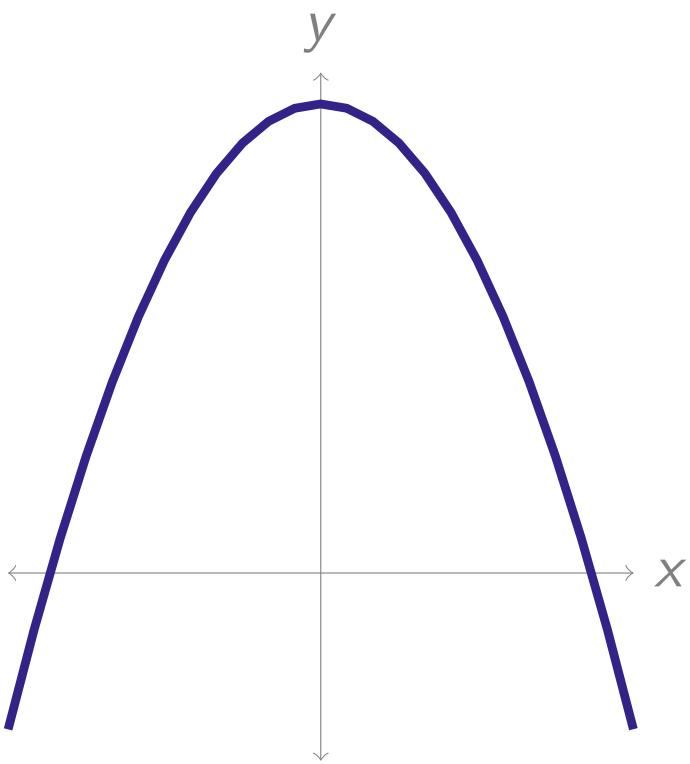

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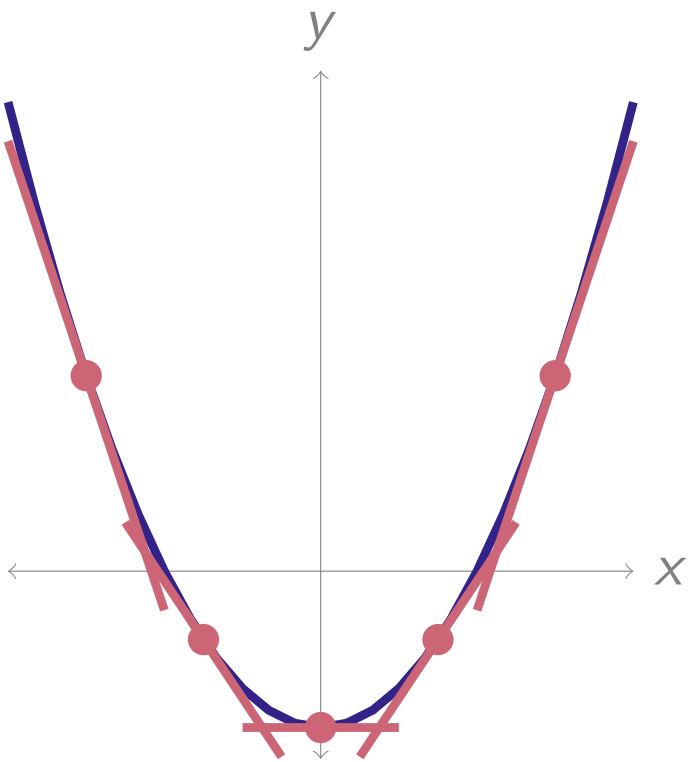

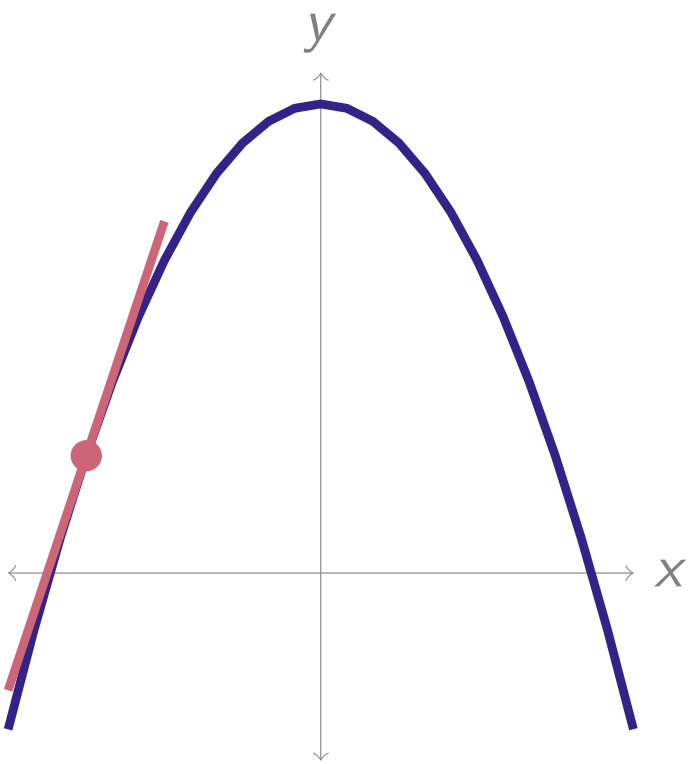

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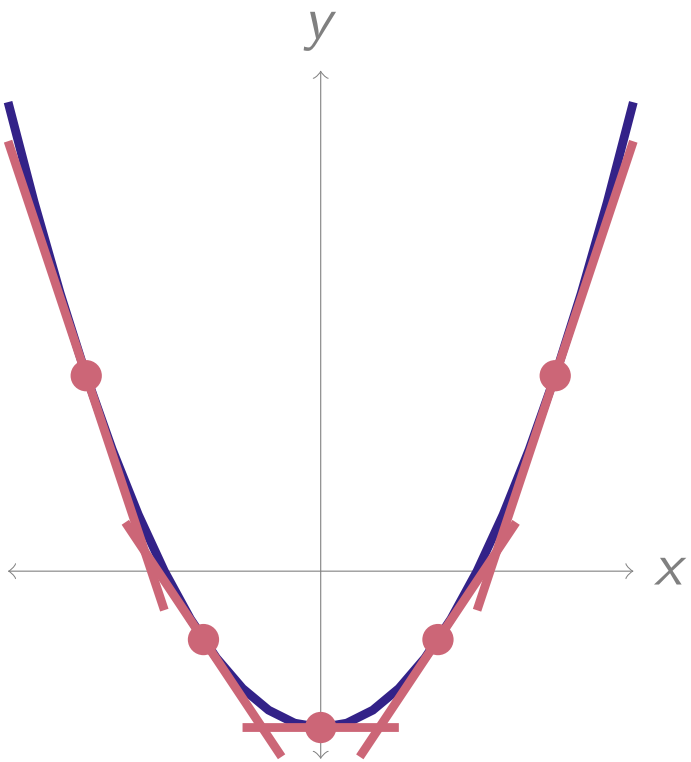

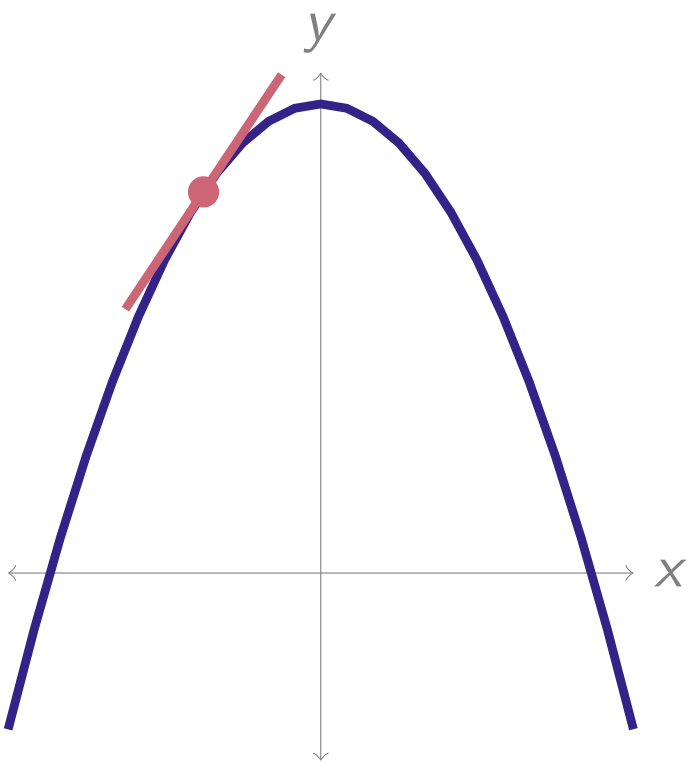

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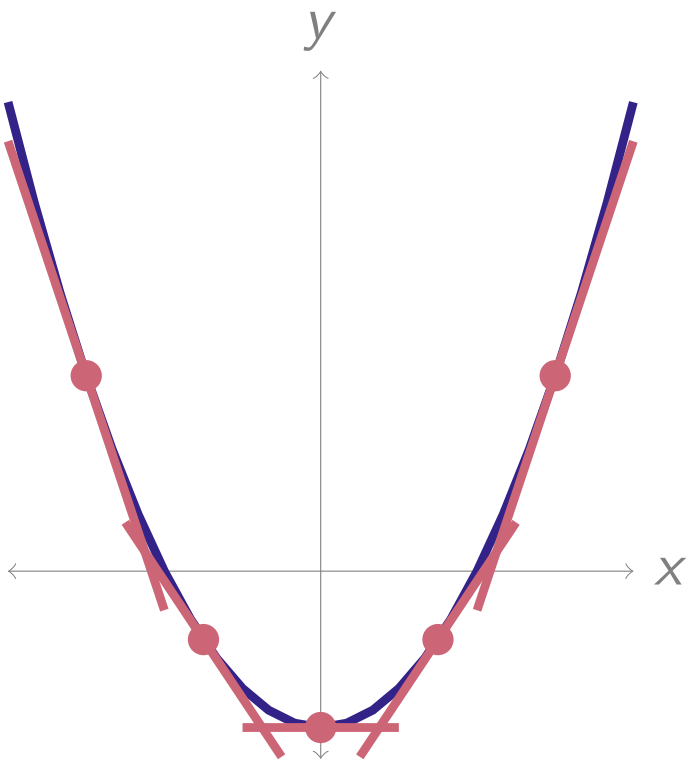

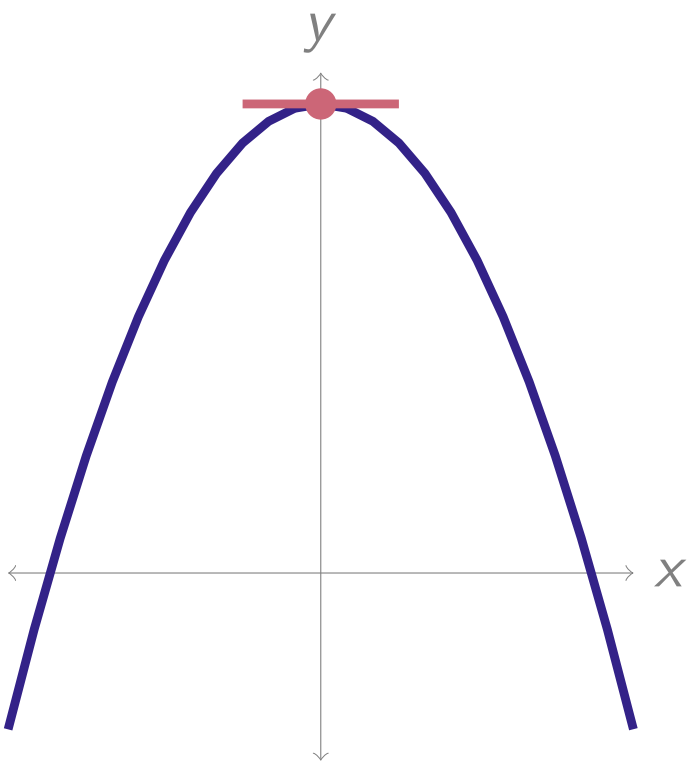

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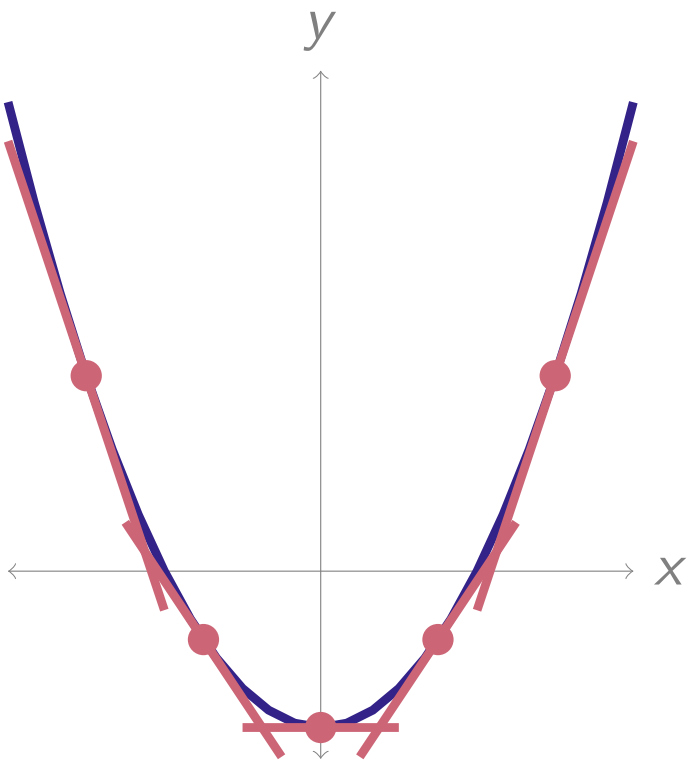

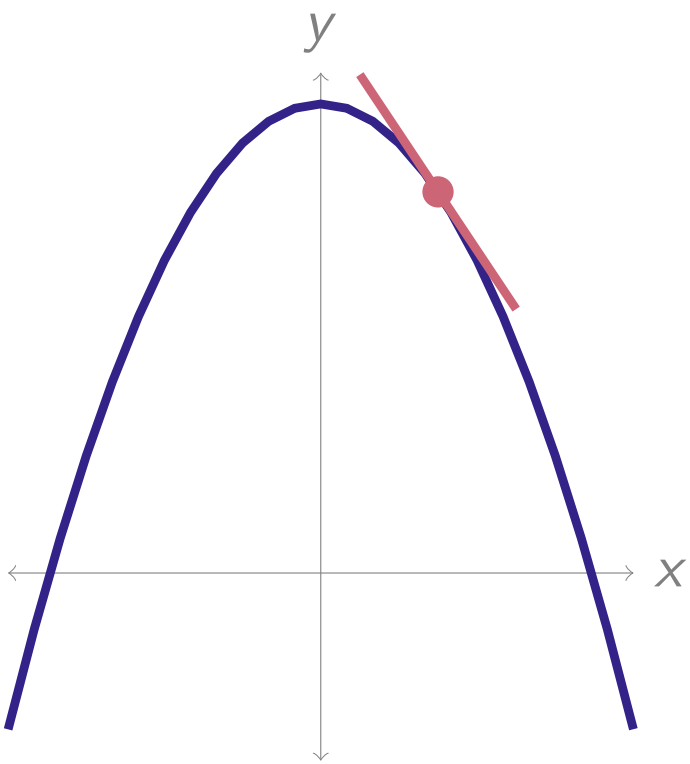

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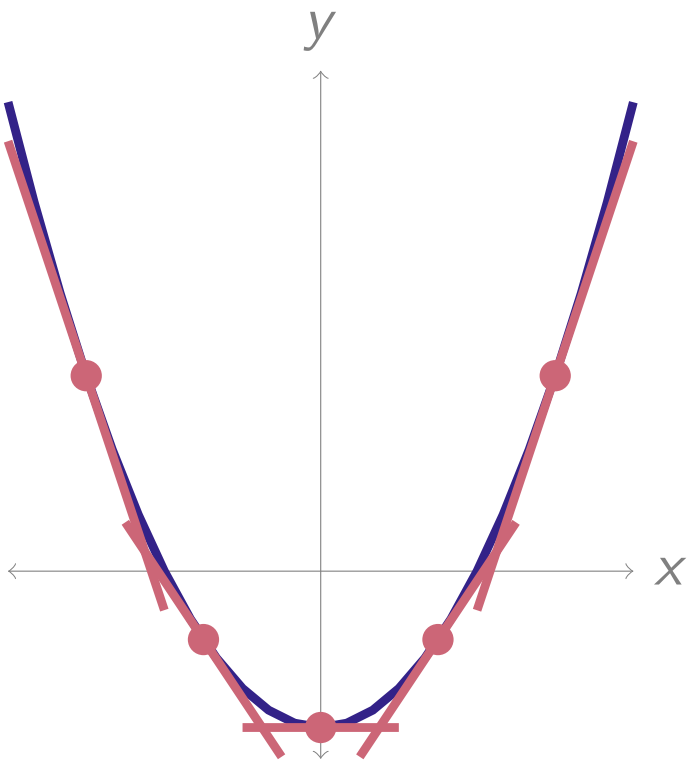

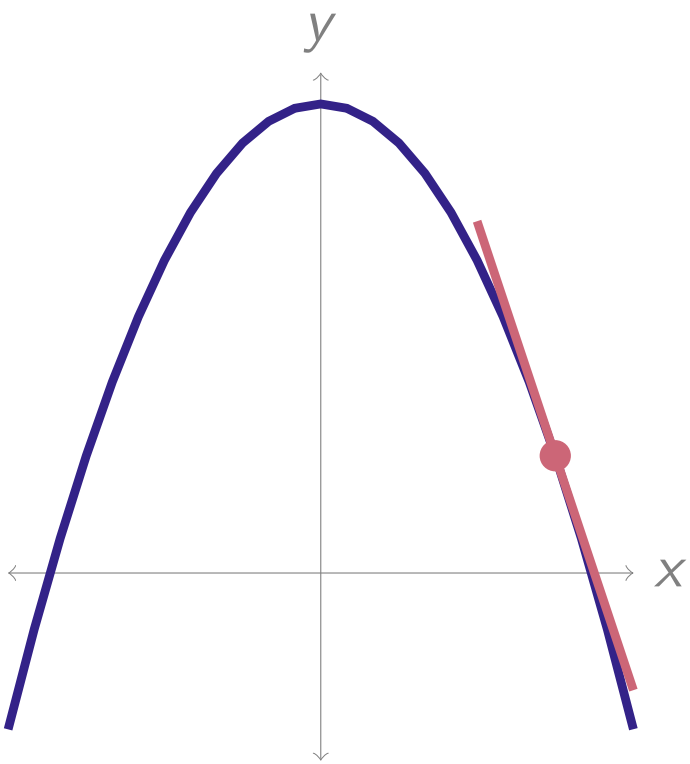

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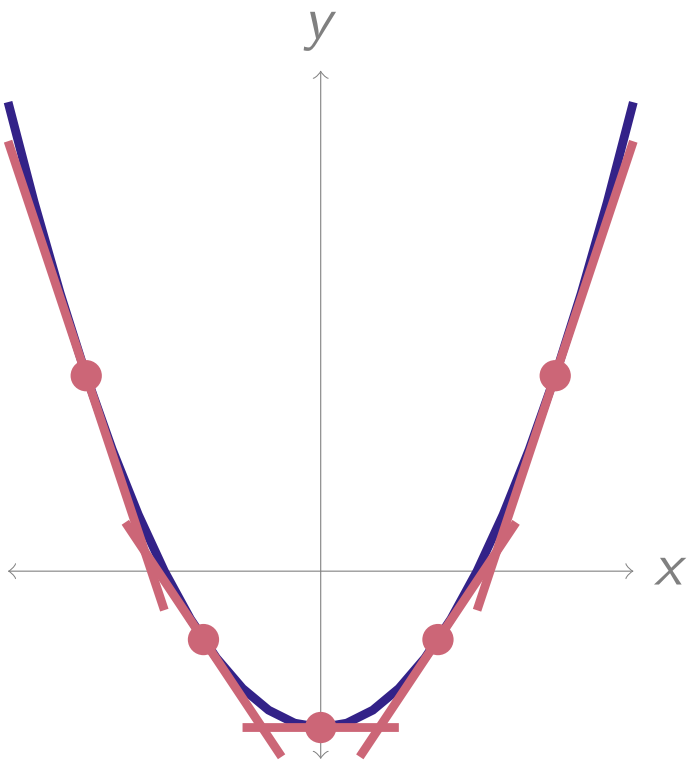

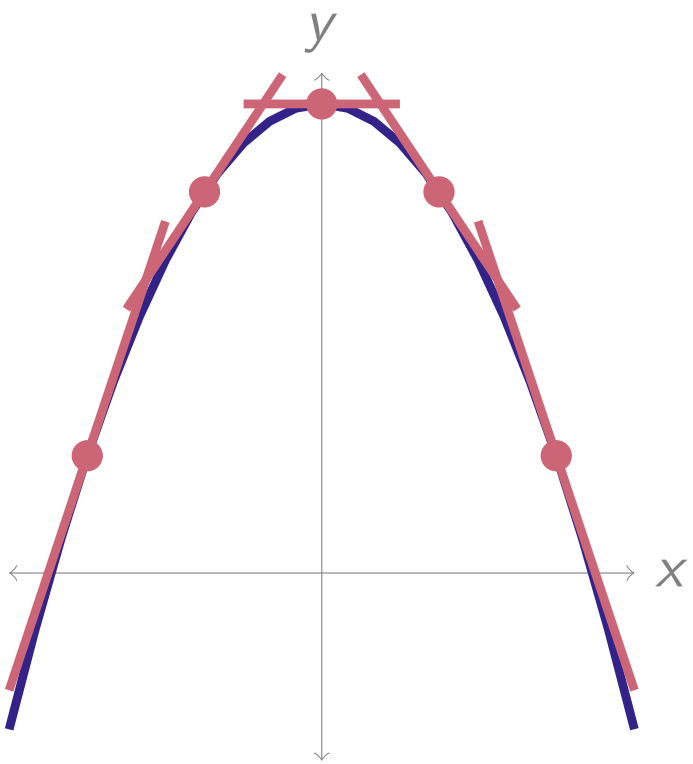

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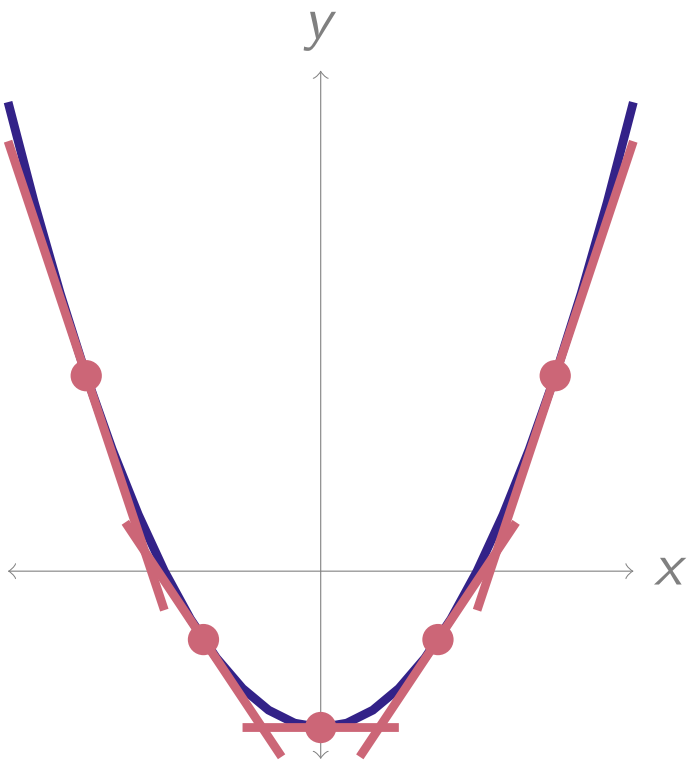

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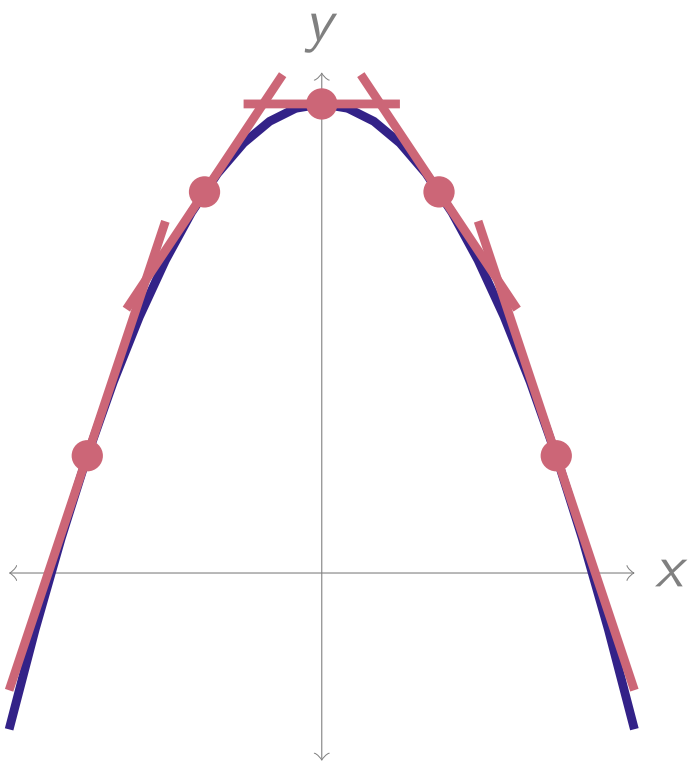

