Overview

Graphing More Accurately

- First and second derivatives
Overview

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- First and second derivatives
- Critical points
- Extrema
Up with hope, down with dope,
Increasing functions have positive slope
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Increasing functions have positive slope

Using our previous method, we can sketch $f(x) = x^3 - 3x^2$: 
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Using our previous method, we can sketch \( f(x) = x^3 - 3x^2 \):

\[
f'(x) > 0 \quad f'(x) < 0 \quad f'(x) > 0
\]
Up with hope, down with dope,
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Using our previous method, we can sketch $f(x) = x^3 - 3x^2$:

$$f'(x) = 3x^2 - 6x$$
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$$0 = 3x(x - 2)$$

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$f'(x) > 0$  $f'(x) < 0$  $f'(x) > 0$
Sketch:

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  Positive: \((-\infty, -2), (0, \infty)\). Negative: \((-2, -1), (-1, 0)\)
Chapter 6: Sketching the graph of a function using calculus tools

6.1 Overall shape of the graph of a function

Sketch:

\[ f(x) = \frac{x^2}{x+1} \]

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\[ f(x) = \frac{x^2}{x + 1} \]

- Blow-up discontinuity: \( x = -1 \)
- \( f'(x) = \frac{x(x+2)}{(x+1)^2} \)
- \( f''(x) = \frac{2}{(x+1)^3} \)
Concavity

- Slopes are increasing: $f''(x) > 0$ → "concave up"
- Slopes are decreasing: $f''(x) < 0$ → "concave down"
6.1 Overall shape of the graph of a function

**Concavity**

- **Slopes are increasing**: $f''(x) > 0$, "concave up"
- **Slopes are decreasing**: $f''(x) < 0$, "concave down"
Concavity

Slopes are increasing
\[ f''(x) > 0 \]
"concave up"

Slopes are decreasing
\[ f''(x) < 0 \]
"concave down"
Concavity

Slopes are increasing
\( f''(x) > 0 \) "concave up"

Slopes are decreasing
\( f''(x) < 0 \) "concave down"
Concavity

Slopes are increasing if $f''(x) > 0$; the graph is "concave up".

Slopes are decreasing if $f''(x) < 0$; the graph is "concave down".
Concavity

Slopes are increasing \( f''(x) > 0 \)  "concave up"

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6.1 Overall shape of the graph of a function

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Slopes are increasing
Concavity

Slopes are increasing
\[ f''(x) > 0 \]
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Chapter 6: Sketching the graph of a function using calculus tools

6.1 Overall shape of the graph of a function

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\[ f''(x) > 0 \]
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Concavity

Where is the function concave up? Concave down?
Concavity

Where is the function concave up? Concave down?

\[ f''(x) > 0 \]
Concavity

Where is the function concave up? Concave down?

\[ f''(x) > 0 \quad \text{concave up} \]

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inflection point

inflection point
Chapter 6: Sketching the graph of a function using calculus tools

6.1 Overall shape of the graph of a function

**Concavity**

Where is the function concave up? Concave down?

\[ f''(x) > 0 \quad f''(x) < 0 \quad f''(x) > 0 \]

inflection point

\[ f'''(x) \text{ changes sign} \]
Sketch:

\[ f(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 - 3x^2 \]
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- Consider behaviour near origin and far away from origin
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- Second derivative: concavity
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- First derivative: increasing and decreasing; find \( f(x) \) at special points
- Second derivative: concavity

\[ f'(x) = x^3 - x^2 - 6x = x(x - 3)(x + 2) \]
Sketch:

\[ f(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 - 3x^2 \]

- Consider behaviour near origin and far away from origin
- First derivative: increasing and decreasing; find \( f(x) \) at special points
- Second derivative: concavity

\[ f'(x) = x^3 - x^2 - 6x = x(x - 3)(x + 2) \]

<table>
<thead>
<tr>
<th></th>
<th>((-\infty, -2))</th>
<th>(-2)</th>
<th>((-2, 0))</th>
<th>0</th>
<th>(0, 3)</th>
<th>3</th>
<th>(3, (\infty))</th>
</tr>
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<tbody>
<tr>
<td>(x)</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0</td>
<td>+</td>
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<td>+</td>
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<tr>
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<tr>
<td>(f(x))</td>
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<td>(-\frac{16}{3})</td>
<td>inc</td>
<td>0</td>
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<td>(-\frac{63}{4})</td>
<td>inc</td>
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Using info from first derivative:
\[ f''(x) = 3x^2 - 2x - 6 \]