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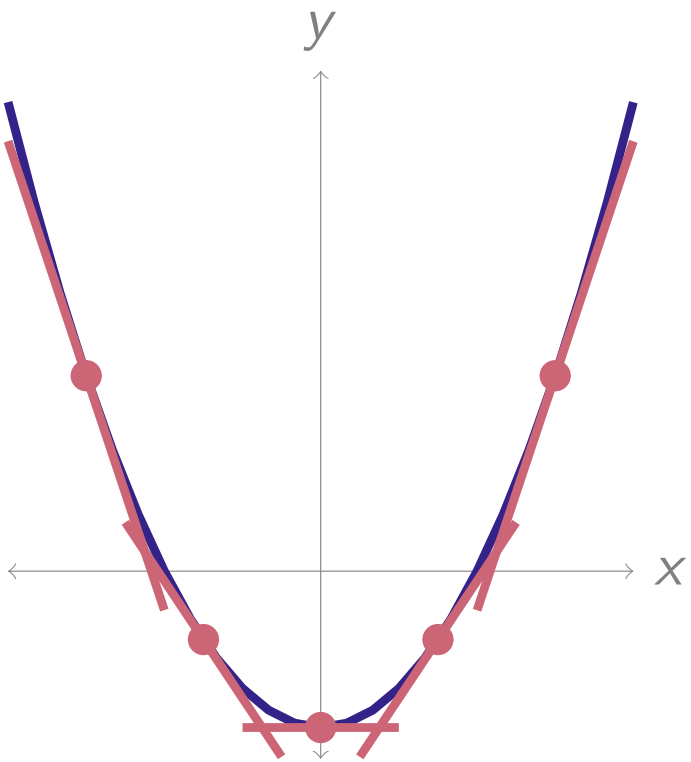

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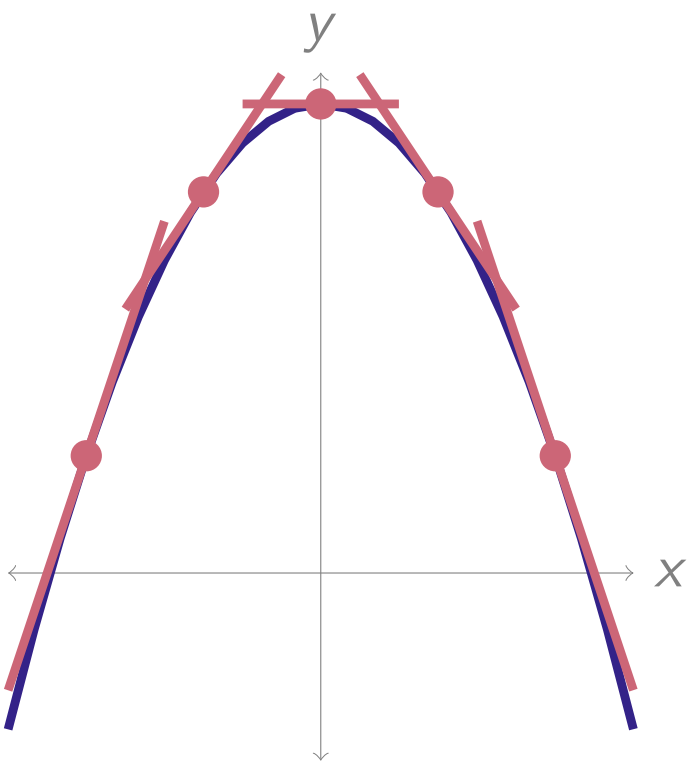

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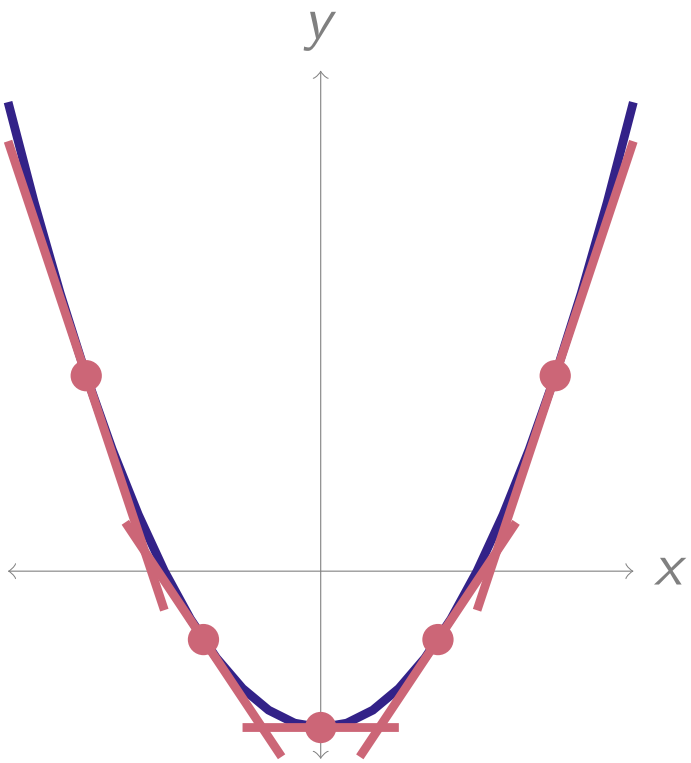

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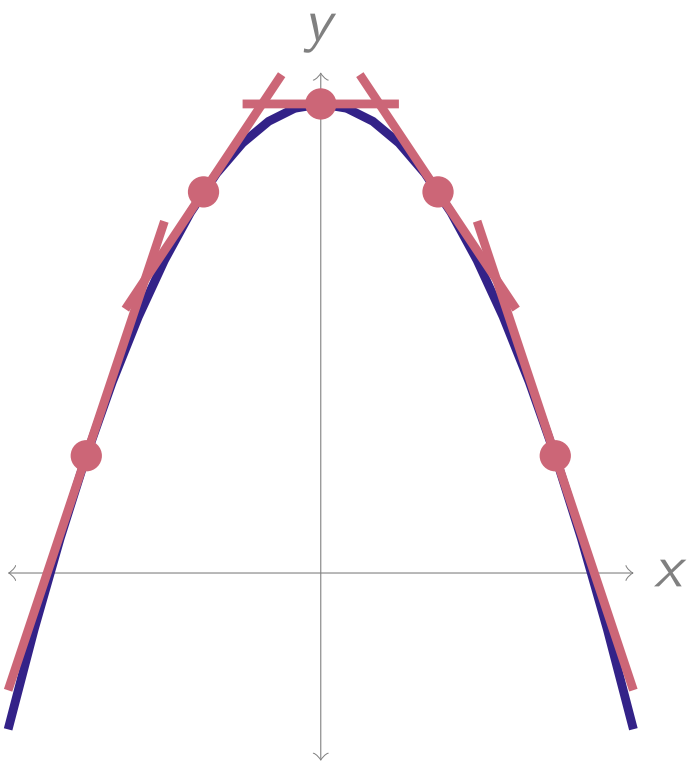

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Mnemonic

Concavity

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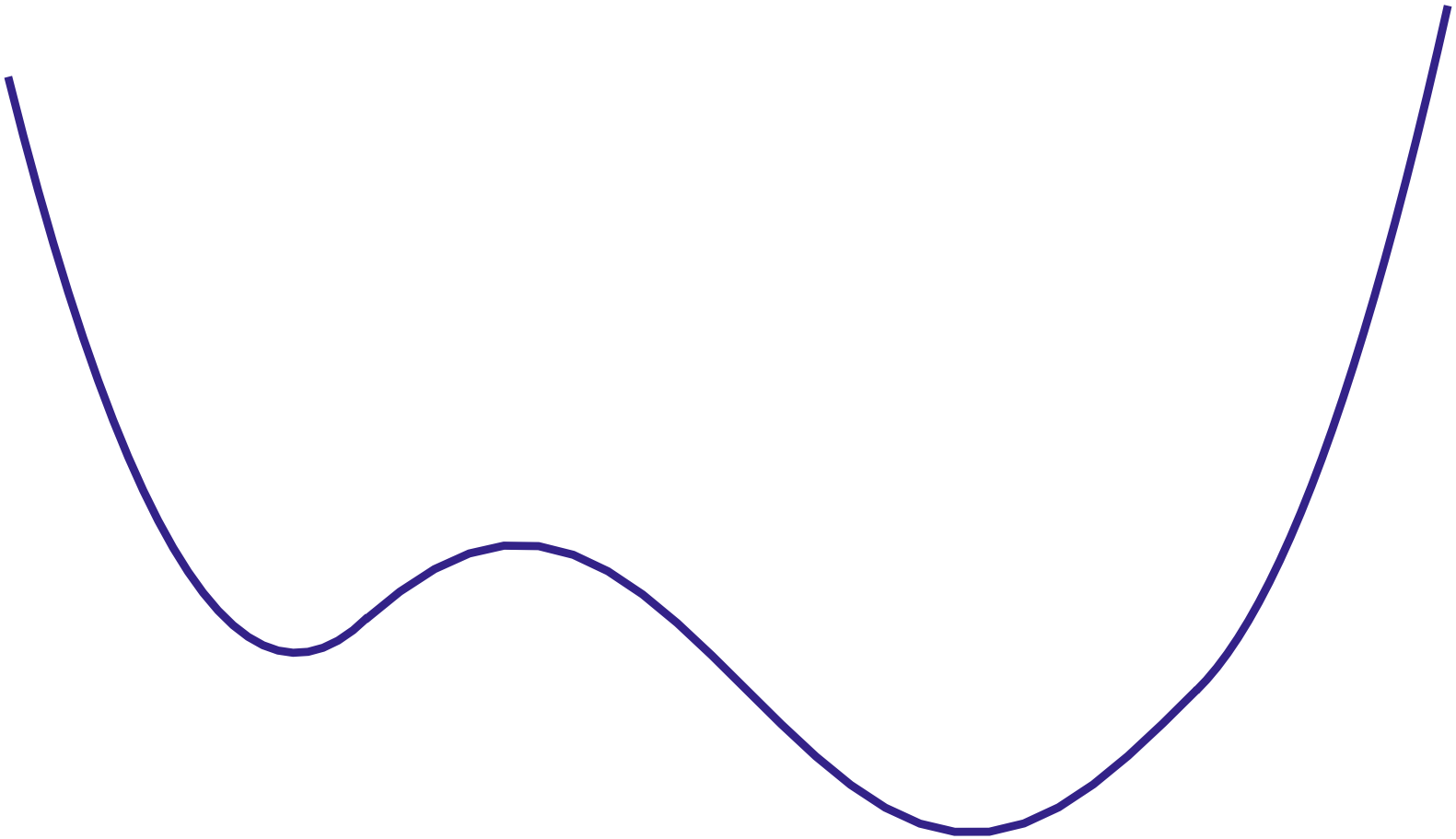

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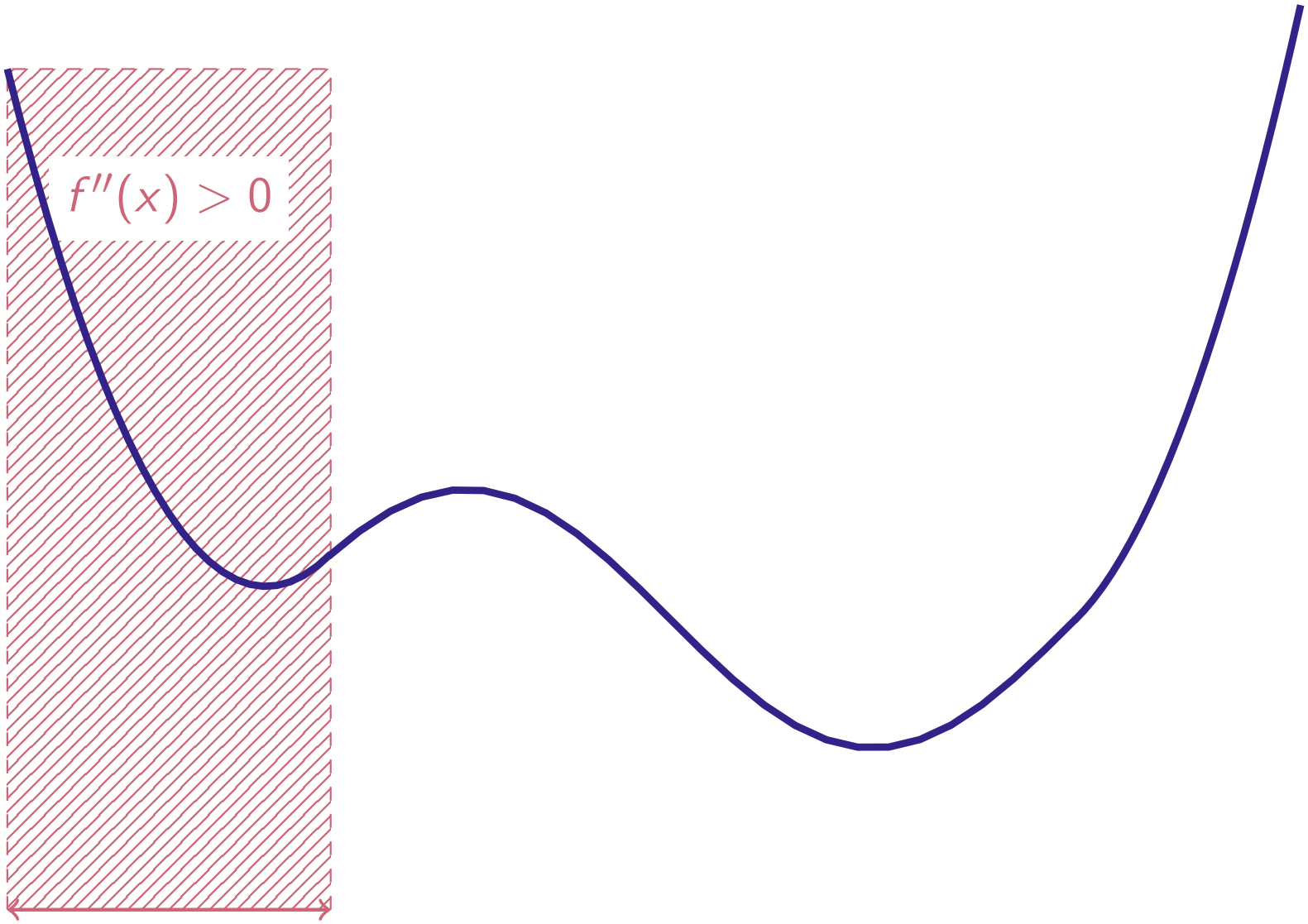

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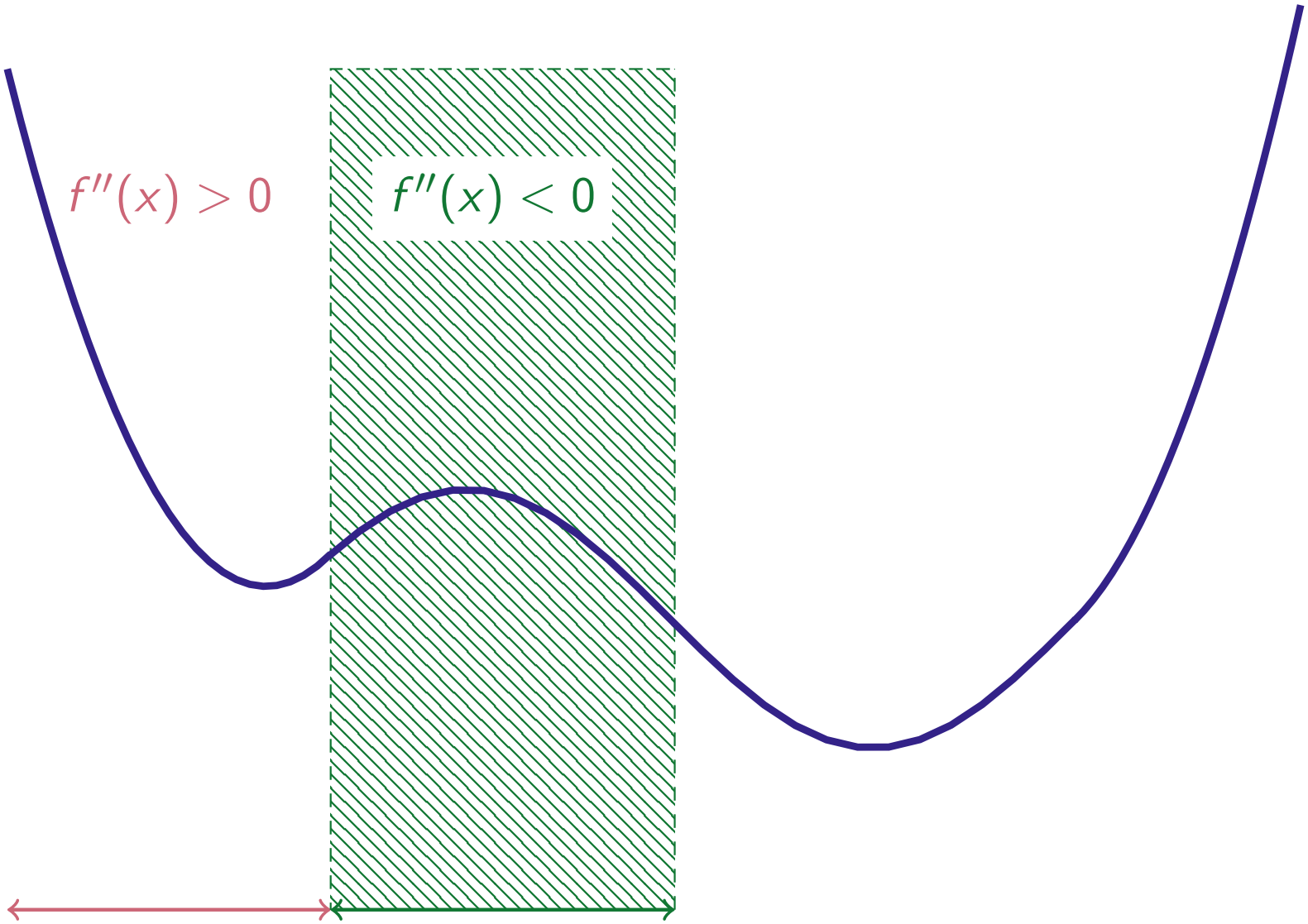

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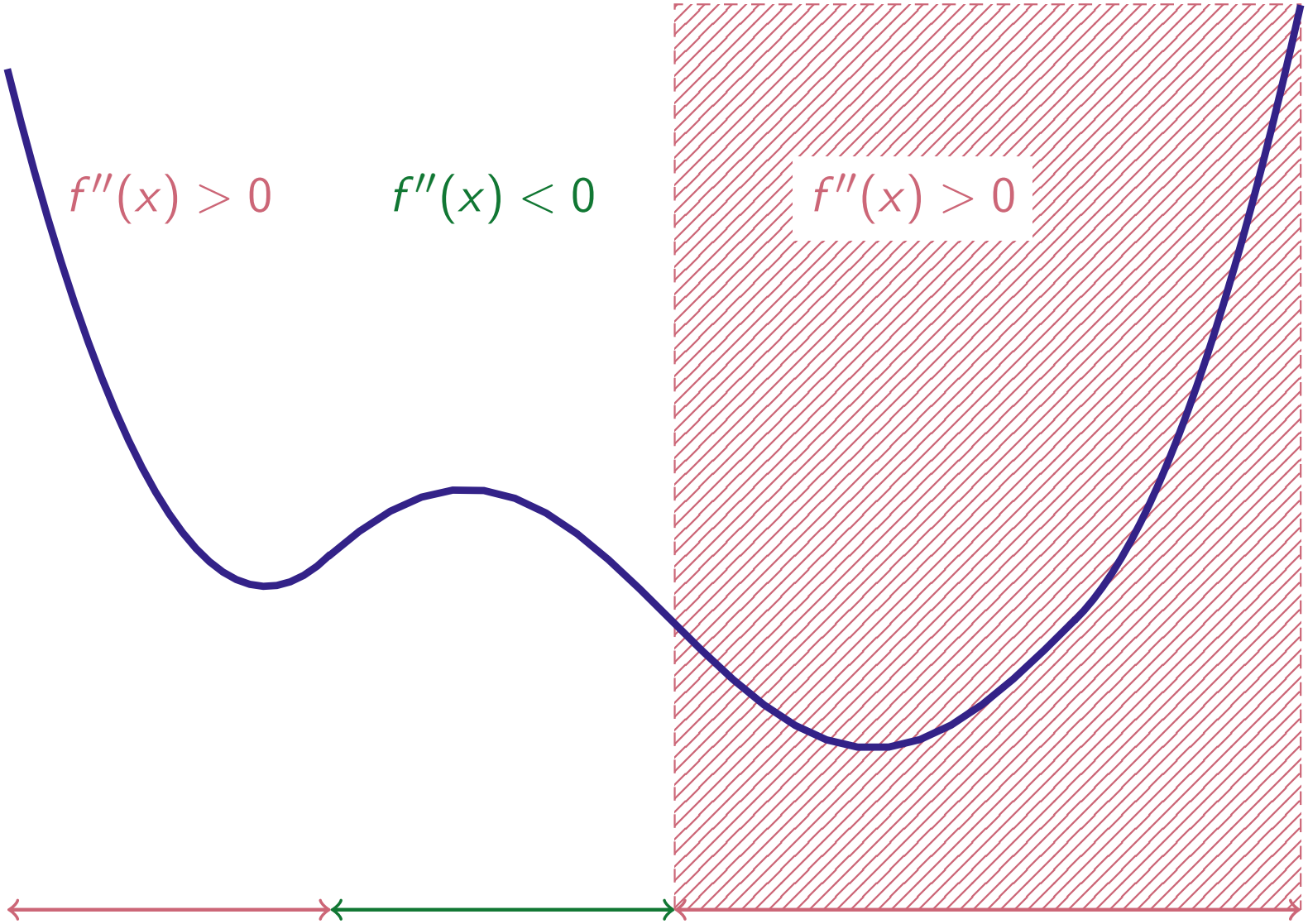
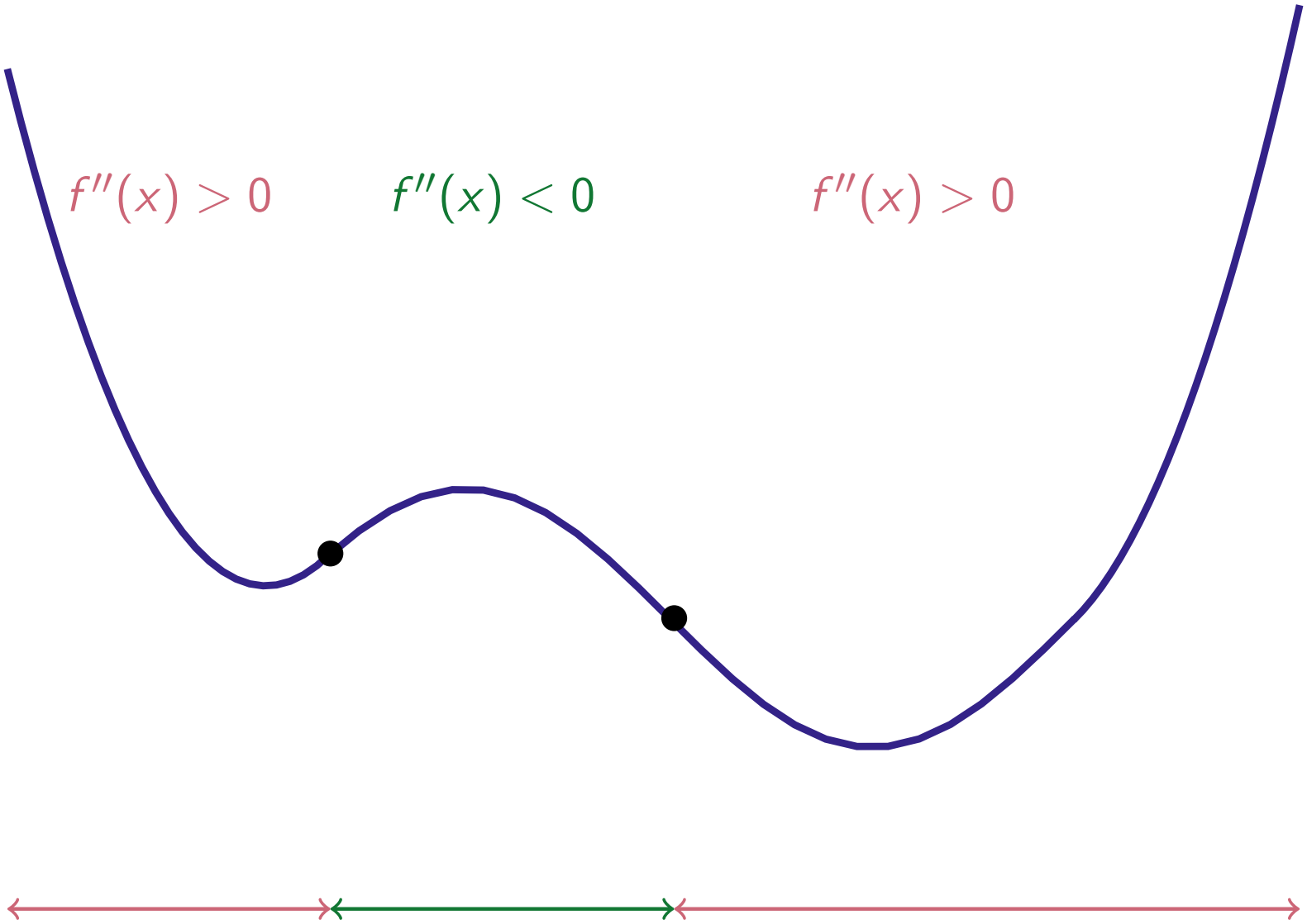


Concavity

Where is the function concave up? Concave down?



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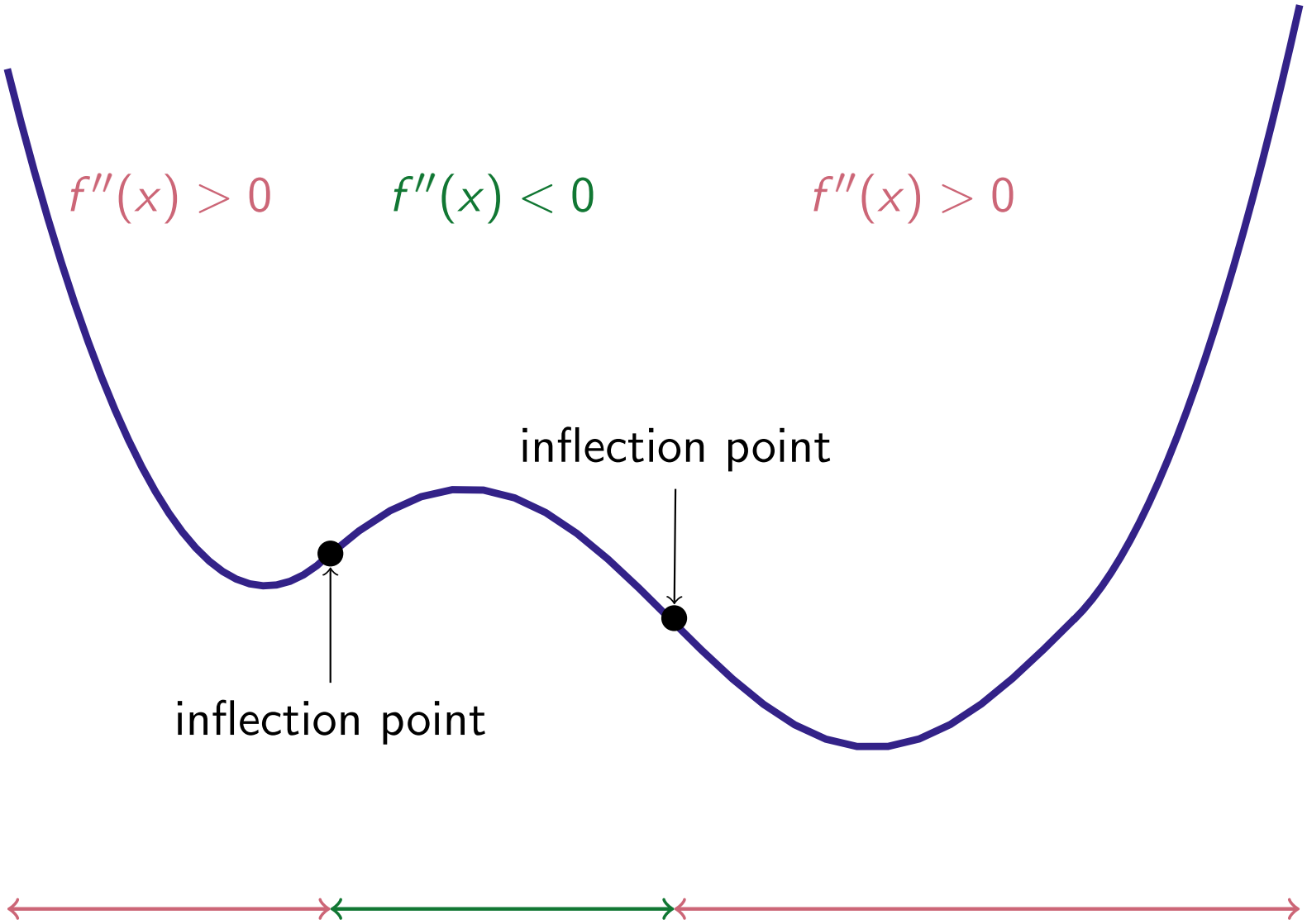

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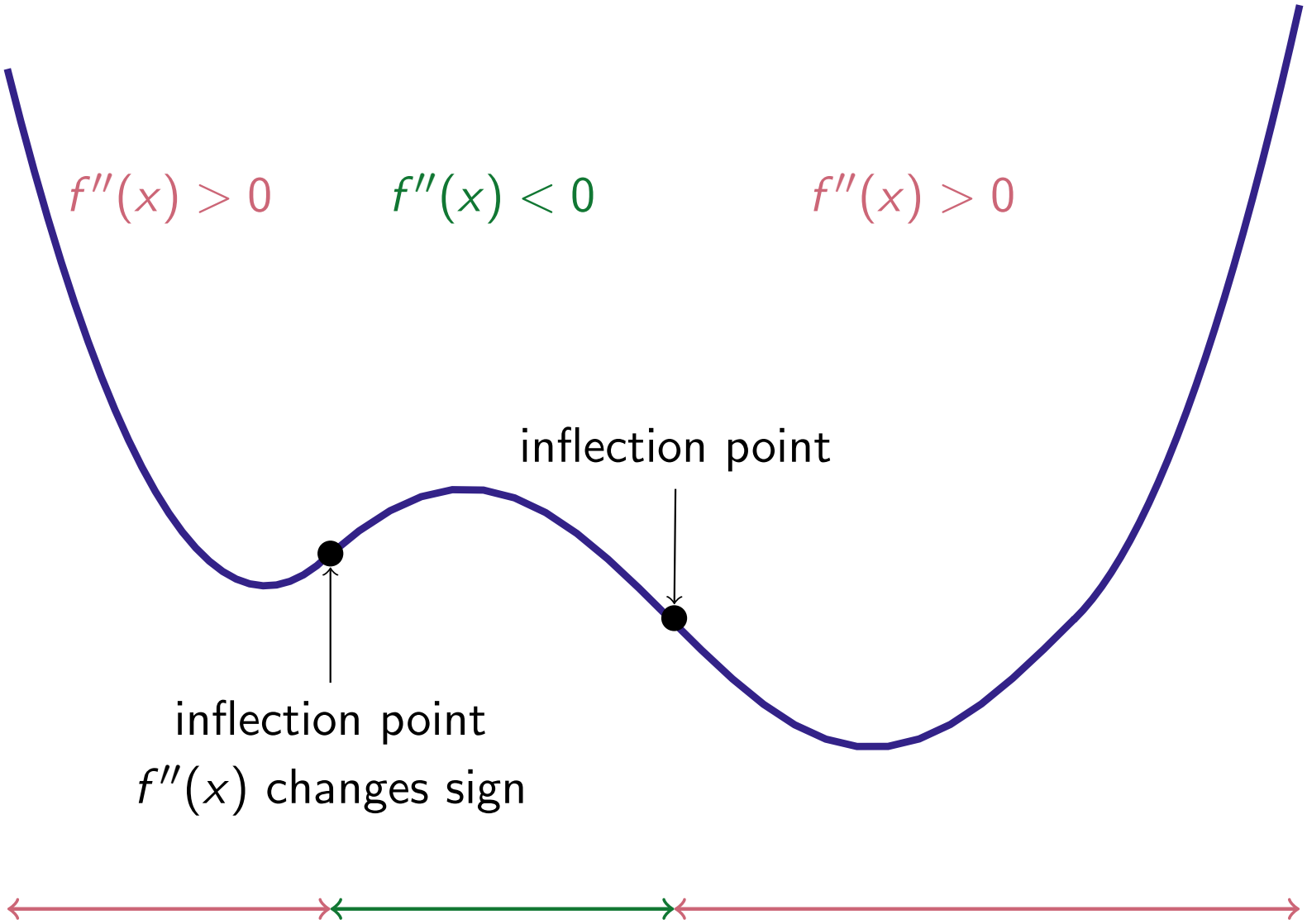

Sketch:

$$f(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 - 3x^2$$

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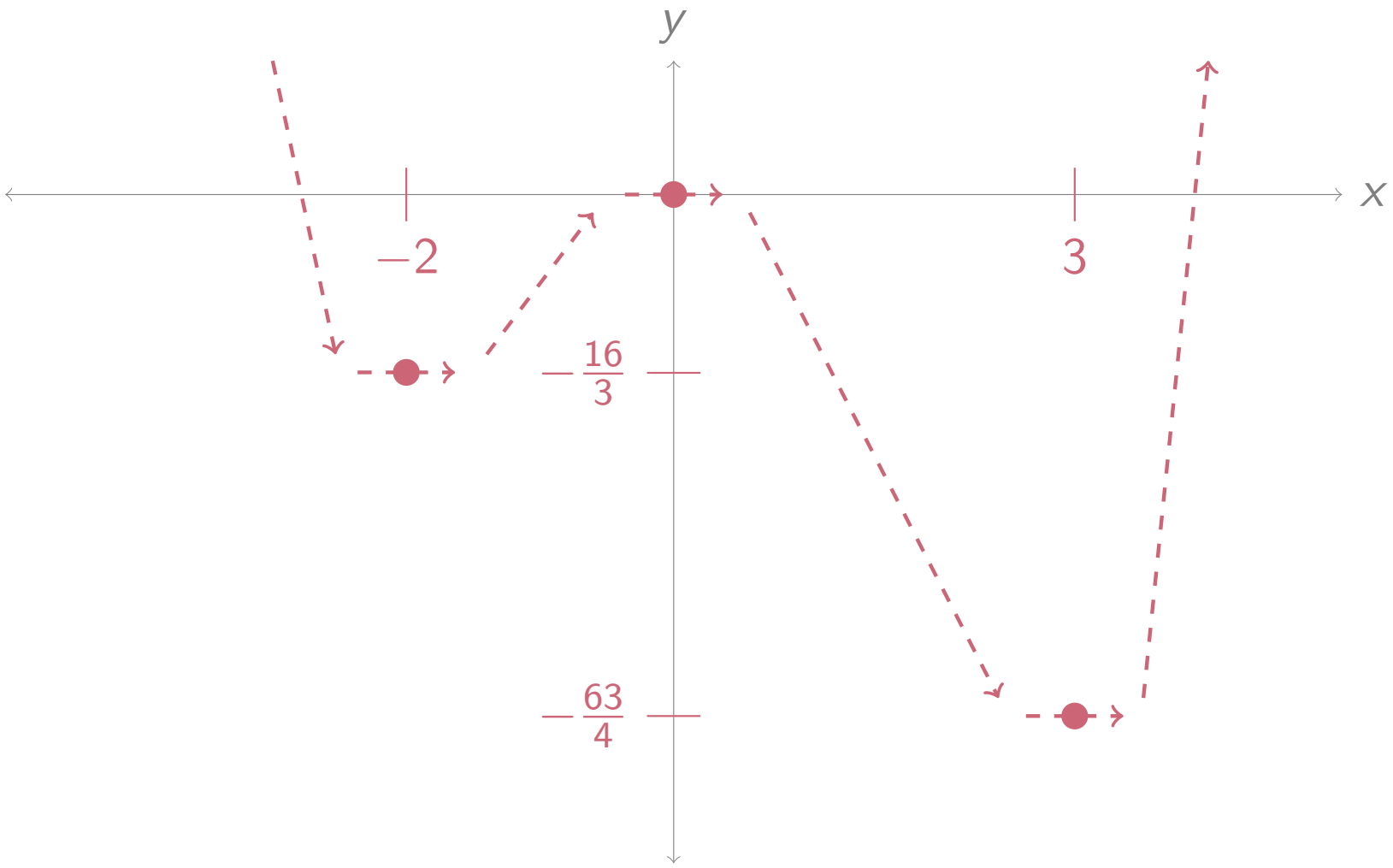
$$f'(x) = x^3 - x^2 - 6x = x(x-3)(x+2)$$

	$(-\infty, -2)$	-2	$(-2, 0)$	0	$(0, 3)$	3	$(3, \infty)$
(x)	$-$	$-$	$-$	0	$+$	$+$	$+$
$(x-3)$	$-$	$-$	$-$	$-$	$-$	0	$+$
$(x+2)$	$-$	0	$+$	$+$	$+$	$+$	$+$
$f'(x)$	$-$	0	$+$	0	$-$	0	$+$
$f(x)$	dec	$\frac{-16}{3}$	inc	0	dec	$\frac{-63}{4}$	inc

Using info from first derivative:



$$f''(x) = 3x^2 - 2x - 6$$

To find inflection points (i.e. where the second derivative changes sign), we find where $f''(x) = 0$ or where it does not exist.
In this case, $f''(x)$ is a quadratic equation. It exists everywhere, and its zeroes are:

$$x = \frac{2 \pm \sqrt{4 - 4(3)(-6)}}{2 \cdot 3} = \frac{1 \pm \sqrt{1 + 18}}{3} = \frac{1 \pm \sqrt{19}}{3}$$

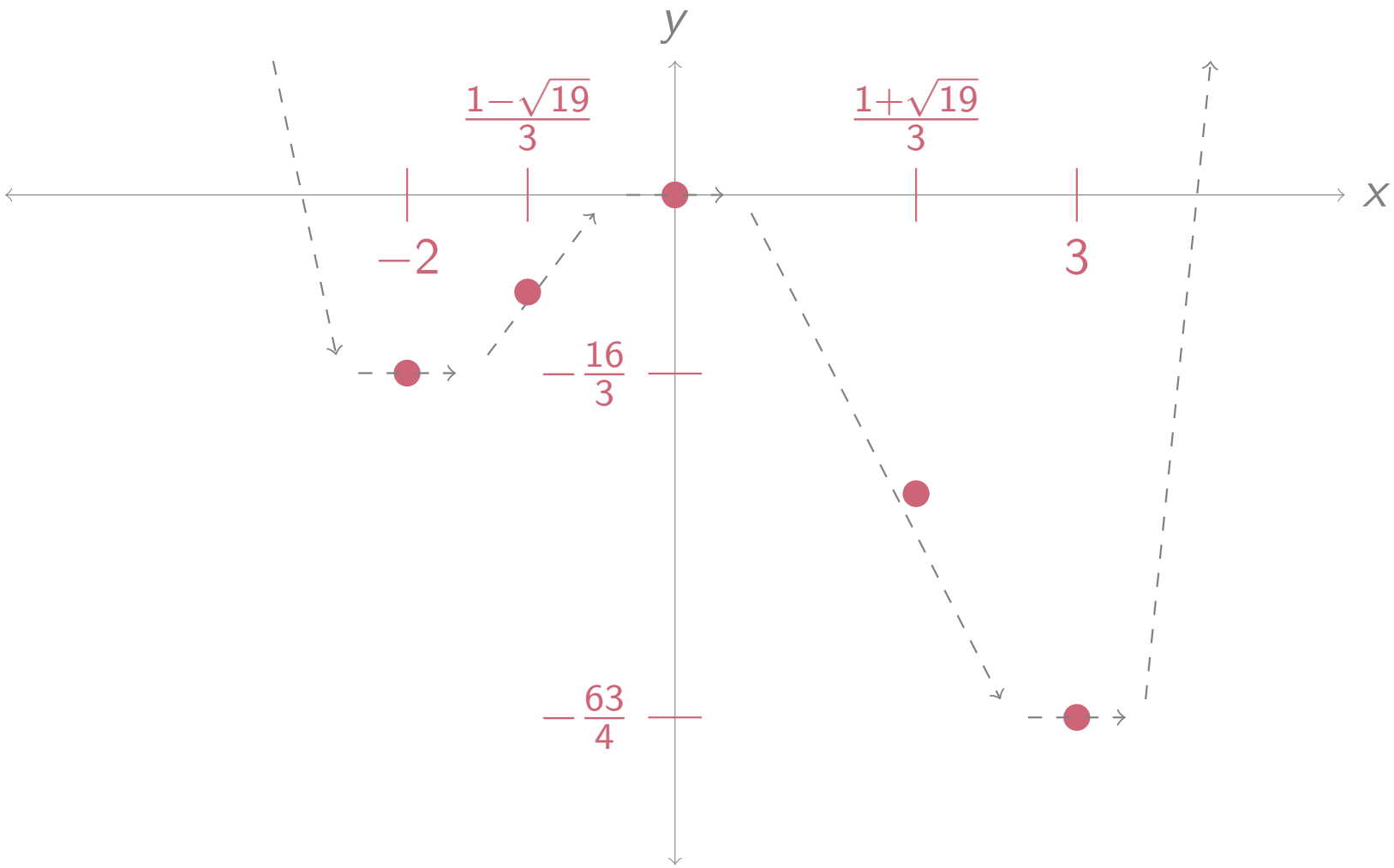
$$x = \frac{1 + \sqrt{19}}{6} \approx 1.8 \quad \text{and} \quad x = \frac{1 - \sqrt{19}}{6} \approx -1.1$$

So, $f(x)$ is concave up when $x < \frac{1-\sqrt{19}}{3}$ and also when $x > \frac{1+\sqrt{19}}{3}$, and it is concave down between these points.
Then $f(x)$ has two inflection points. These occur at:

$$x = \frac{1 - \sqrt{19}}{3} \approx -1.1, \qquad y \approx -1.4$$

$$x = \frac{1 + \sqrt{19}}{6} \approx 1.8, \qquad y \approx -5$$

Using info from second derivative:



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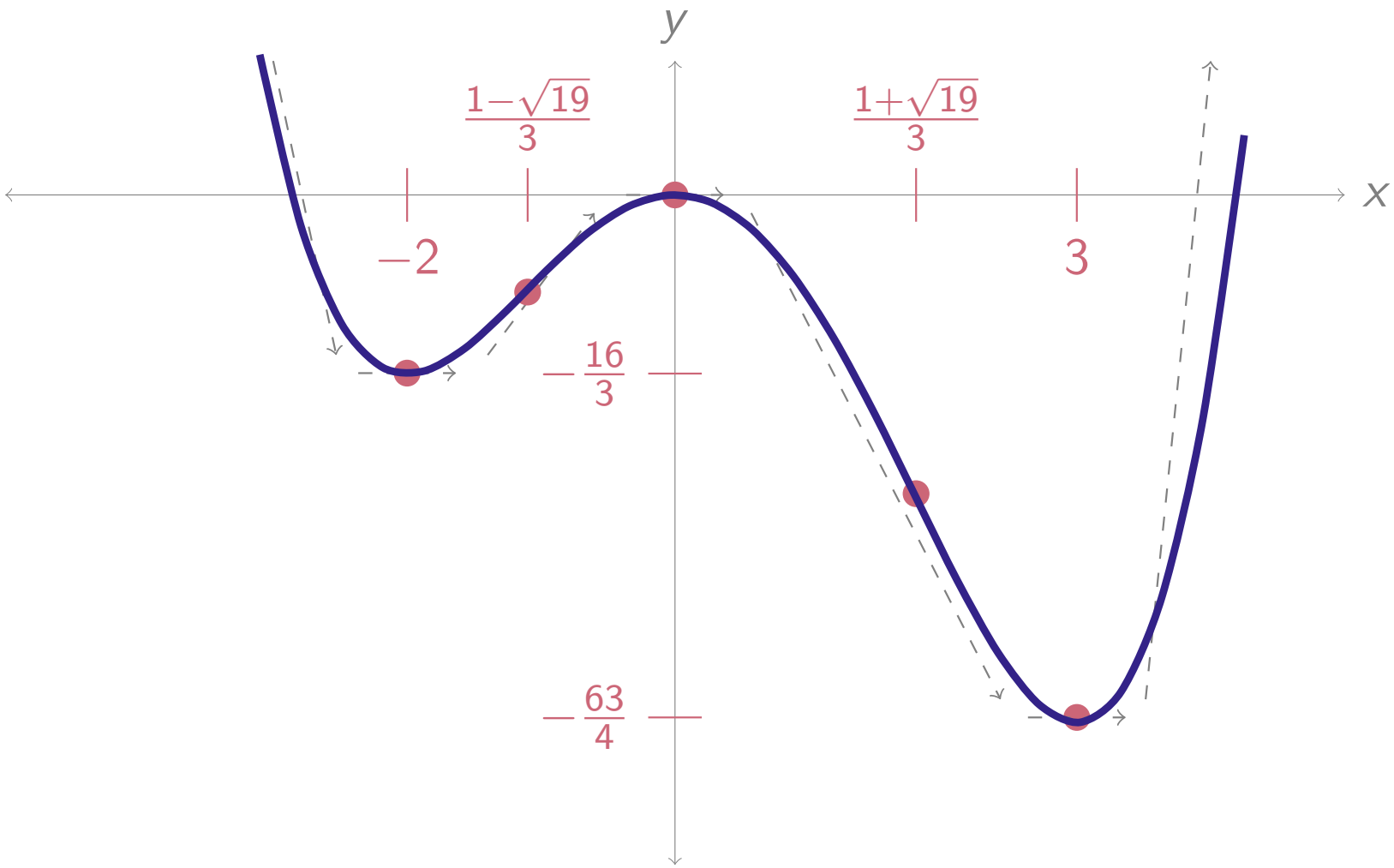
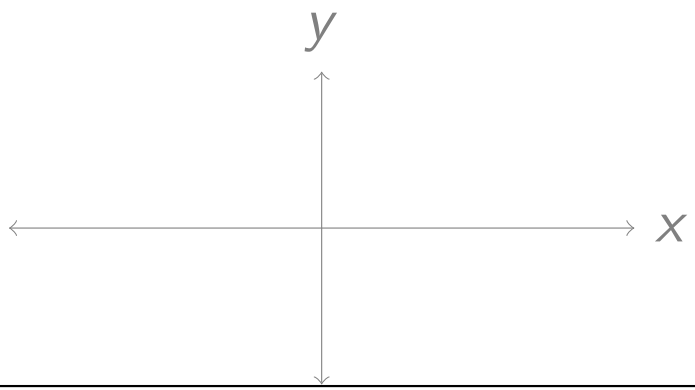
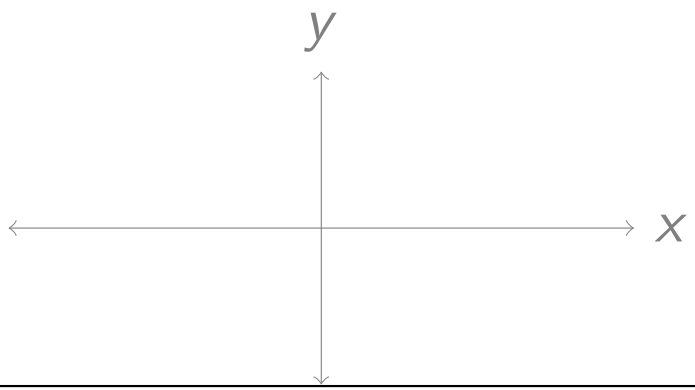
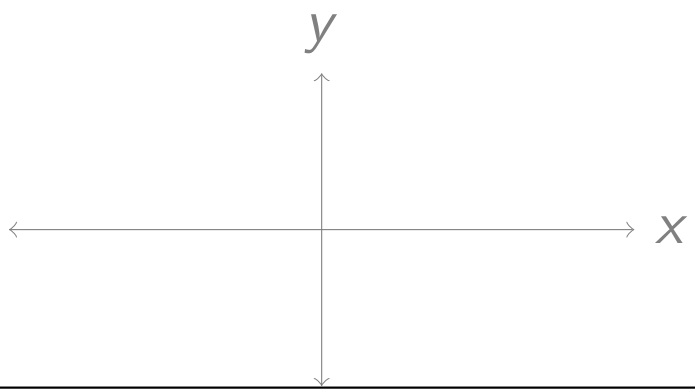
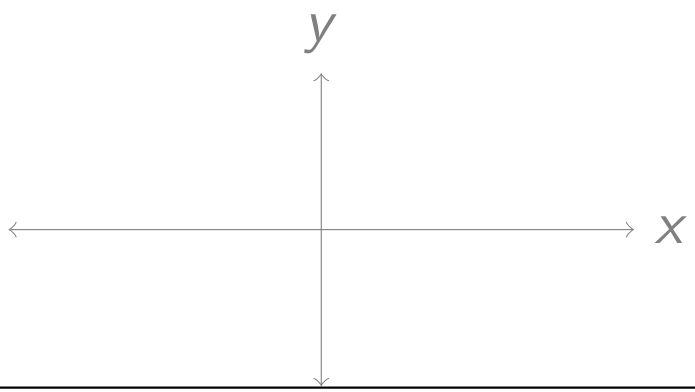


Sketch graphs with the following properties, or explain that none exist.

	concave up	concave down
increasing		
decreasing		

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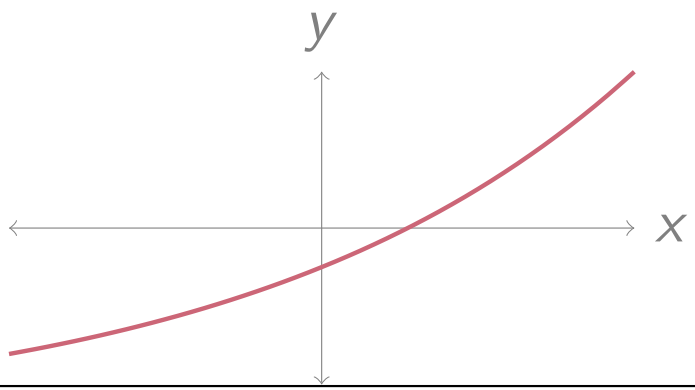
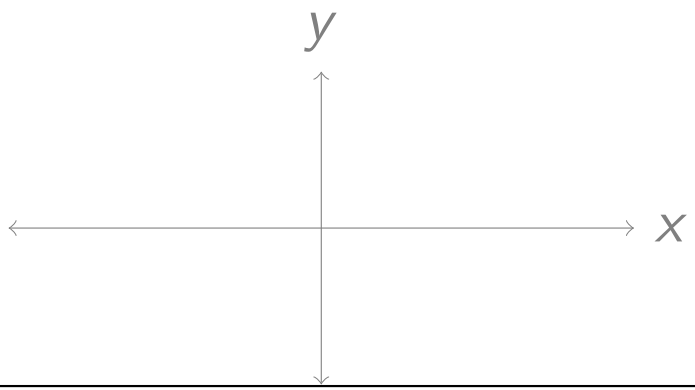
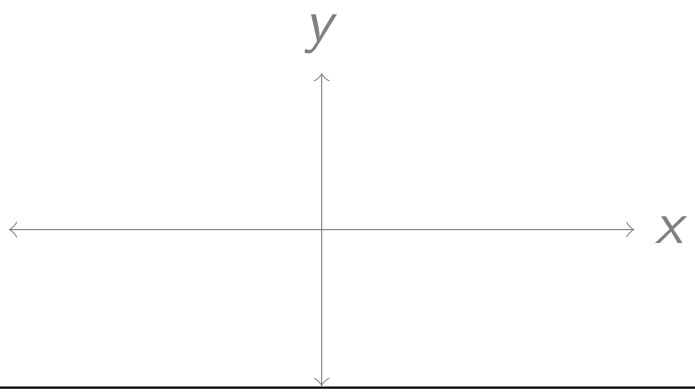
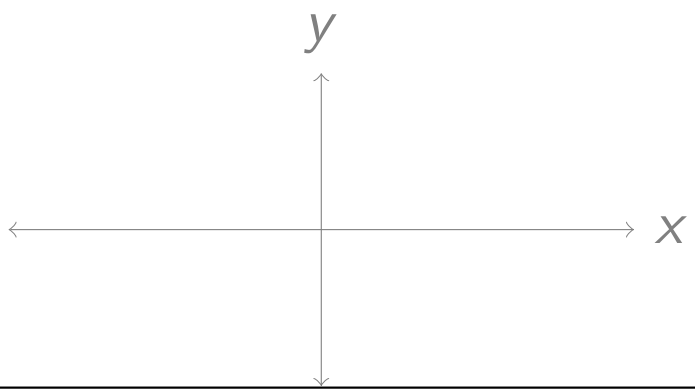
increasing		
decreasing		

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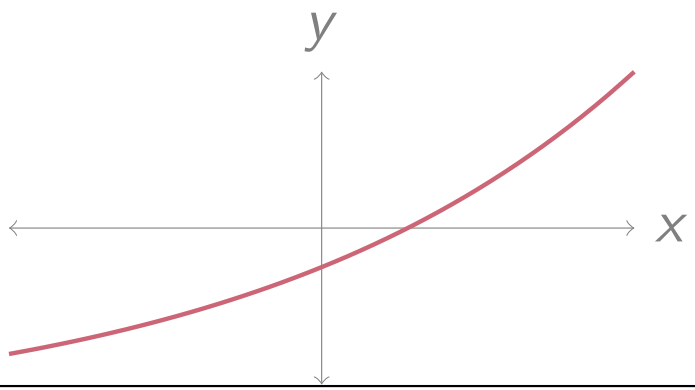
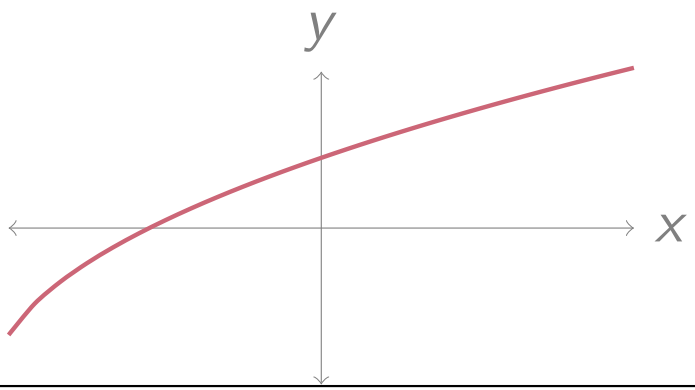
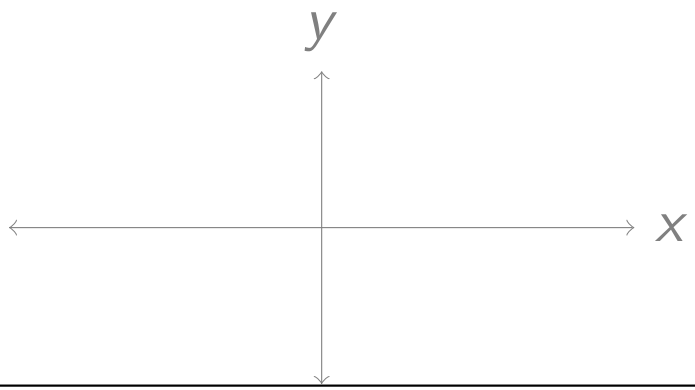
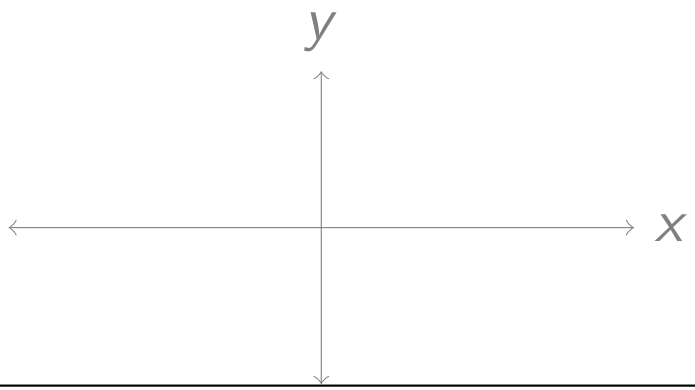
increasing		
decreasing		

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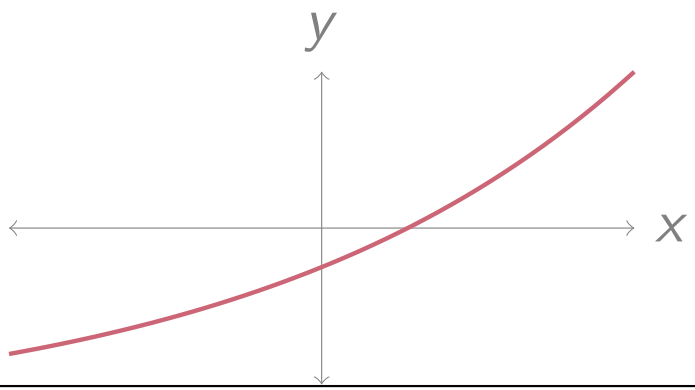
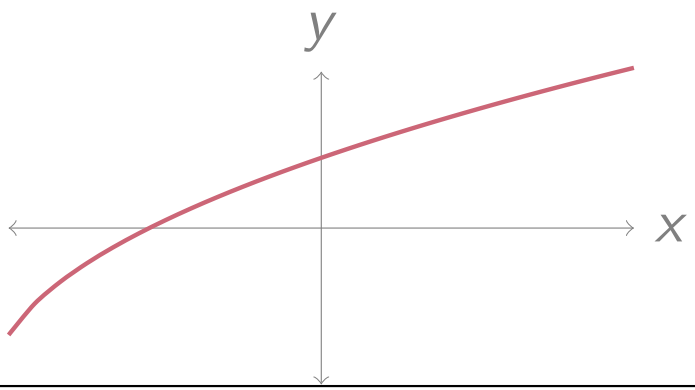
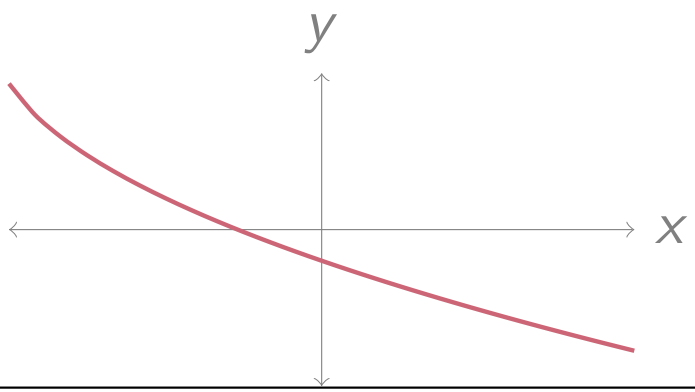
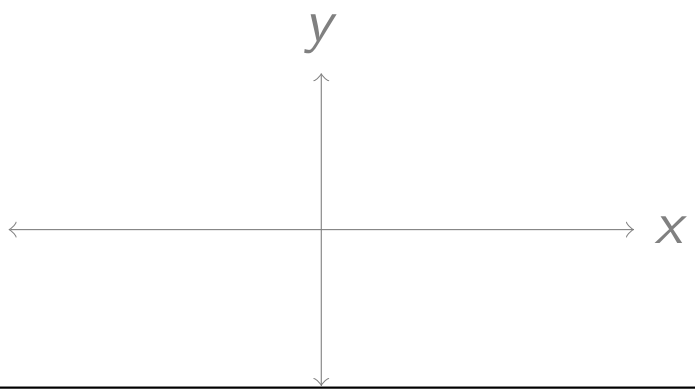
increasing		
decreasing		

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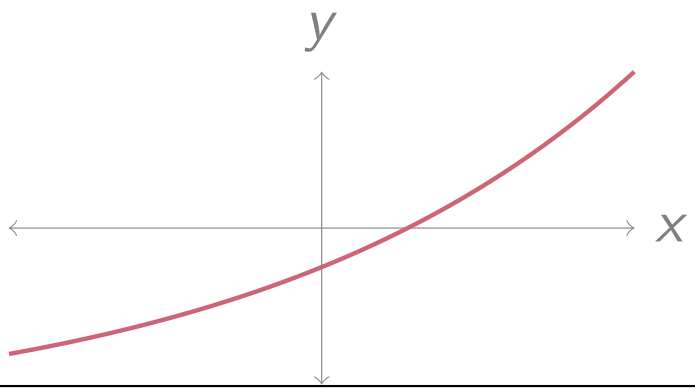
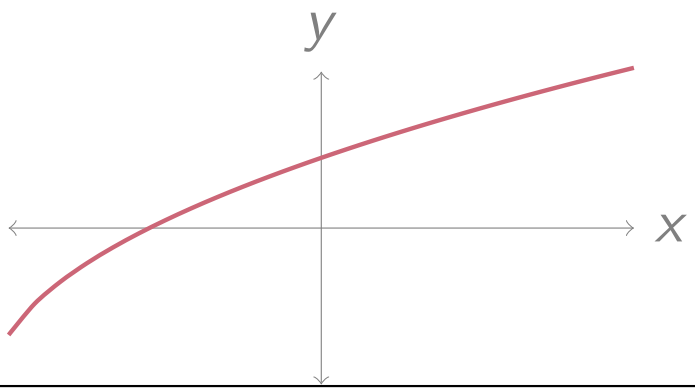
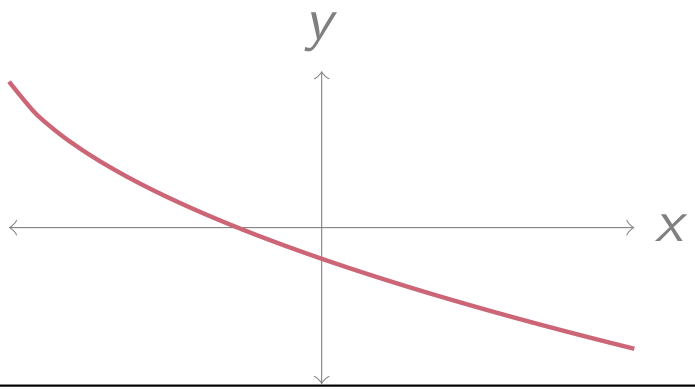
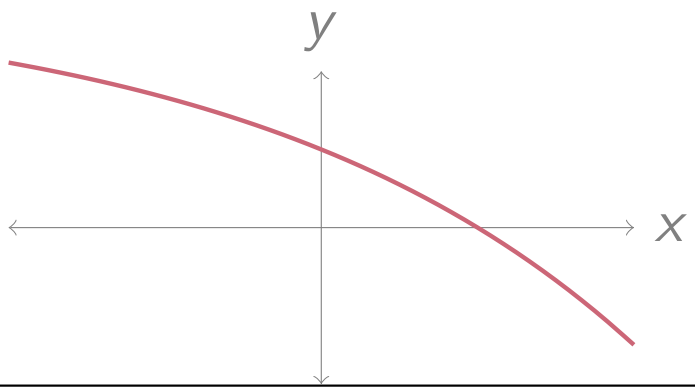
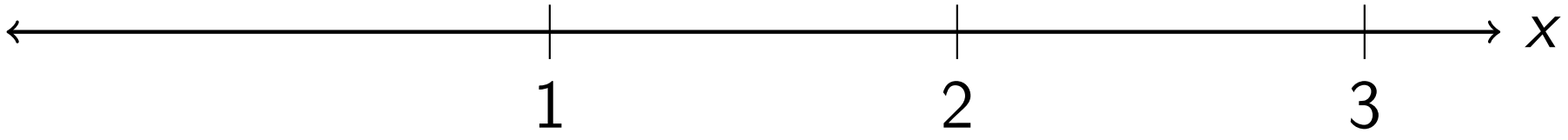
increasing		
decreasing		

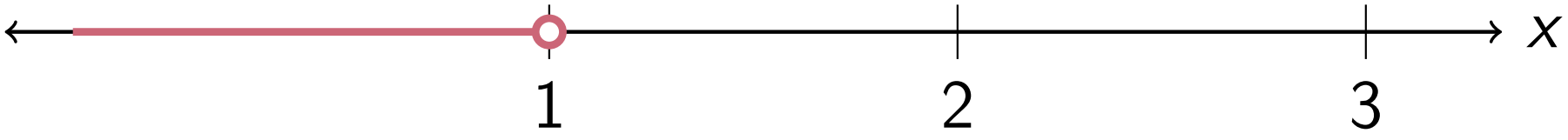
Sign Changes in a Factored Function

$$f(x) = (x-1)\ (x-2)\ (x-3)$$



Sign Changes in a Factored Function

$$f(x) = \underset{-}{(x-1)} \quad (x-2) \quad (x-3)$$



Sign Changes in a Factored Function

$$f(x) = \underset{+}{(x-1)} \quad (x-2) \quad (x-3)$$

Sign Changes in a Factored Function

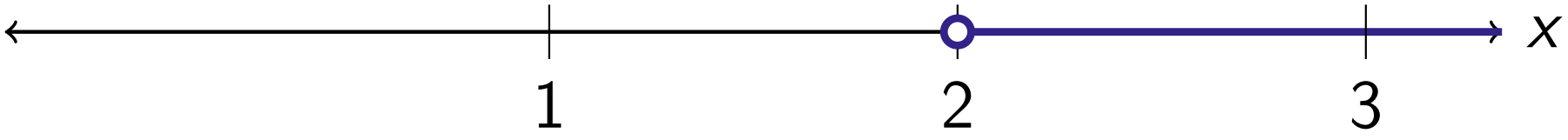
$$f(x) = \quad (x-1) \quad (x-2) \quad (x-3)$$

$$-$$

Sign Changes in a Factored Function

$$f(x) = \quad (x-1) \quad (x-2) \quad (x-3)$$

$$+$$



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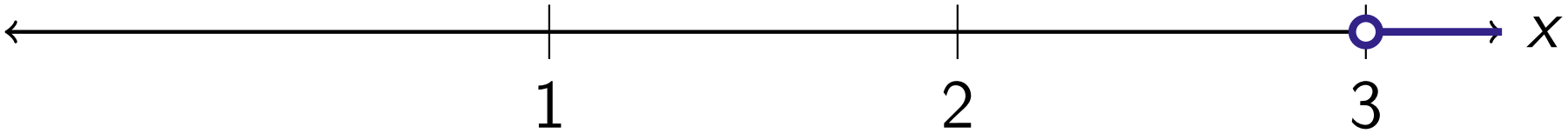
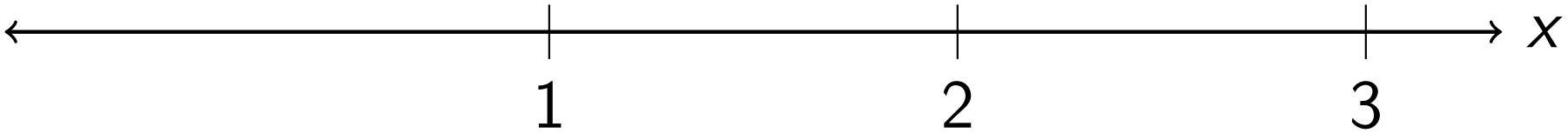


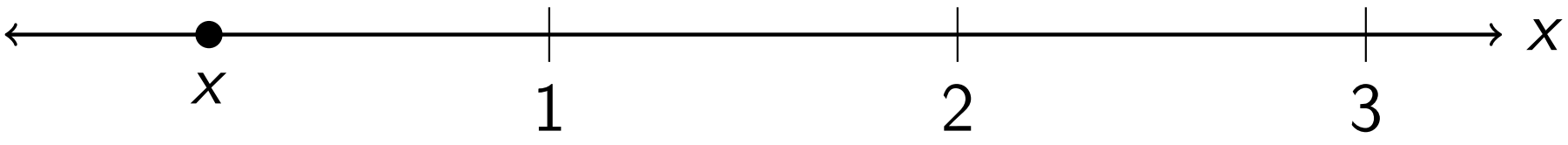
Sign Changes in a Factored Function

$f(x) = \quad (x-1) \quad (x-2) \quad (x-3)$ Sign of product:



Sign Changes in a Factored Function

$$f(x) = \underset{-}{(x-1)} \quad \underset{-}{(x-2)} \quad \underset{-}{(x-3)}$$ Sign of product: $-$



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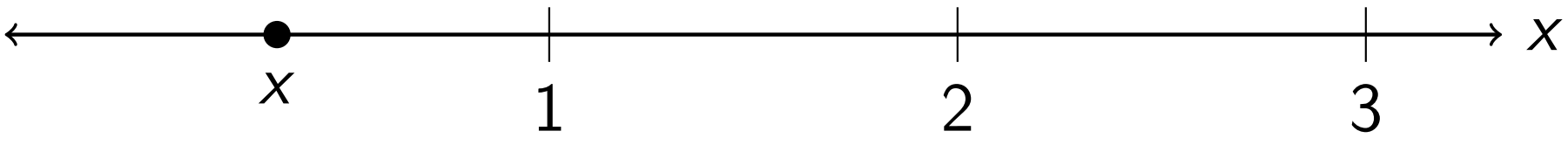

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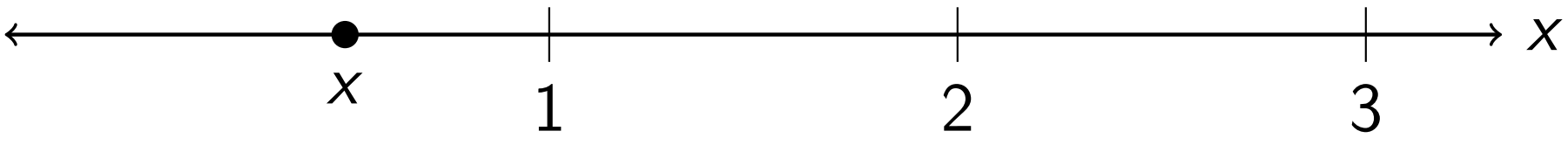

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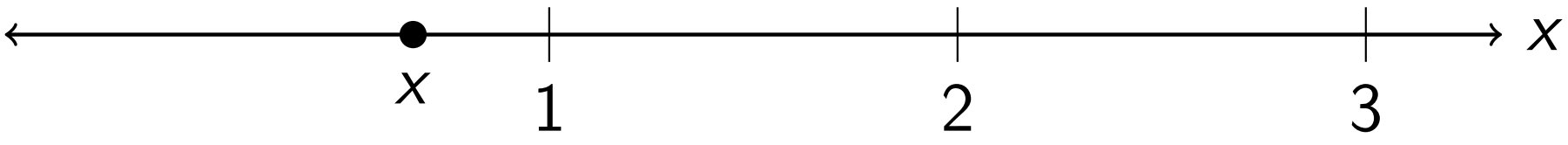

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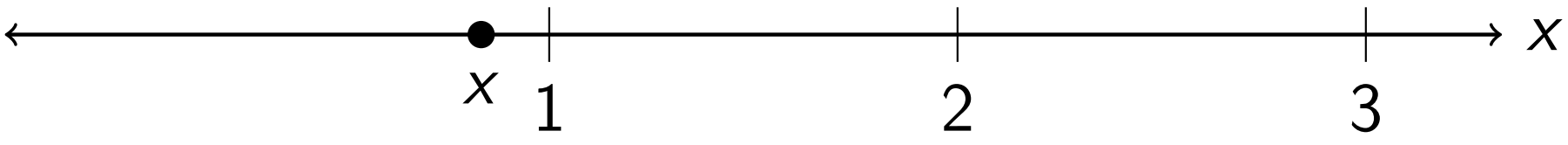
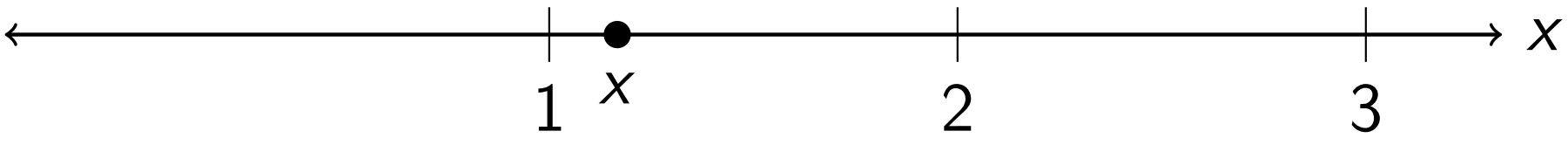


Sign Changes in a Factored Function

$$f(x) = \underset{+}{(x-1)} \quad \underset{-}{(x-2)} \quad \underset{-}{(x-3)} \quad \text{Sign of product:} \quad +$$



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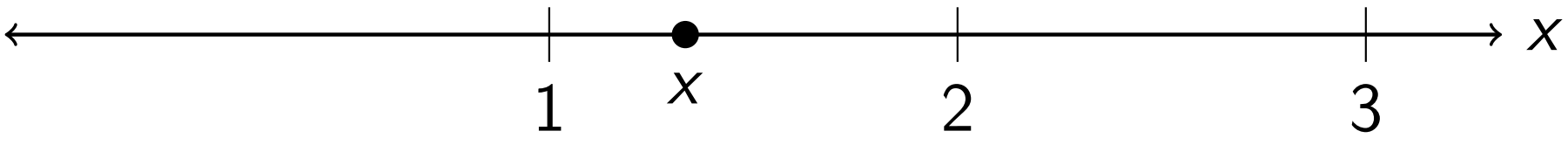

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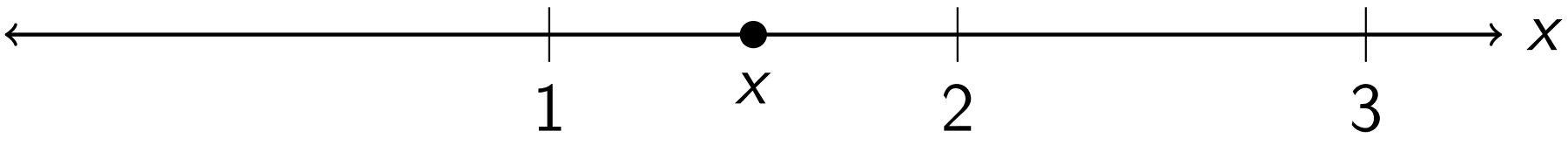

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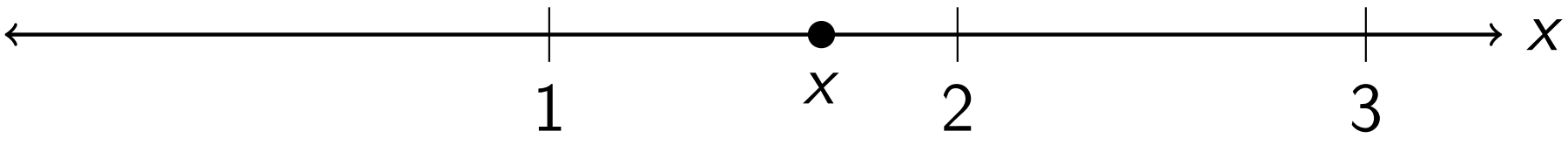

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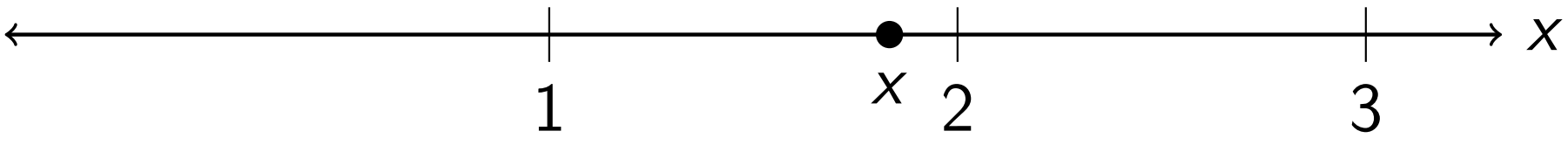

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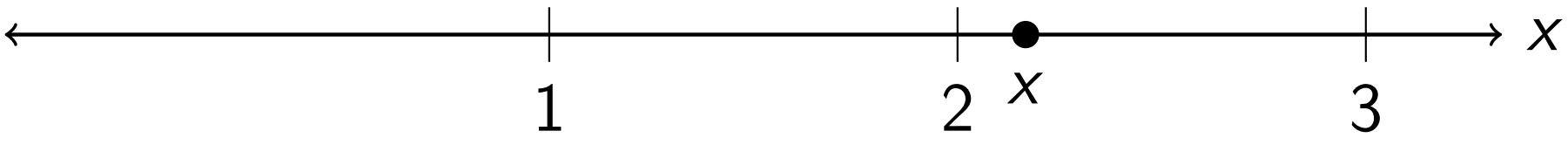

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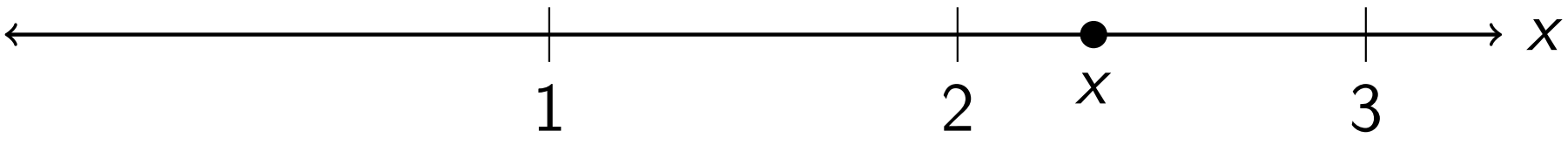

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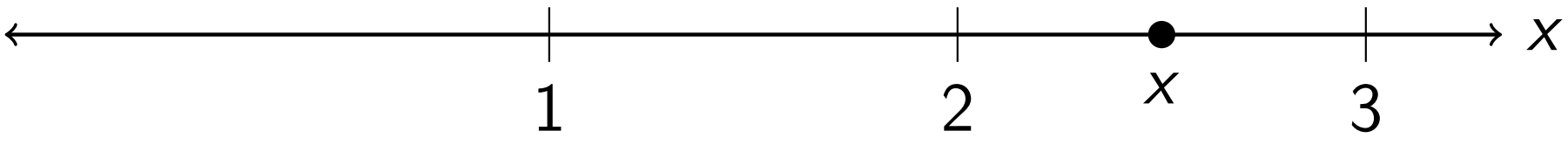

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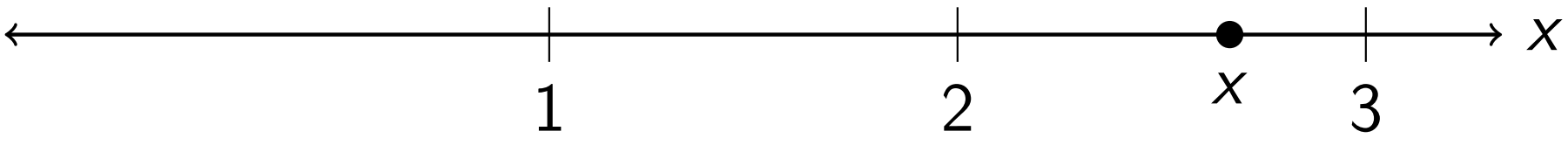
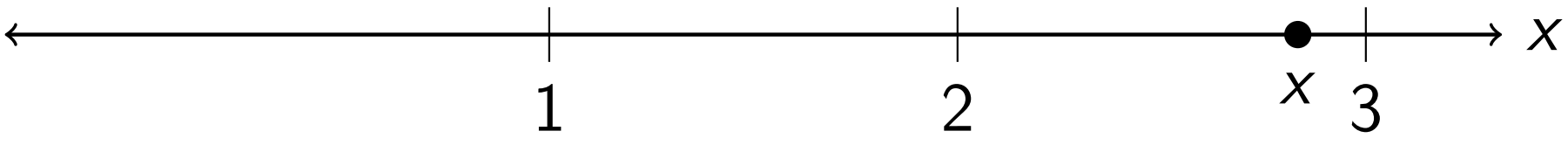


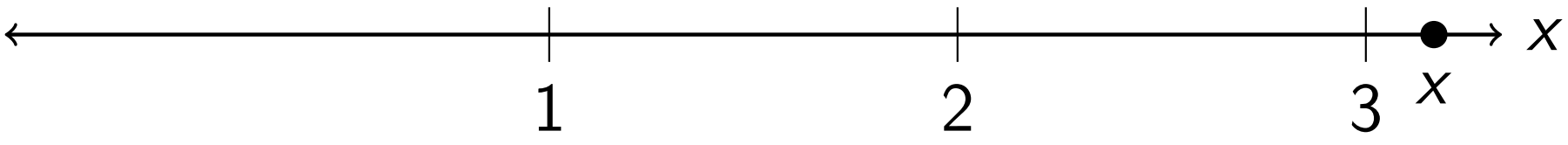
Sign Changes in a Factored Function

$$f(x) = \underset{+}{(x-1)} \quad \underset{+}{(x-2)} \quad \underset{-}{(x-3)} \quad \text{Sign of product: } -$$



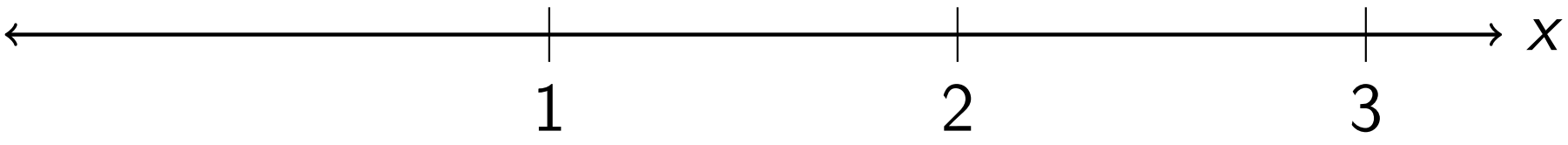
Sign Changes in a Factored Function

$$f(x) = \underset{+}{(x-1)} \quad \underset{+}{(x-2)} \quad \underset{+}{(x-3)} \quad \text{Sign of product: } +$$



Sign Changes in a Factored Function

$f(x) = \quad (x-1) \quad (x-2)^2 \quad (x-3)$ Sign of product:



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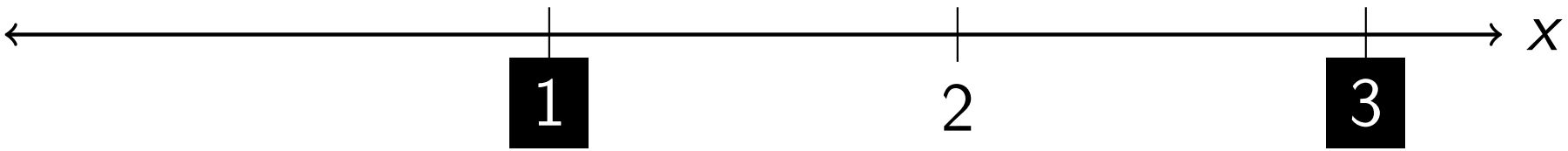
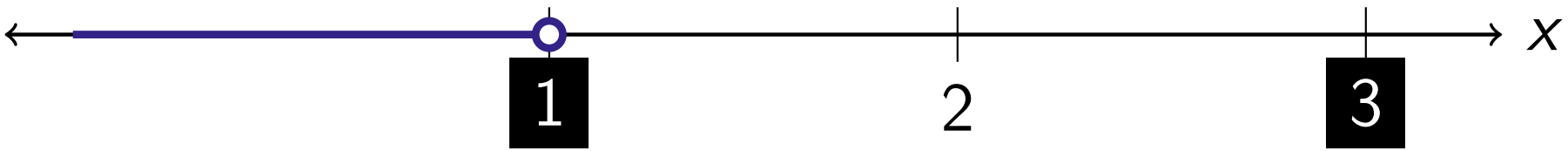
Sign of product:

$$+$$



Sign Changes in a Factored Function

$$f(x) = \underset{-}{(x-1)} \quad \underset{+}{(x-2)^2} \quad \underset{-}{(x-3)} \quad \text{Sign of product:} \quad +$$



Sign Changes in a Factored Function

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Sign Changes in a Factored Function

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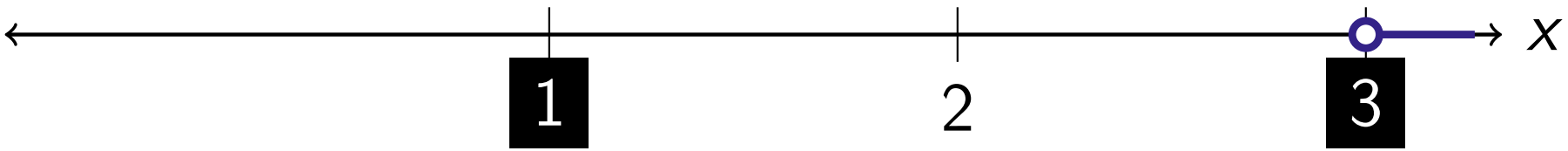
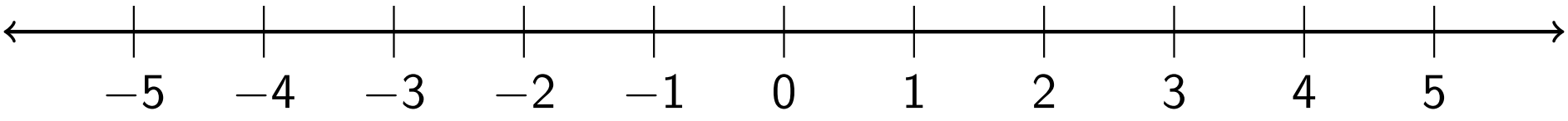


Sign Changes in a Factored Form

Example 1:

$$f(x) = (x-3)(x-1)^2 x(x+2)^3(x+5)^4$$

Where is $f(x)$ positive? Where is it negative?



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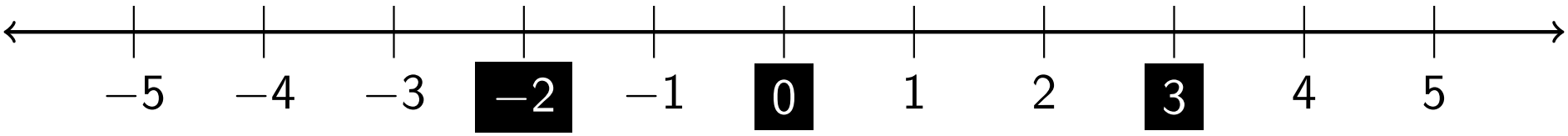

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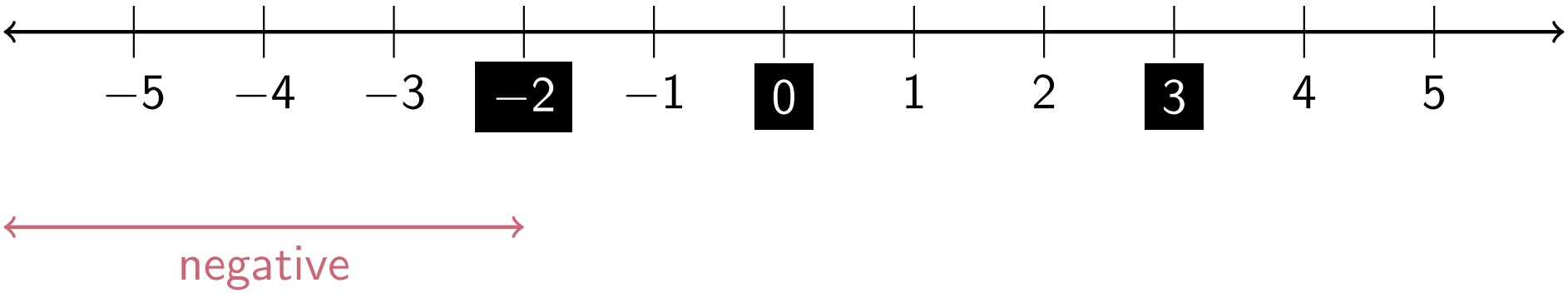

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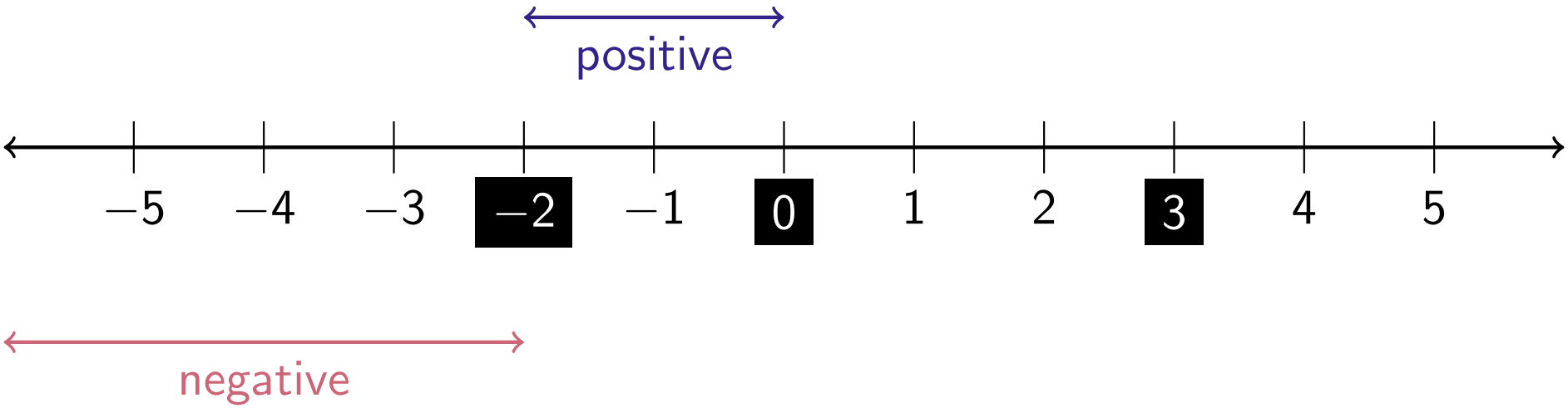

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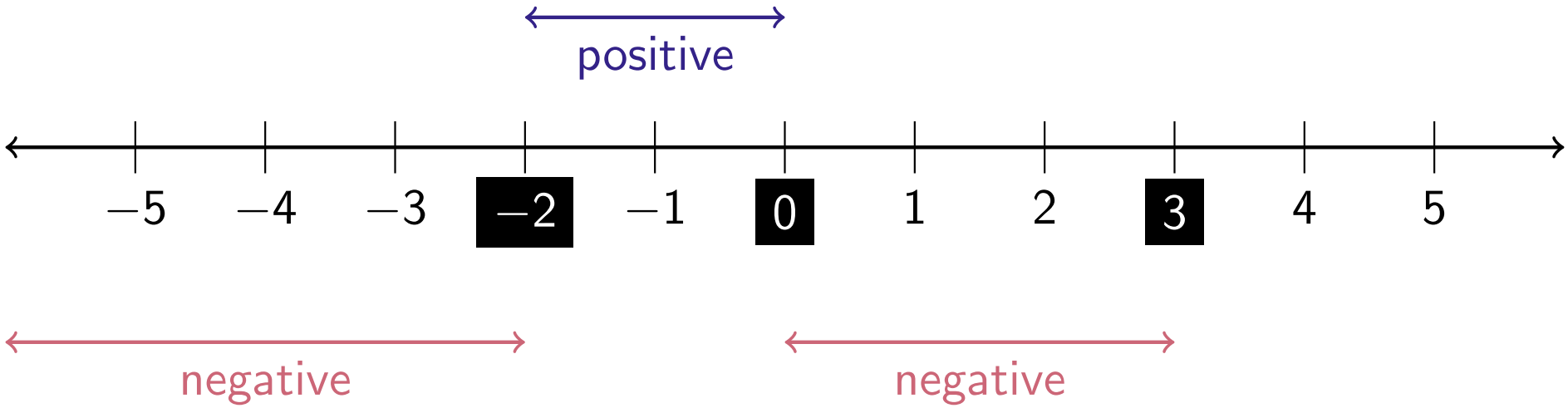

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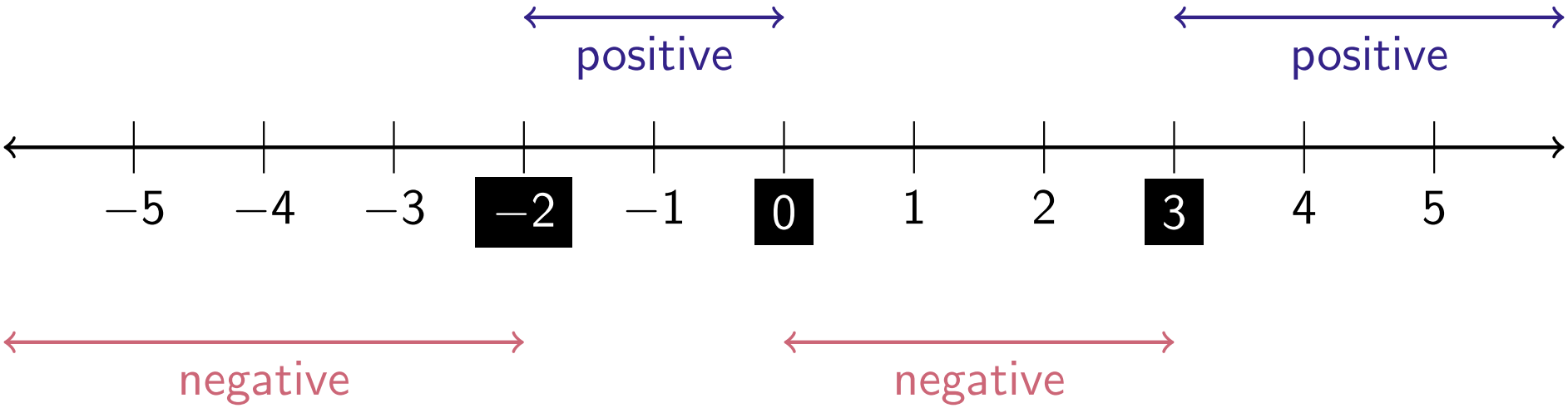

Special Points:

- Roots / zeroes

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- maxima, minima
- CP that is not an extremum

Each of the following functions have a critical point at $x = 0$.
Match the derivatives with their graphs.

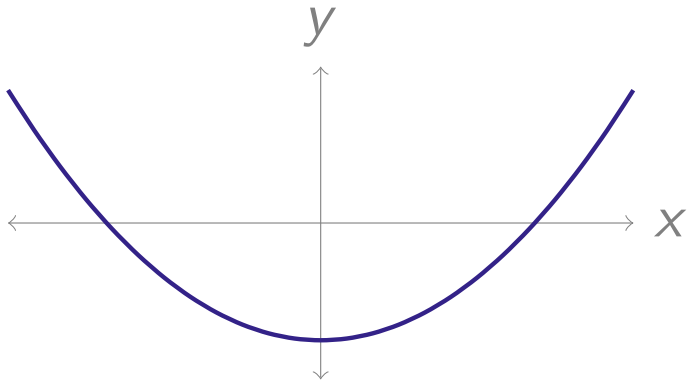
(a) $f'(x)$ negative when $x < 0$
$f'(x)$ negative when $x > 0$

(b) $f'(x)$ positive when $x < 0$
$f'(x)$ positive when $x > 0$

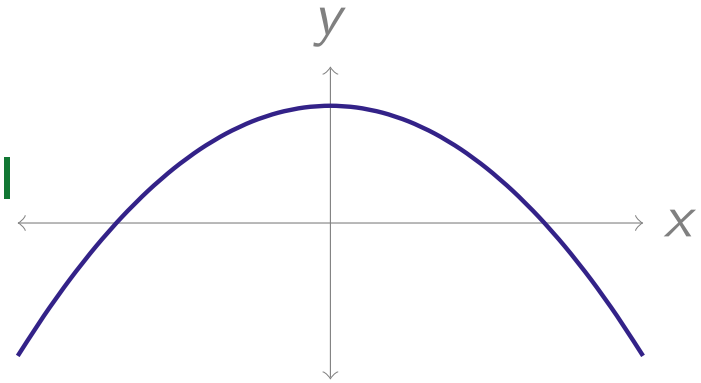
(c) $f'(x)$ negative when $x < 0$
$f'(x)$ positive when $x > 0$

(d) $f'(x)$ positive when $x < 0$
$f'(x)$ negative when $x > 0$

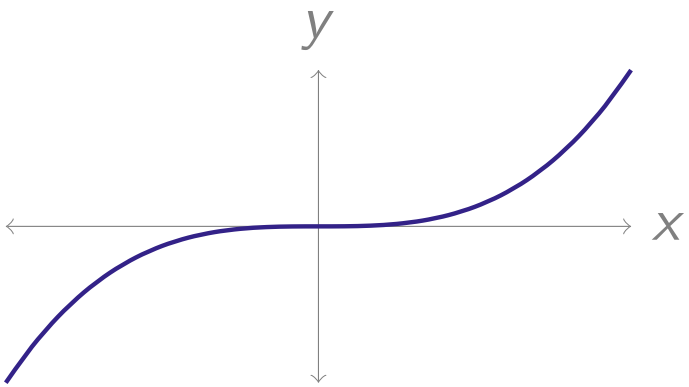
I



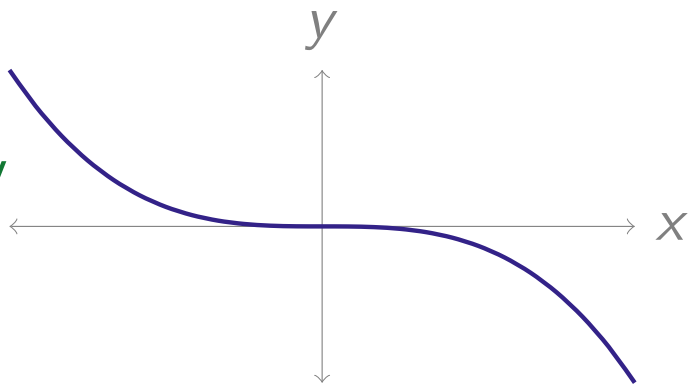
II



III



IV



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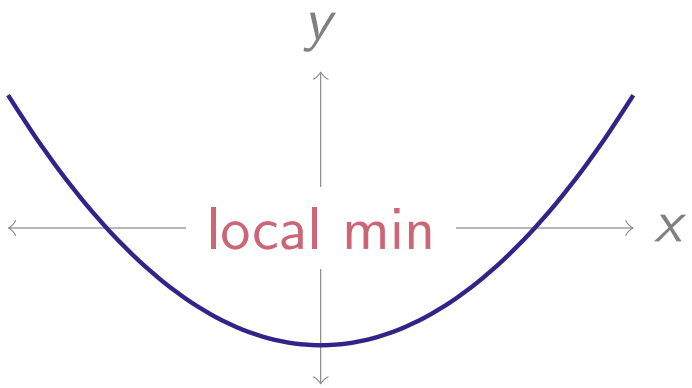

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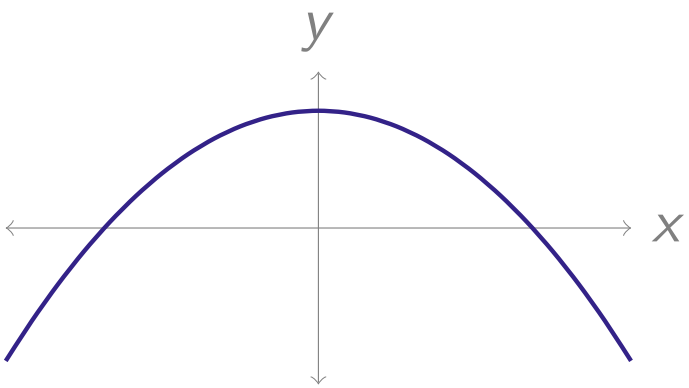

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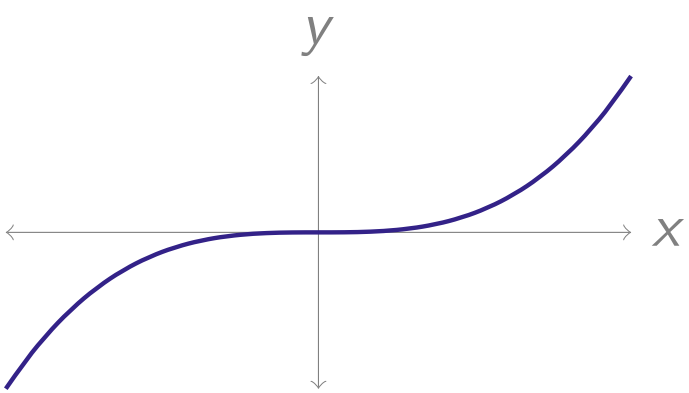

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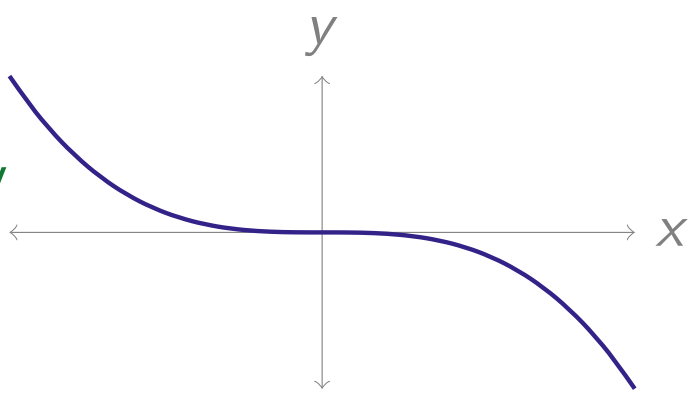

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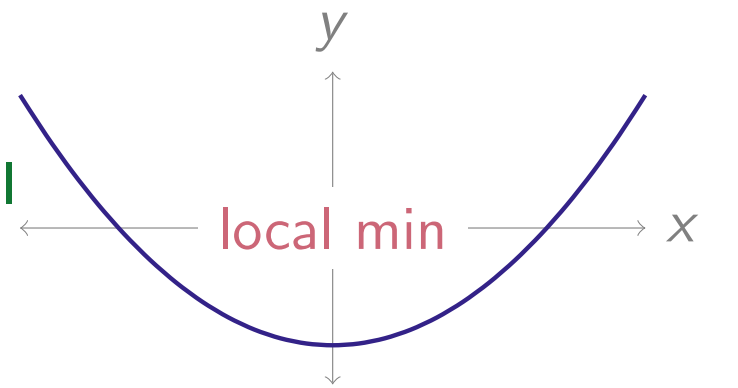

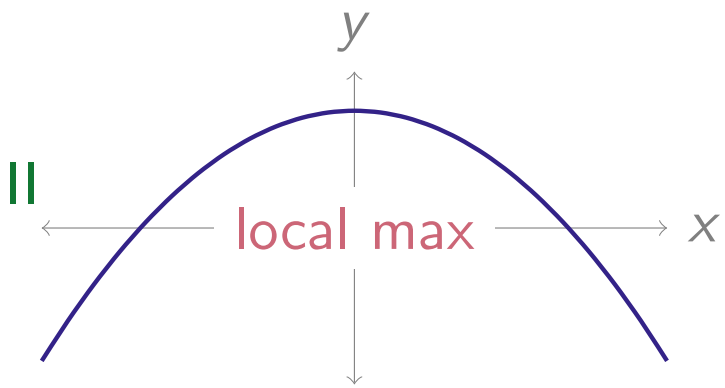

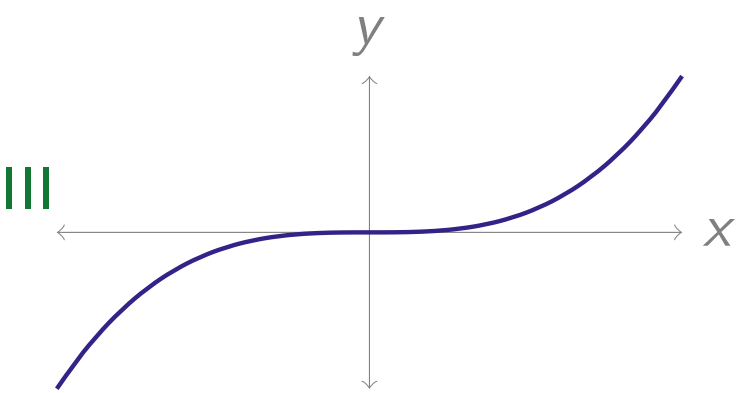

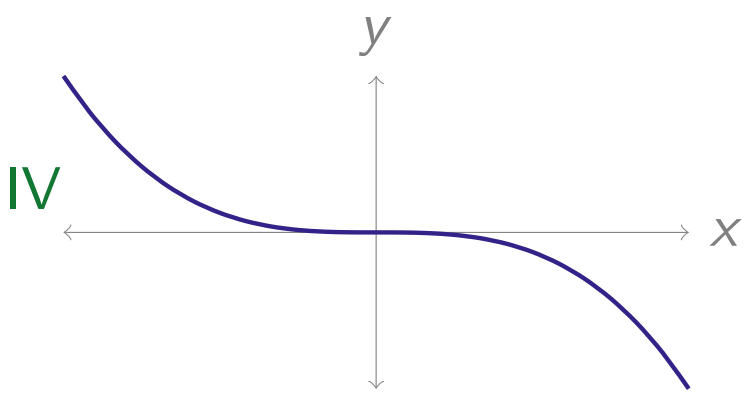

Match the second derivatives with their graphs.

(a) $f''(x)$ negative

(b) $f''(x)$ positive

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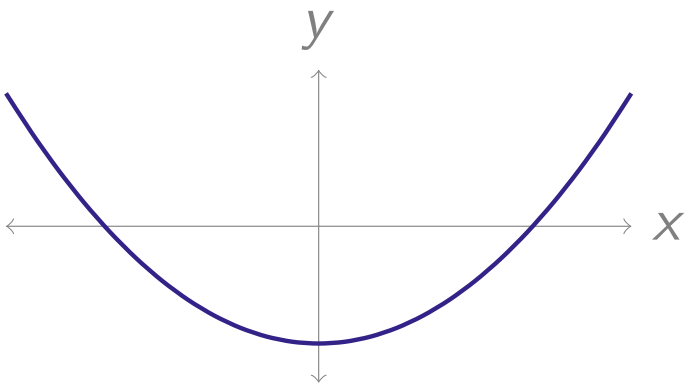

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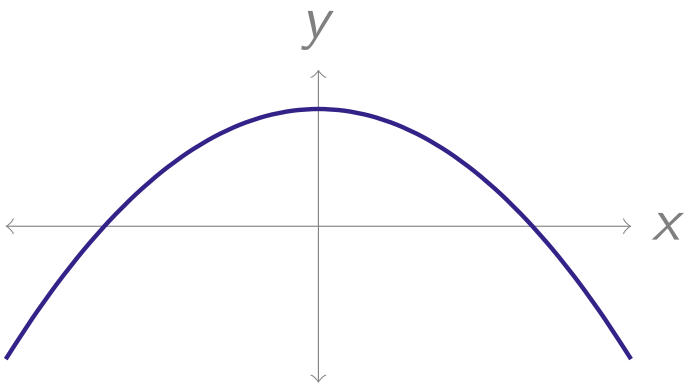

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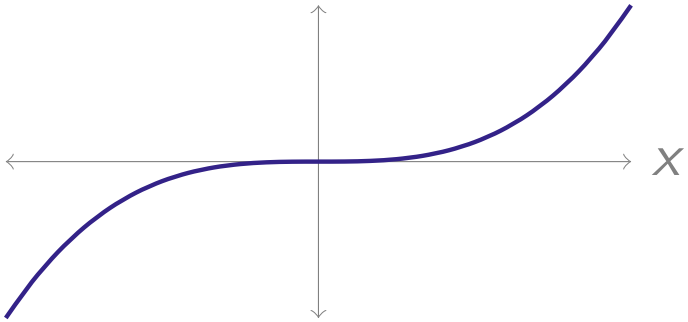

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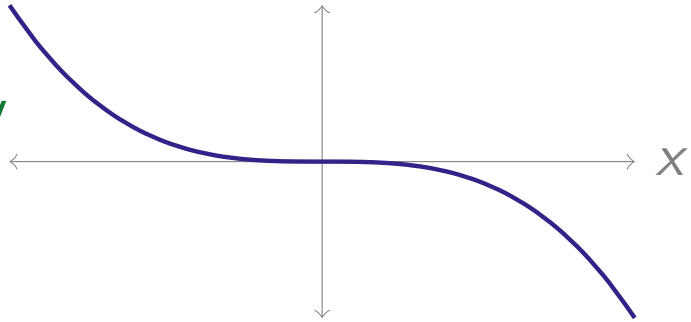

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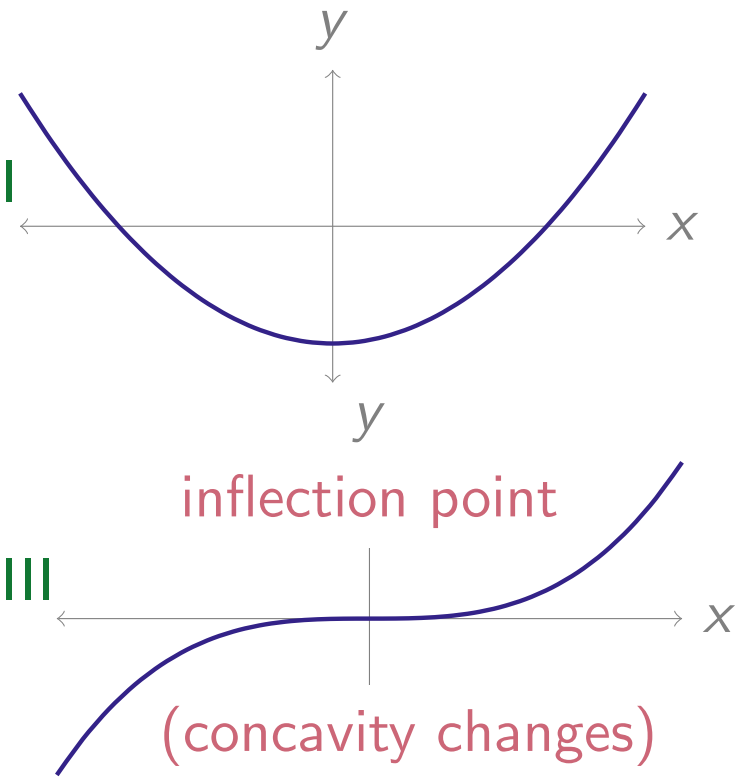

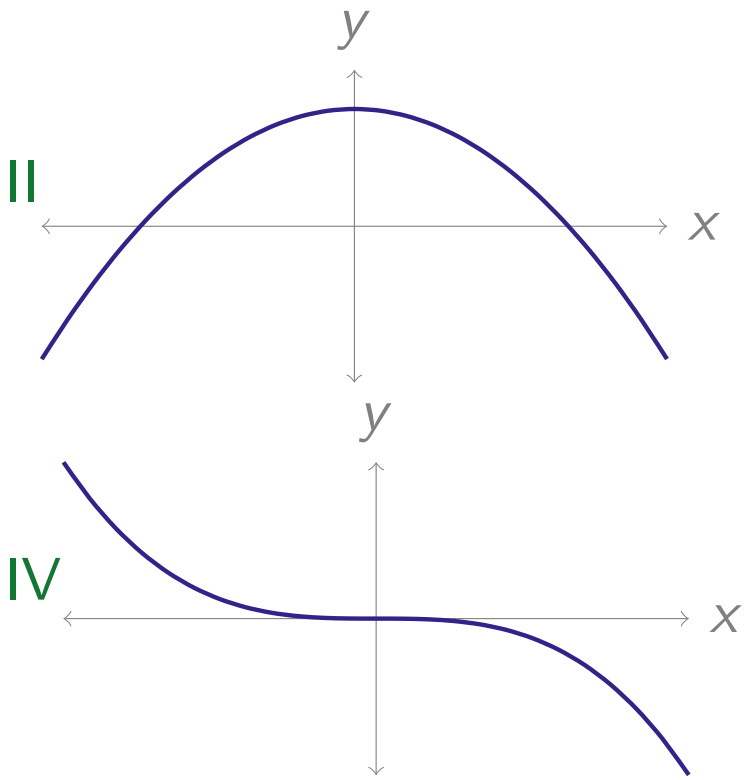

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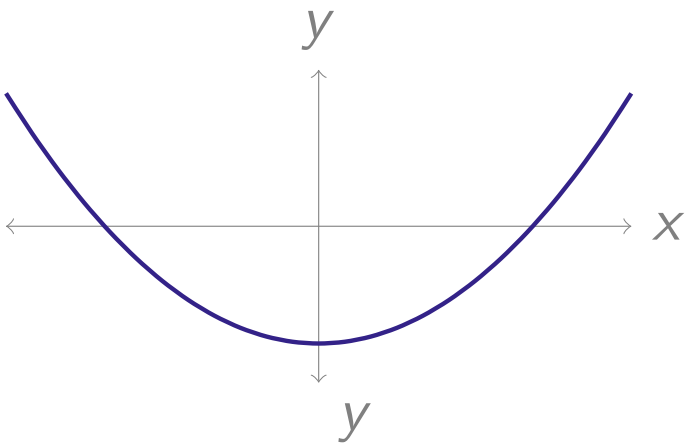

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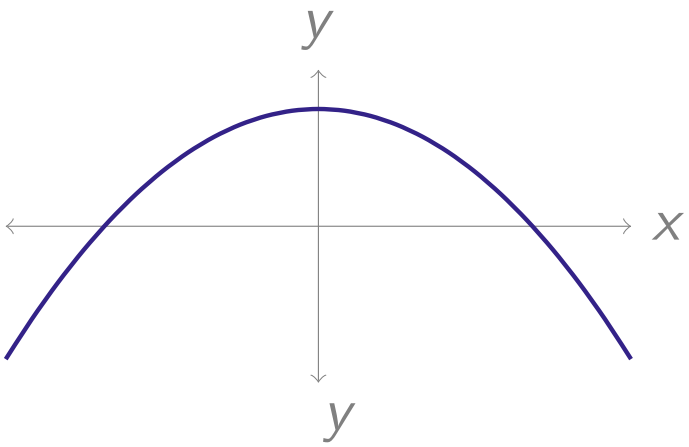
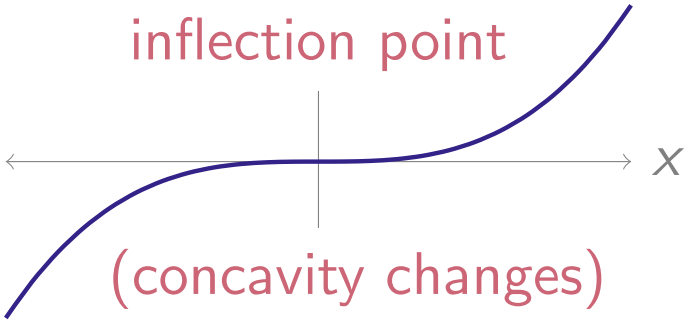


inflection point

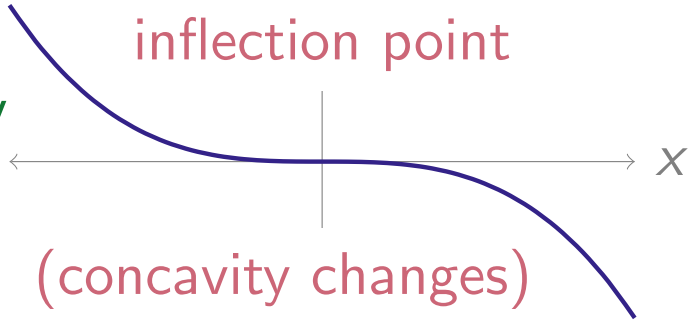
III



(concavity changes)

inflection point

IV



(concavity changes)

Suppose $x = a$ is a critical point of the continuous function $f(x)$. That is, $f'(a) = 0$ or $f'(x)$ does not exist at a.

(a) $f'(a)$ changes from neg to pos

(b) $f'(a)$ changes from pos to neg

(c) $f''(a) = 0$

(d) $f''(a)$ changes from neg to pos

(e) $f''(a)$ changes from pos to neg

(f) $f''(a)$ positive and $f'(a) = 0$

(g) $f''(a)$ negative and $f'(a) = 0$

I. inflection point

II. local max

III. local min

IV. not a local extrema

V. could be local max, local min, or neither

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(f) $f''(a)$ positive and $f'(a) = 0$

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(a) - III (b) - II (c) - V (d),(e) - I,V (f) - III (g) - II

Tests

The information from the previous questions forms a method for finding local extrema.

Tests

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- First, find all critical points (where $f'(x) = 0$ or $f'(x)$ doesn't exist). Local extrema can ONLY occur at critical points.

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- First, find all critical points (where $f'(x) = 0$ or $f'(x)$ doesn't exist). Local extrema can ONLY occur at critical points.
- Now, we have to classify those critical points: they could be local maxes, local mins, or neither. Two options for testing:

Tests

The information from the previous questions forms a method for finding local extrema.

- First, find all critical points (where $f'(x) = 0$ or $f'(x)$ doesn't exist). Local extrema can ONLY occur at critical points.
- Now, we have to classify those critical points: they could be local maxes, local mins, or neither. Two options for testing:
- Using the second derivative:
- Using the first derivative:

Tests

The information from the previous questions forms a method for finding local extrema.

- First, find all critical points (where $f'(x) = 0$ or $f'(x)$ doesn't exist). Local extrema can ONLY occur at critical points.
- Now, we have to classify those critical points: they could be local maxes, local mins, or neither. Two options for testing:
- Using the second derivative:
  - If $f''(x) > 0$, the CP is a local
  - If $f''(x) < 0$, the CP is a local
  - If $f''(x) = 0$,
- Using the first derivative:

Tests

The information from the previous questions forms a method for finding local extrema.

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  - If $f''(x) = 0$, we need to look at the first derivative to decide
- Using the first derivative:

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- Using the second derivative:
  - If $f''(x) > 0$, the CP is a local min
  - If $f''(x) < 0$, the CP is a local max
  - If $f''(x) = 0$, we need to look at the first derivative to decide
- Using the first derivative:
  - If $f'(x)$ does not change signs at the CP, then
  - If $f'(x)$ changes from increasing to decreasing, the CP is a local
  - If $f'(x)$ changes from decreasing to increasing, the CP is a local

Tests

The information from the previous questions forms a method for finding local extrema.

- First, find all critical points (where $f'(x) = 0$ or $f'(x)$ doesn't exist). Local extrema can ONLY occur at critical points.
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  - If $f''(x) > 0$, the CP is a local min
  - If $f''(x) < 0$, the CP is a local max
  - If $f''(x) = 0$, we need to look at the first derivative to decide
- Using the first derivative:
  - If $f'(x)$ does not change signs at the CP, then it's neither a local max nor a local min–it's just a momentary flat point
  - If $f'(x)$ changes from increasing to decreasing, the CP is a local
  - If $f'(x)$ changes from decreasing to increasing, the CP is a local

Tests

The information from the previous questions forms a method for finding local extrema.

- First, find all critical points (where $f'(x) = 0$ or $f'(x)$ doesn't exist). Local extrema can ONLY occur at critical points.
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  - If $f''(x) > 0$, the CP is a local min
  - If $f''(x) < 0$, the CP is a local max
  - If $f''(x) = 0$, we need to look at the first derivative to decide
- Using the first derivative:
  - If $f'(x)$ does not change signs at the CP, then it's neither a local max nor a local min–it's just a momentary flat point
  - If $f'(x)$ changes from increasing to decreasing, the CP is a local max
  - If $f'(x)$ changes from decreasing to increasing, the CP is a local

Tests

The information from the previous questions forms a method for finding local extrema.

- First, find all critical points (where $f'(x) = 0$ or $f'(x)$ doesn't exist). Local extrema can ONLY occur at critical points.
- Now, we have to classify those critical points: they could be local maxes, local mins, or neither. Two options for testing:
- Using the second derivative:
  - If $f''(x) > 0$, the CP is a local min
  - If $f''(x) < 0$, the CP is a local max
  - If $f''(x) = 0$, we need to look at the first derivative to decide
- Using the first derivative:
  - If $f'(x)$ does not change signs at the CP, then it's neither a local max nor a local min–it's just a momentary flat point
  - If $f'(x)$ changes from increasing to decreasing, the CP is a local max
  - If $f'(x)$ changes from decreasing to increasing, the CP is a local min

$$f(x) = \frac{1}{4}x^4 - 2x^3 + \frac{9}{2}x^2 + 1$$

1. Find all critical points of $f(x)$ (that is, find where $f'(x) = 0$)
2. Are they local maxima, local minima, or neither?

Challenge: sketch the function.

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To find CPs, we set $f'(x) = 0$.

$$f'(x) = x^3 - 6x^2 + 9x = x(x^2 - 6x + 9) = x(x-3)^2$$

So the CPs are at $x = 0$ and $x = 3$

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To classify them, note $f''(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3)$, so $f''(0) > 0$ and $f''(3) = 0$.

Then at $x = 0$ there is a LOCAL MIN, and we need more information for $x = 3$.

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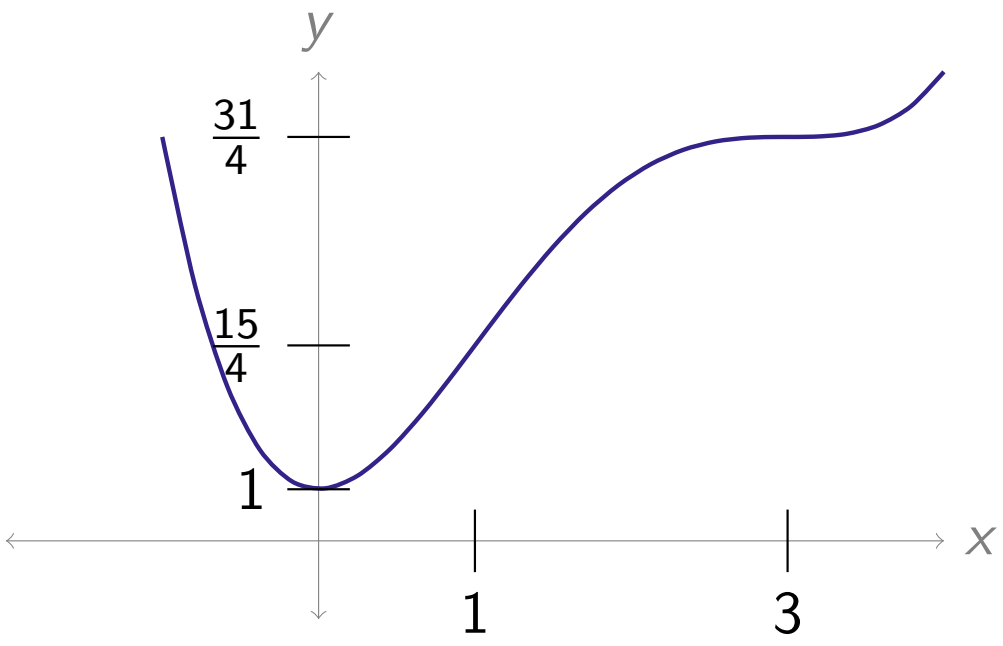
To classify them, note $f''(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3)$, so $f''(0) > 0$ and $f''(3) = 0$.

Then at $x = 0$ there is a LOCAL MIN, and we need more information for $x = 3$. Since $f'(x)$ is positive both just before and just after $x = 3$, there is neither a local max nor a local min there.

Since $f''(x)$ changes signs at $x = 3$, there is an inflection point there.

$$f(x) = \frac{1}{4}x^4 - 2x^3 + \frac{9}{2}x^2 + 1$$

Sketch.



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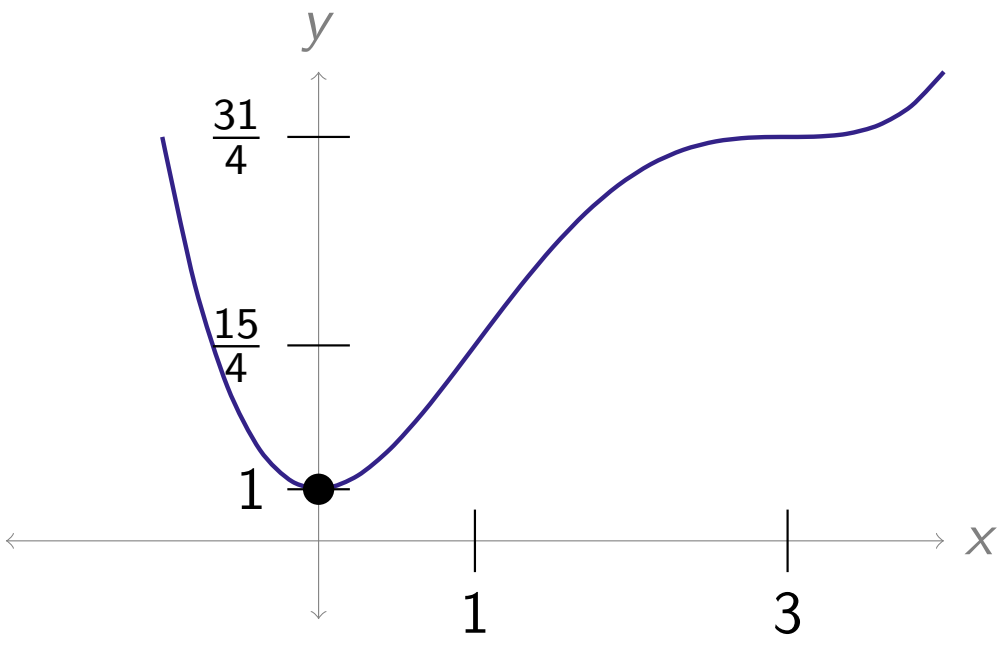

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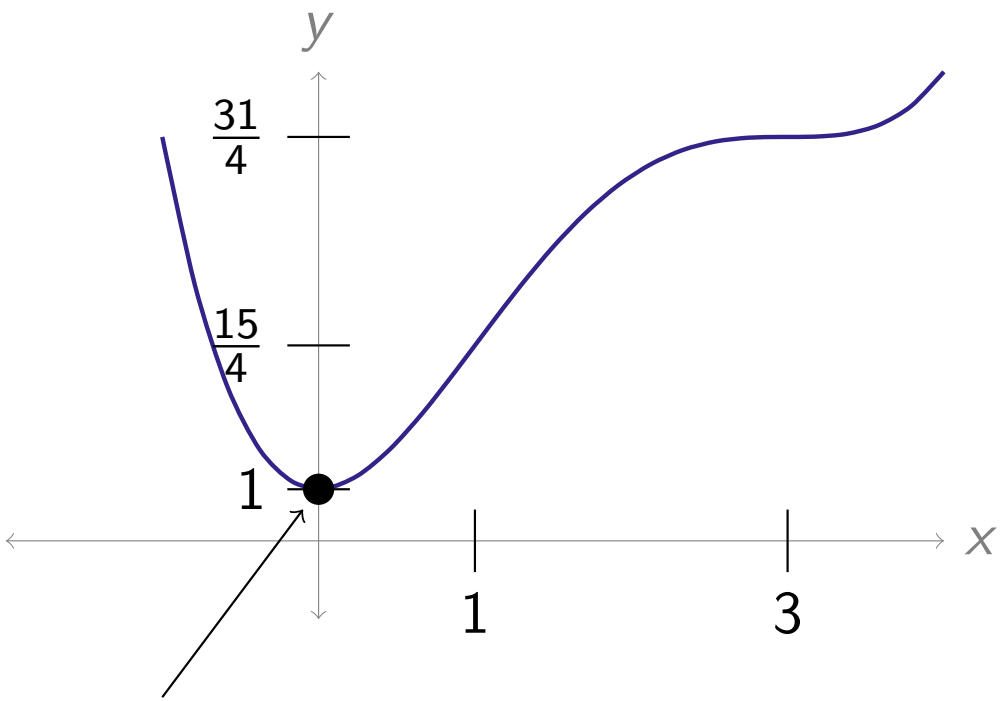
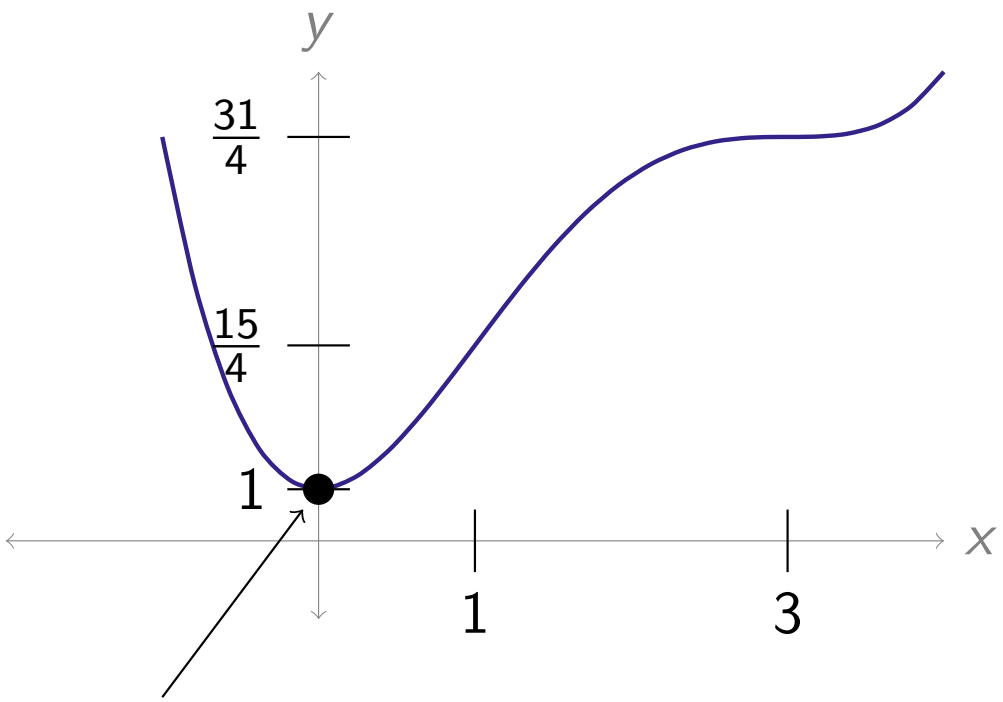


local min at $(0, 1)$

$$f(x) = \frac{1}{4}x^4 - 2x^3 + \frac{9}{2}x^2 + 1$$

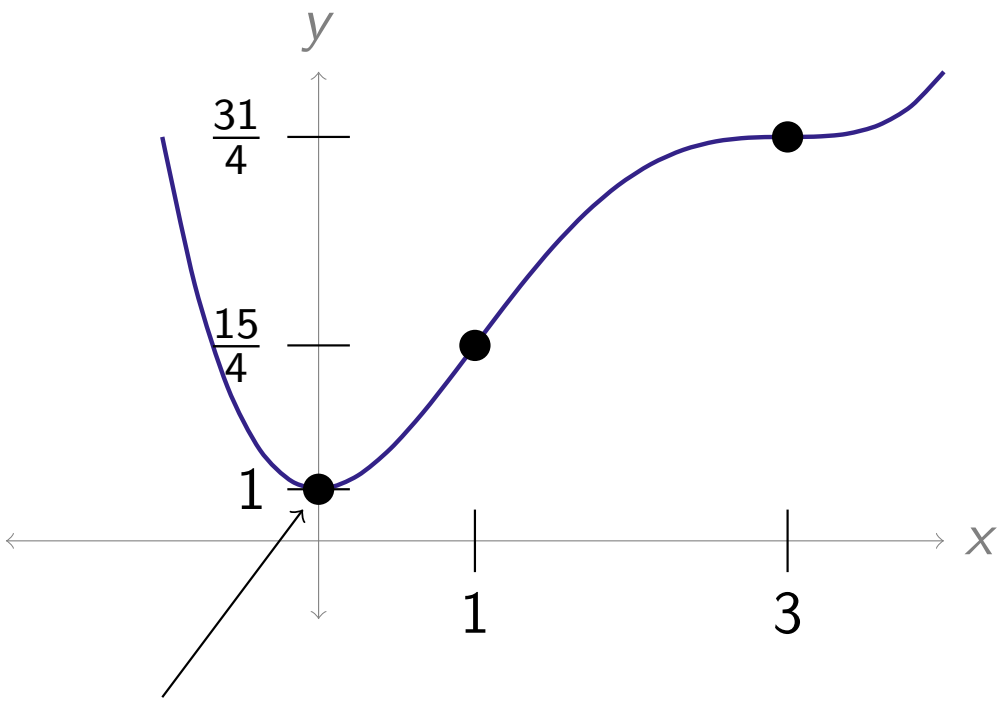
Sketch.



local min at $(0, 1)$
also **global** min

$$f(x) = \frac{1}{4}x^4 - 2x^3 + \frac{9}{2}x^2 + 1$$

Sketch.



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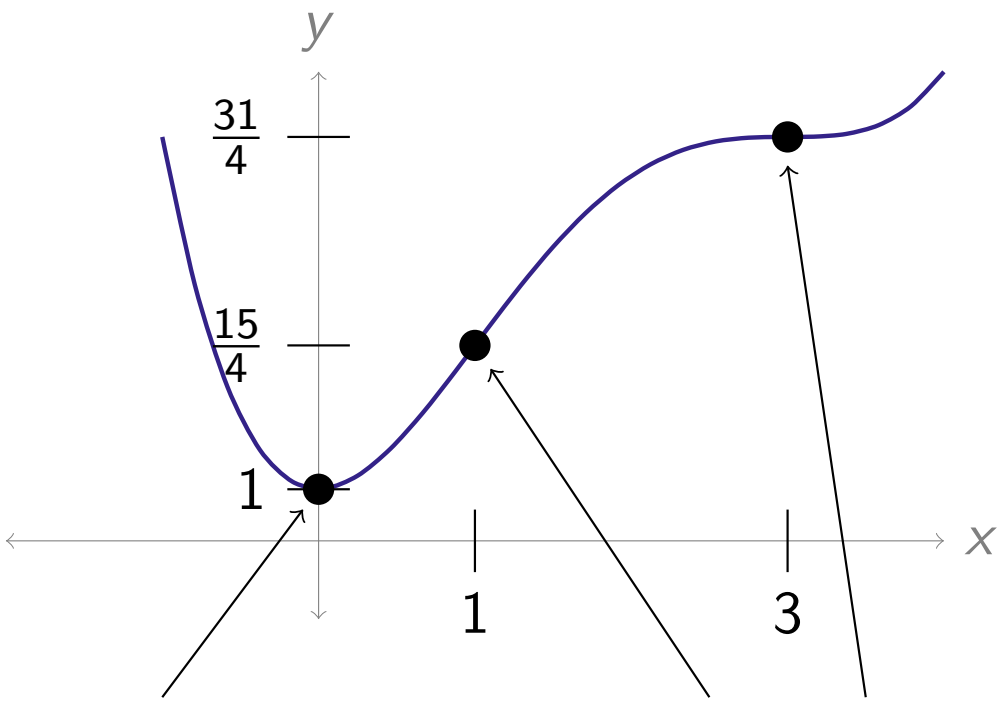
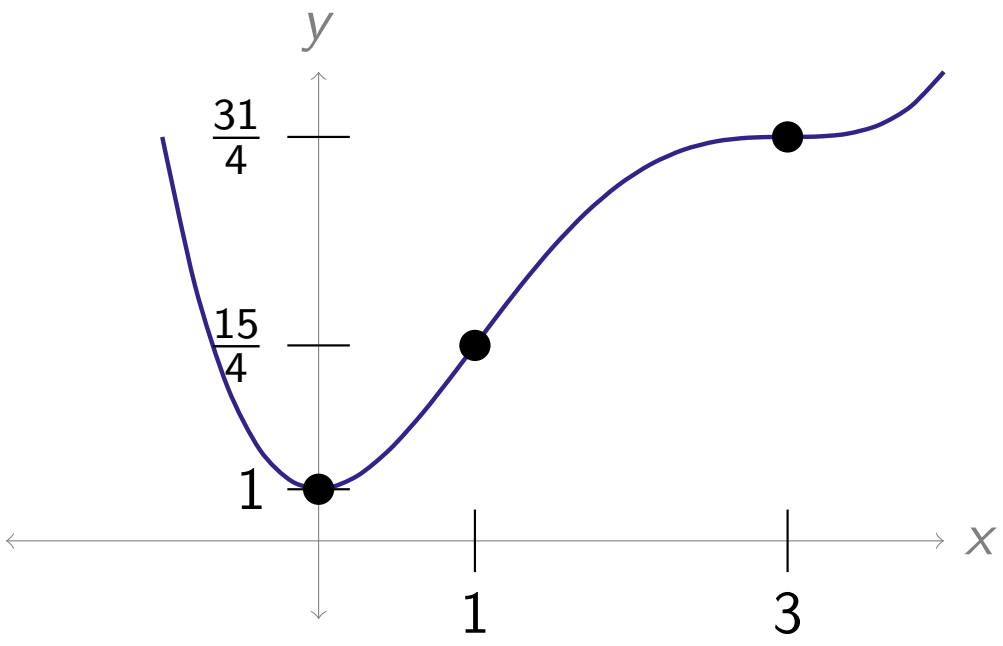


local min at $(0, 1)$
also **global** min

inflection points at $(3, \frac{31}{4})$
and $(1, \frac{11}{4})$

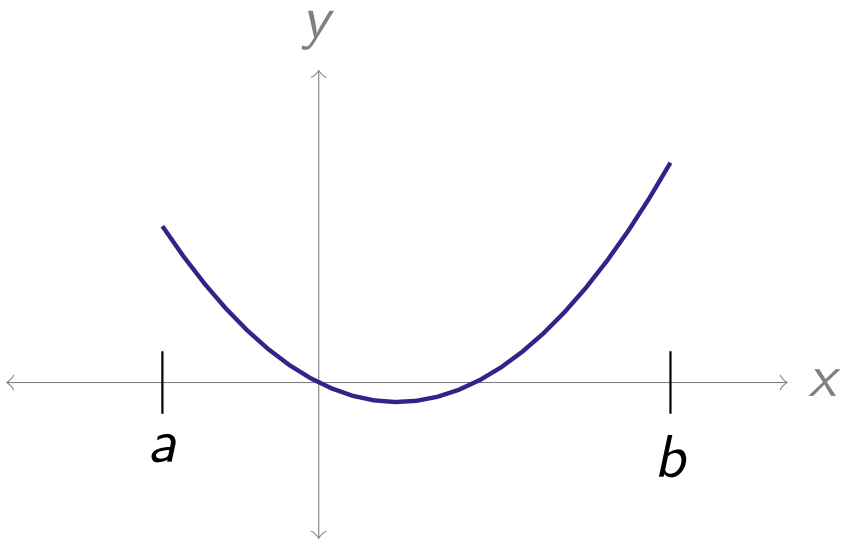
$$f(x) = \frac{1}{4}x^4 - 2x^3 + \frac{9}{2}x^2 + 1$$

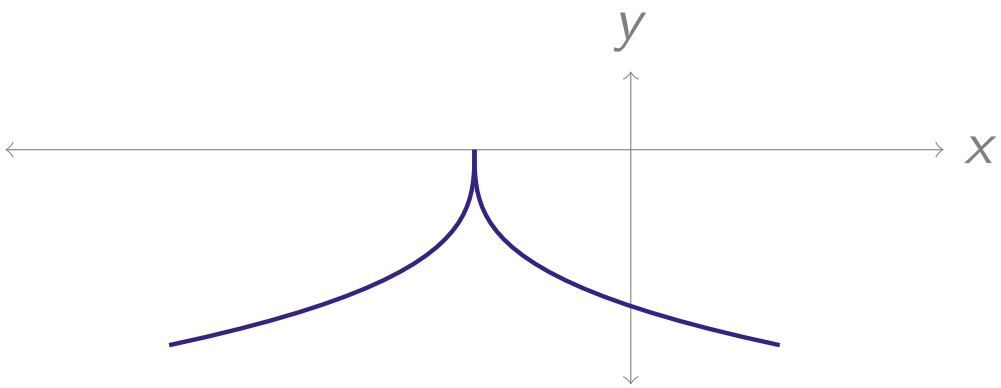
Sketch.



Global extrema over a closed interval

Where do global extrema occur?





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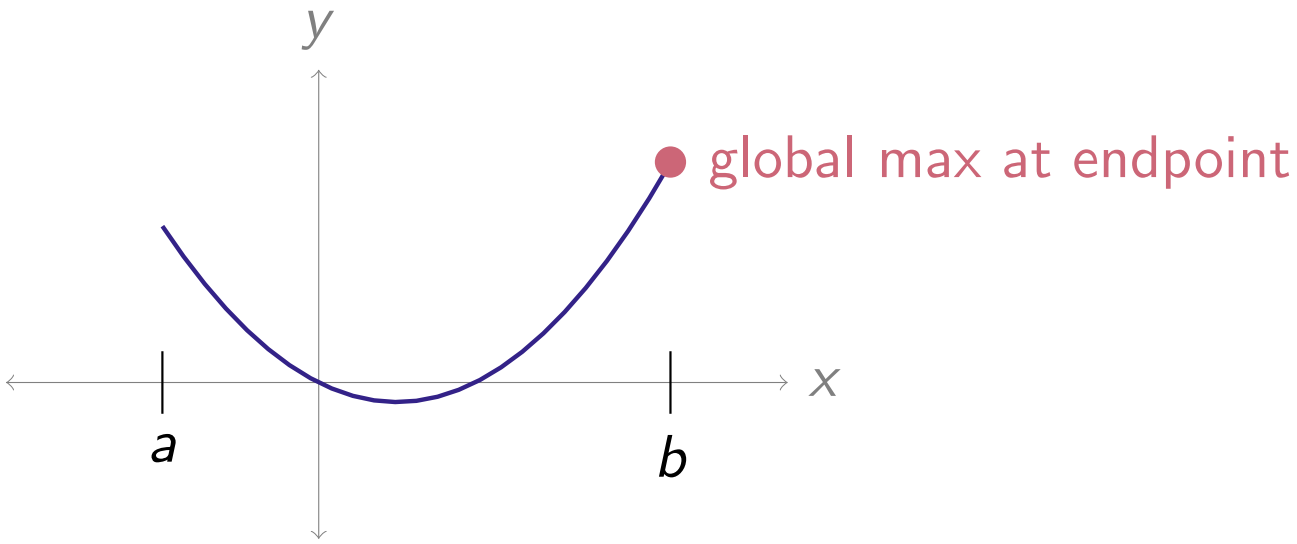

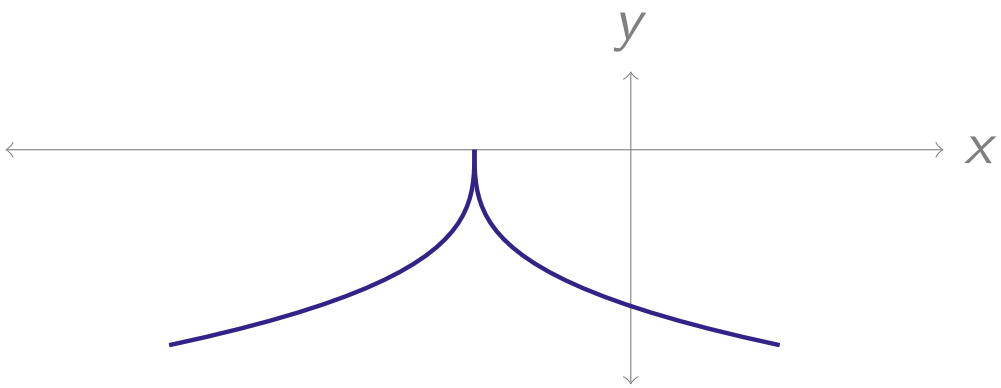

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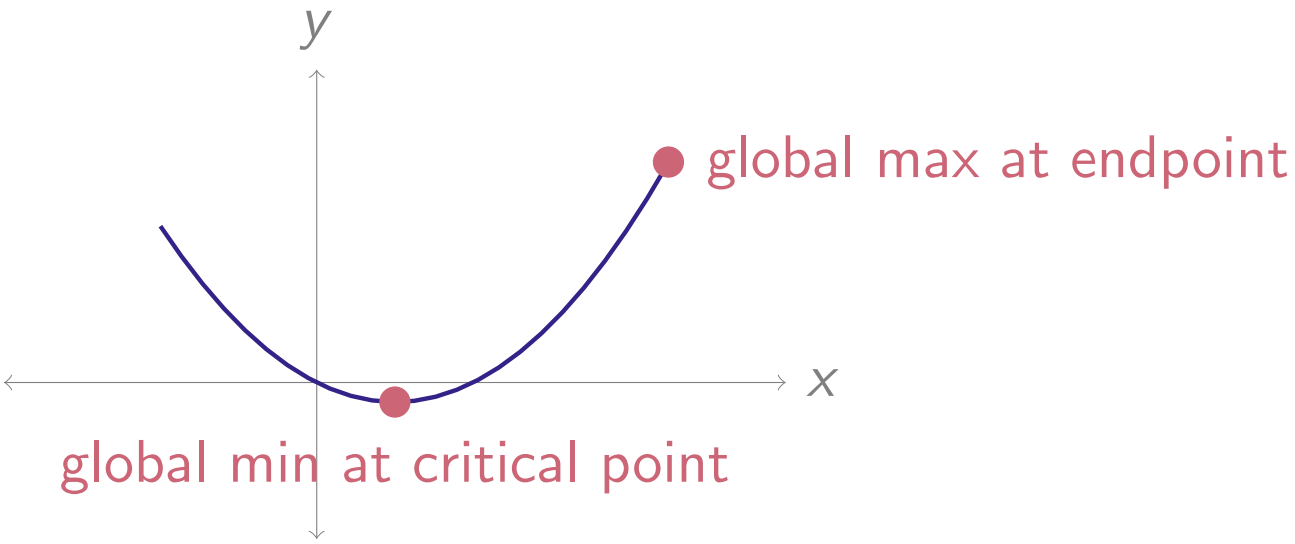

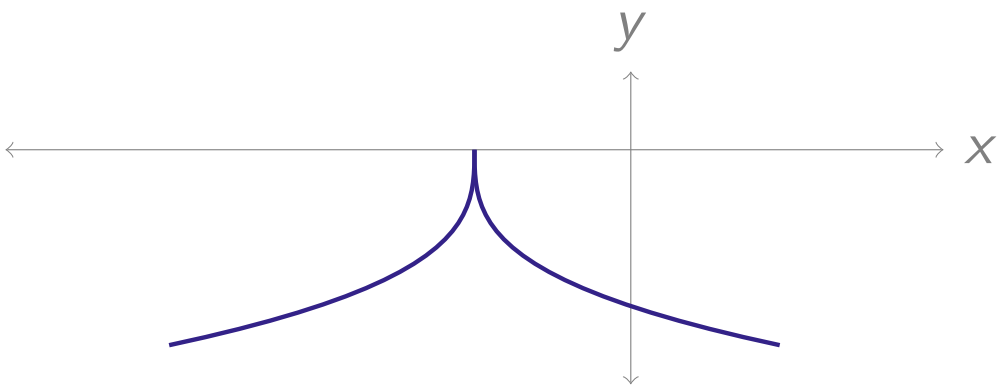

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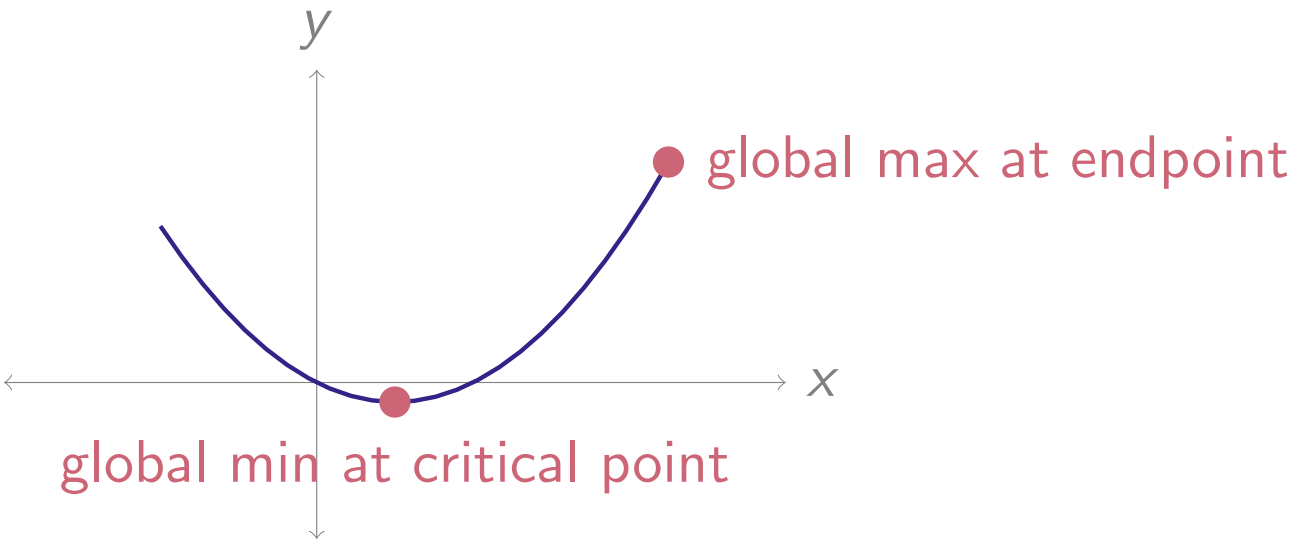

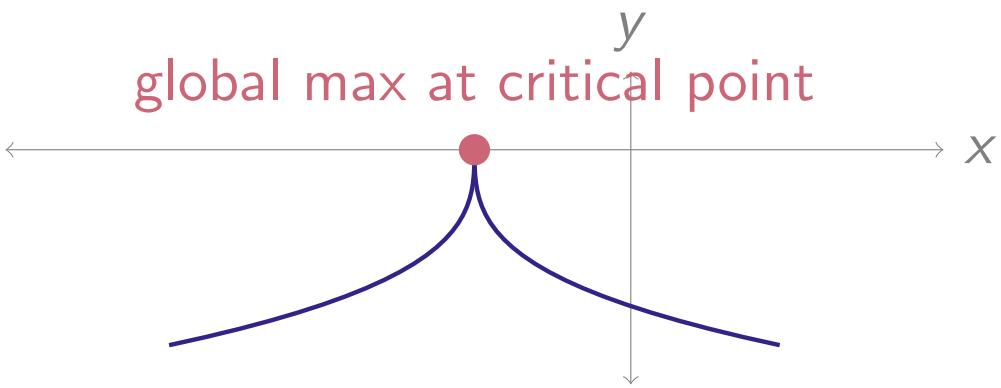

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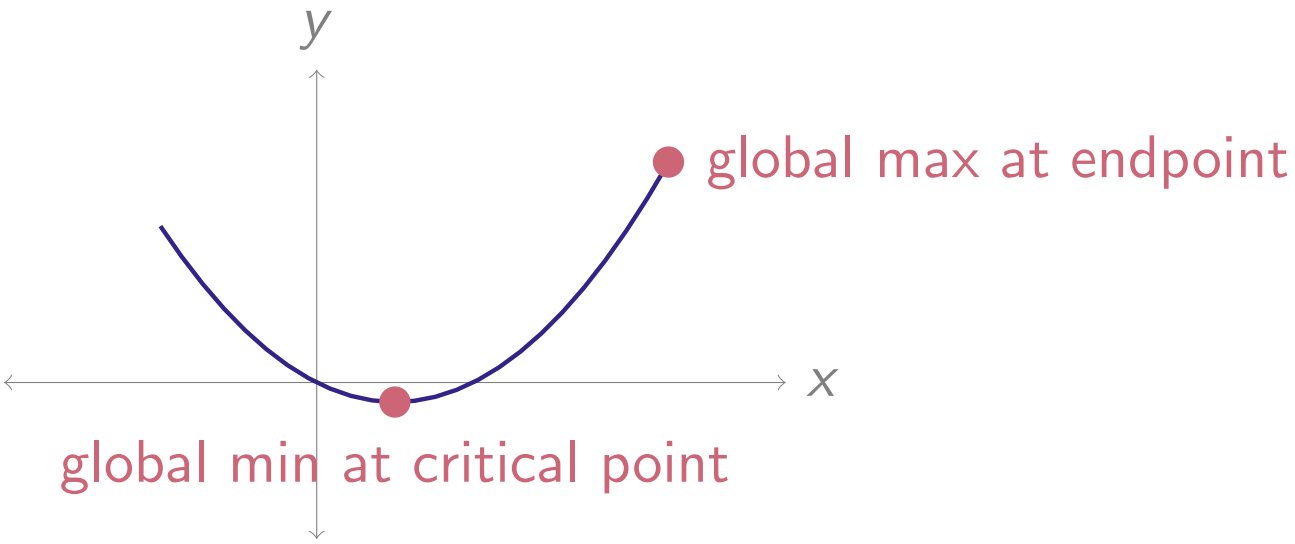
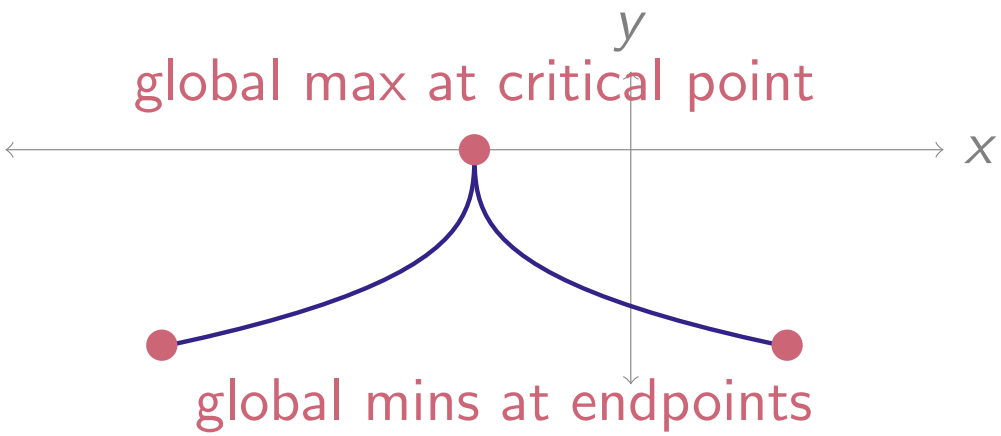
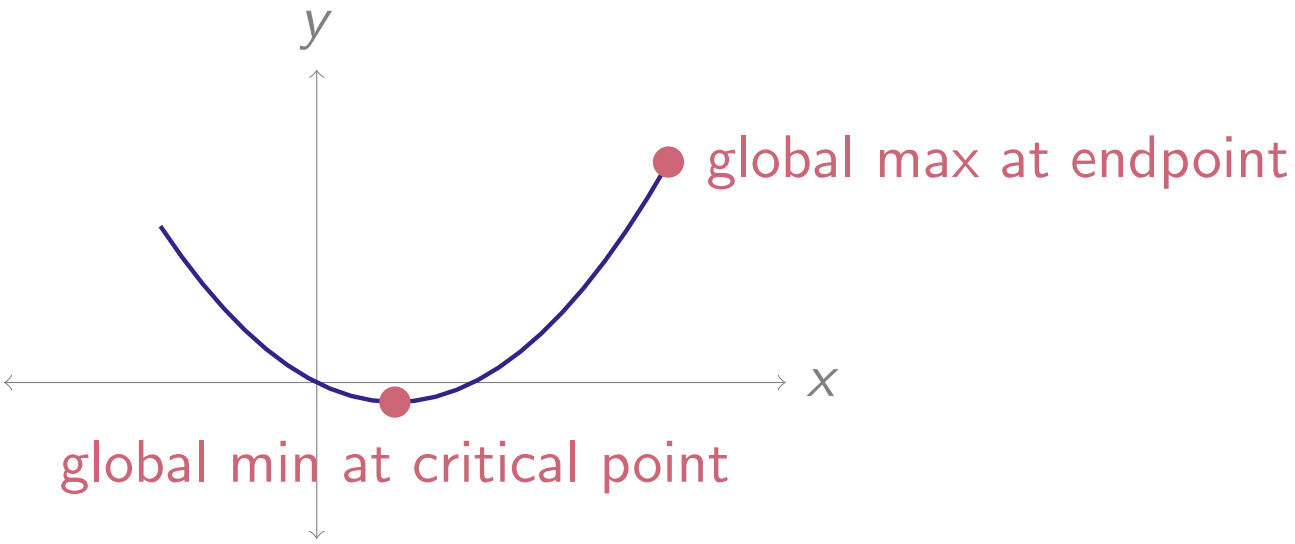
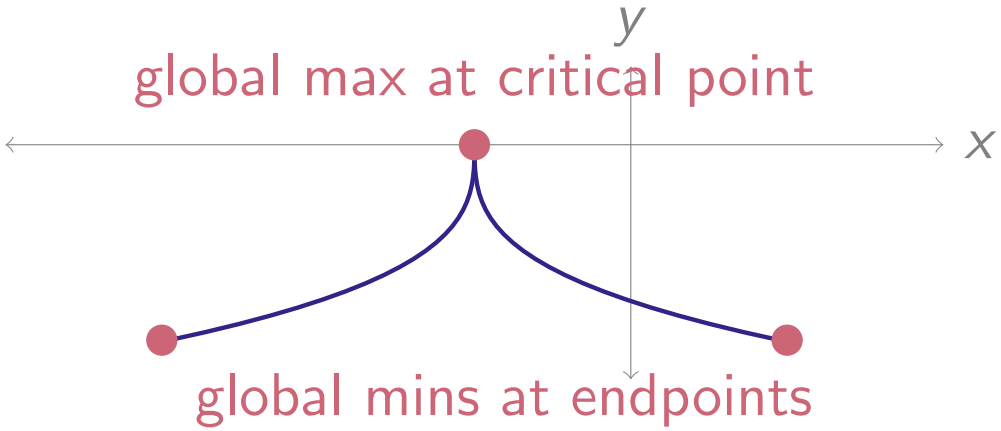




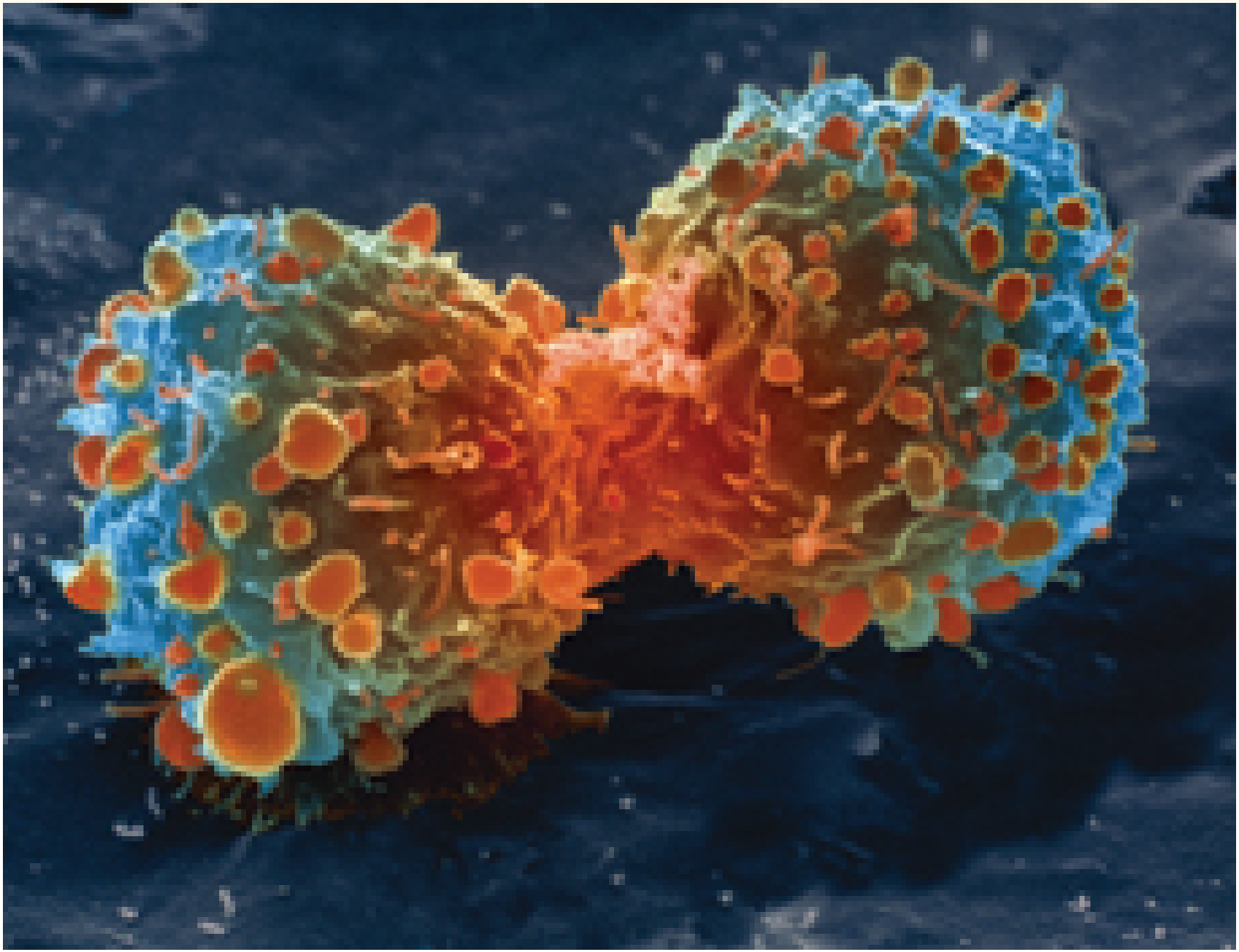
Global extrema over a closed interval

Where do global extrema occur? endpoints or critical points





Cell Division



Cell Division

A cell of age t hours has a probability[1] of dividing given by the function

$$P(t) = \frac{at}{t^3 + 16},$$

where a is the constant $\frac{9\sqrt[6]{3}}{2\pi}$.

From time $t = 0$ to time $t = 10$, when is the cell likeliest to divide? When is it least likely to divide?

[1] Actually, we're giving a probability density function... it's a little more complicated than what we said. Still, a higher value of $P(t)$ means a cell age that is more reproduce-y, so we just want to find the max and min of $P(t)$ when $0 \leq t \leq 10$.

Cell Division

We're looking for the absolute maximum and minimum of $P(t)$ on the interval $[0, 10]$. This can occur at the endpoints (that is, $t = 0$ and $t = 10$) or at critical points.
To find the critical points, we take the first derivative.

$$P'(t) = \frac{(t^3 + 16)(a) - at(3t^2)}{(t^3 + 16)^2} = a\frac{16 - 2t^3}{(t^3 + 16)^2}$$

We find this is undefined only at negative numbers–not in our interval, so we don't care. It's zero when $16 - 2t^3 = 0$, i.e. when $t = 2$. So, our sole critical point is at $t = 2$.
Now, we compare $P(0)$, $P(2)$, and $P(10)$ to see which one is largest and which smallest.

- $P(0) = 0$: this must be smallest, since probabilities can't be negative
- $P(2) = \frac{2a}{2^3+16} = \frac{a}{12}$
- $P(10) = \frac{10a}{10^3+16} = \frac{10a}{1016} = \frac{a}{101.6}$

Since $P(2) > P(10) > P(0)$, the cell is likeliest to divide when $t = 2$, and least likely to divide when $t = 10$.

Sketch

$$f(x) = \frac{(x-1)^2}{x^3}$$

Include the following, if they exist:

- Roots (zeroes)
- Discontinuities
- Asymptotes
- Local and global extrema (maxes and mins)
- Inflection points and concavity

Sketch

$$f(x) = \frac{(x-1)^2}{x^3}$$

$$f'(x) = \frac{-(x-1)(x-3)}{x^4}$$

$$f''(x) = \frac{2(x^2-6x+6)}{x^5}$$

Sketch

$f(x) = \frac{(x-1)^2}{x^3}$

- $f(1) = 0$
- blow-up discontinuity at $x = 0$
- horizontal asymptote $y = 0$ both sides

$f'(x) = \frac{-(x-1)(x-3)}{x^4}$

$f''(x) = \frac{2(x^2-6x+6)}{x^5}$

Sketch

$f(x) = \frac{(x-1)^2}{x^3}$

- $f(1) = 0$
- blow-up discontinuity at $x = 0$
- horizontal asymptote $y = 0$ both sides

$f'(x) = \frac{-(x-1)(x-3)}{x^4}$

- CPs at $x = 1, 3$
- $f(x)$ increasing on $(1, 3)$, decreasing elsewhere (except $x = 0$)
- So, local min at $x = 1$, local max at $x = 3$

$f''(x) = \frac{2(x^2-6x+6)}{x^5}$

Sketch

$f(x) = \frac{(x-1)^2}{x^3}$

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$f'(x) = \frac{-(x-1)(x-3)}{x^4}$

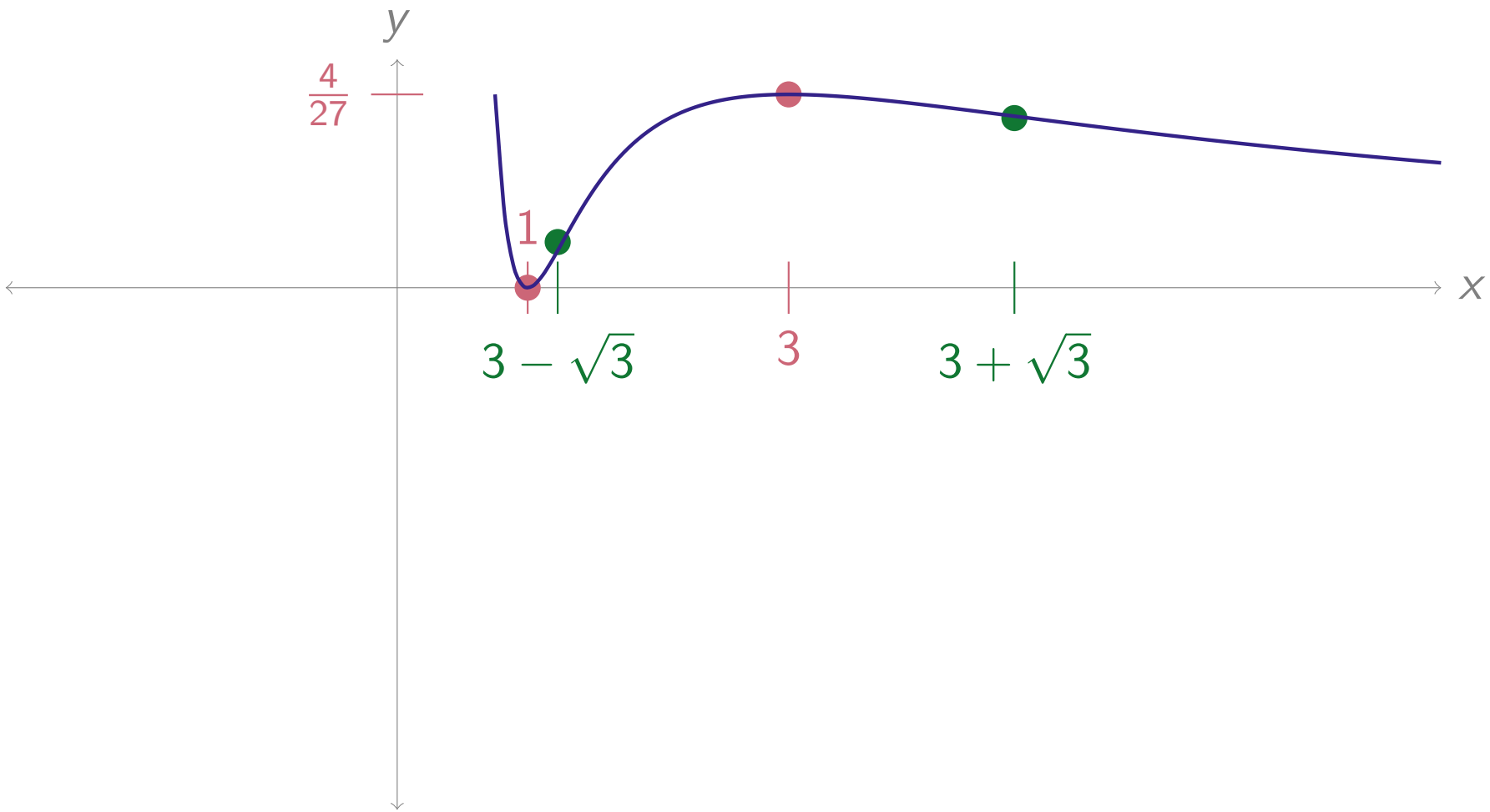
- CPs at $x = 1, 3$
- $f(x)$ increasing on $(1, 3)$, decreasing elsewhere (except $x = 0$)
- So, local min at $x = 1$, local max at $x = 3$

$f''(x) = \frac{2(x^2-6x+6)}{x^5}$

- $f''(x) = 0$ when $x = 3 \pm \sqrt{3}$ (quadratic formula)
- concave down on $(3 - \sqrt{3}, 3 + \sqrt{3})$ and $(-\infty, 0)$;
- concave up on $(0, 3 - \sqrt{3})$ and $(3 + \sqrt{3}, \infty)$
- So, inflection points at $3 \pm \sqrt{3}$

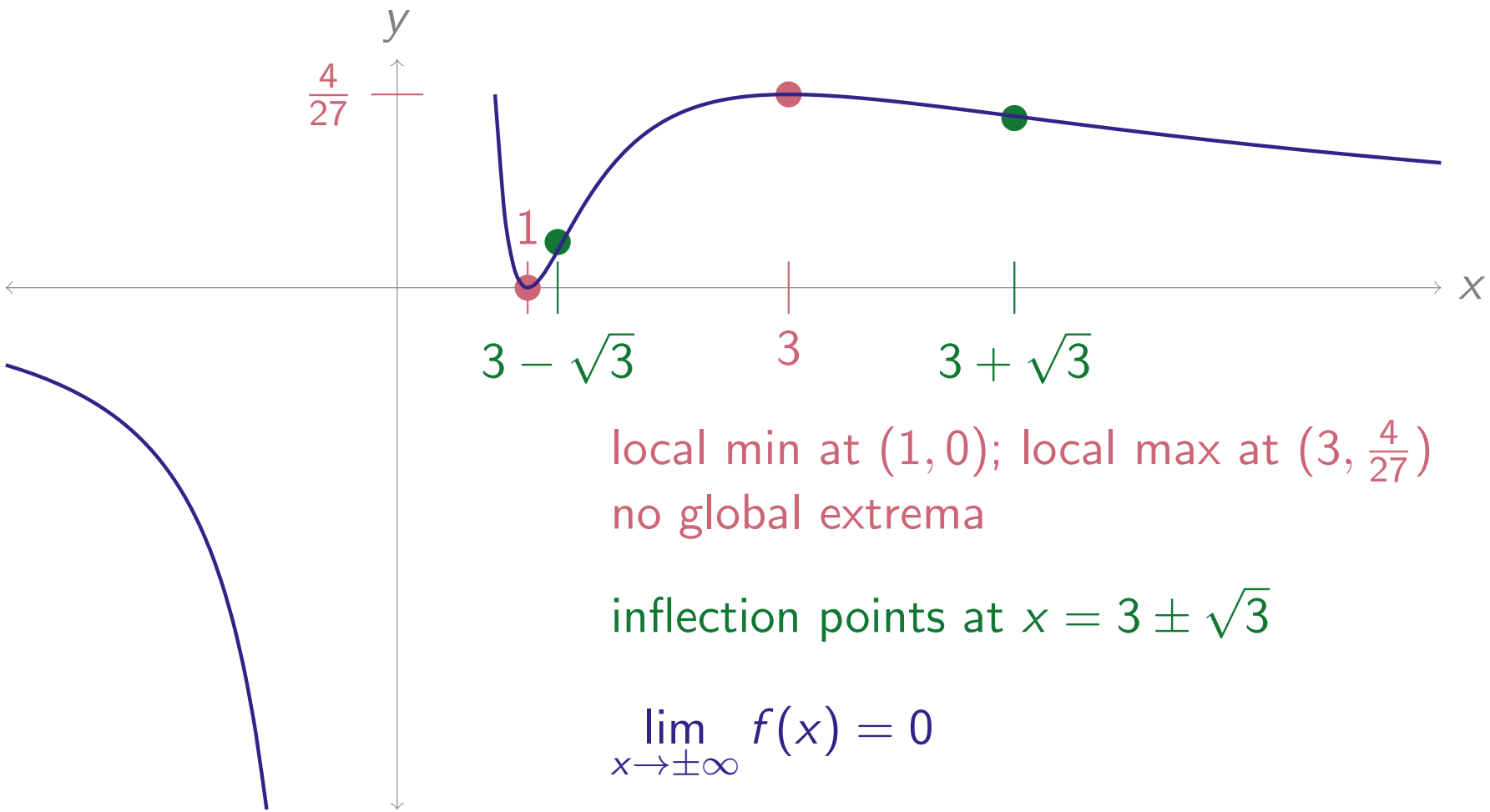
Sketch

$$f(x) = \frac{(x-1)^2}{x^3}$$



Sketch

$$f(x) = \frac{(x-1)^2}{x^3}$$



local min at $(1, 0)$; local max at $(3, \frac{4}{27})$
no global extrema

inflection points at $x = 3 \pm \sqrt{3}$

$$\lim_{x \to \pm\infty} f(x) = 0$$

Sketch:

$$f(x) = \frac{10x}{x^3 + 16}$$

Sketch:

$$f(x) = \frac{10x}{x^3 + 16}$$

- $f'(x) = \frac{20(8-x^3)}{(x^3+16)^2}$
- $f''(x) = \frac{60x^2(x^3-32)}{(x^3+16)^3}$

Sketch:

$$f(x) = \frac{10x}{x^3 + 16}$$

- Roots: $f(0) = 0$
- When $x \approx 0$, $f(x) \approx \frac{5}{8}x$ (straight line)
- Vertical asymptote: $x = -\sqrt[3]{16}$
- Horizontal asymptote: $y = 0$
- CP: $(-2, 5/6)$
- Increasing: $(-\infty, -2) \cup (-2, 2)$; Decreasing: $(2, \infty)$
- Inflection point: $x = 2^{5/3}$
- Concave up: $(-\infty, -2) \cup (2^{5/3}, \infty)$; Concave down: $(-2, 2^{5/3})$
- Local max: $x = 2$, $y = 5/6$
- No global extrema, no local min

Sketch:

$$f(x) = \frac{10x}{x^3 + 16}$$