To find inflection points (i.e. where the second derivative changes sign), we find where \( f''(x) = 0 \) or where it does not exist.
In this case, \( f''(x) \) is a quadratic equation. It exists everywhere, and its zeroes are:

\[
x = \frac{2 \pm \sqrt{4 - 4(3)(-6)}}{2 \cdot 3} = \frac{1 \pm \sqrt{1 + 18}}{3} = \frac{1 \pm \sqrt{19}}{3}
\]

\[
x = \frac{1 + \sqrt{19}}{6} \approx 1.8 \quad \text{and} \quad x = \frac{1 - \sqrt{19}}{6} \approx -1.1
\]

So, \( f(x) \) is concave up when \( x < \frac{1 - \sqrt{19}}{3} \) and also when \( x > \frac{1 + \sqrt{19}}{3} \), and it is concave down between these points.
Then \( f(x) \) has two inflection points. These occur at:

\[
x = \frac{1 - \sqrt{19}}{3} \approx -1.1, \quad y \approx -1.4
\]

\[
x = \frac{1 + \sqrt{19}}{6} \approx 1.8, \quad y \approx -5
\]
Using info from second derivative:
Using info from second derivative:
Sketch graphs with the following properties, or explain that none exist.

<table>
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<tr>
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Sign Changes in a Factored Function

\[ f(x) = (x - 1) \ (x - 2) \ (x - 3) \]
Sign Changes in a Factored Function

\[ f(x) = (x - 1)(x - 2)(x - 3) \]
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Sign of product:
Sign Changes in a Factored Function

\[ f(x) = (x - 1) \ (x - 2) \ (x - 3) \]  

Sign of product:  

\[ - \quad - \quad - \]

\[ x \quad 1 \quad 2 \quad 3 \quad x \]
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Sign of product: − − −
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Sign of product: $-$

\[ x \ 1 \ 2 \ 3 \]
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Sign of product: \(-\) \(-\) \(-\) \(-\)

\[ x \quad 1 \quad 2 \quad 3 \]

\[ x \]
Sign Changes in a Factored Function

\[ f(x) = (x - 1) \ (x - 2) \ (x - 3) \]

Sign of product: +

\[ + \quad - \quad - \]

\[ \begin{array}{c}
1 \\
\bullet \\
2 \\
\end{array} \]

\[ x \quad 3 \]
Sign Changes in a Factored Function

\[ f(x) = (x - 1) \ (x - 2) \ (x - 3) \]

Sign of product: +

\[ + \quad - \quad - \]

\[ x \]

1  x  2  3
Sign Changes in a Factored Function

\[ f(x) = (x - 1) \ (x - 2) \ (x - 3) \]

Sign of product: \( + \)

\[ + \quad - \quad - \]

Diagram:

- \( x = 1 \)
- \( x = 2 \)
- \( x = 3 \)
Sign Changes in a Factored Function

\[ f(x) = (x - 1)(x - 2)(x - 3) \]

Sign of product: \(+\)

\[ + \quad - \quad - \]
Sign Changes in a Factored Function

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Sign Changes in a Factored Function

\[ f(x) = (x - 1) (x - 2) (x - 3) \]

Sign of product: \(-\)

\[
\begin{array}{ccc}
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1 & 2 & 3
\end{array}
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Sign of product: $-$

\[ + \quad + \quad - \]

\[ \xrightarrow{1} \quad \xrightarrow{2} \quad \xrightarrow{3} \]

\[ x \]
Sign Changes in a Factored Function

\[ f(x) = (x - 1)(x - 2)(x - 3) \]

Sign of product: +

\[ \begin{array}{cccc}
+ & + & + & \end{array} \]
Sign Changes in a Factored Function

\[ f(x) = (x - 1) \ (x - 2)^2 \ (x - 3) \]

Sign of product:
Sign Changes in a Factored Function

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Sign of product: 

\[ + \]
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\[ + \quad + \quad - \]
Sign Changes in a Factored Function

\[ f(x) = (x - 1)(x - 2)^2(x - 3) \]

Sign of product: + + + +
Example 1:

\[ f(x) = (x - 3)(x - 1)^2x(x + 2)^3(x + 5)^4 \]

Where is \( f(x) \) positive? Where is it negative?
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Special Points:
  - Roots / zeroes
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- Critical point (i.e. \( f'(x) = 0 \) or \( f'(x) \) DNE)
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- Critical point (i.e. \( f'(x) = 0 \) or \( f'(x) \) DNE)
- Maxima, minima
Special Points:

- Roots / zeroes
- critical point (i.e. $f'(x) = 0$ or $f'(x)$ DNE)
- maxima, minima
- CP that is not an extremum
Each of the following functions have a critical point at $x = 0$. Match the derivatives with their graphs.

(a) \( f'(x) \) negative when \( x < 0 \)
(b) \( f'(x) \) positive when \( x < 0 \)
(c) \( f'(x) \) negative when \( x < 0 \)
(d) \( f'(x) \) positive when \( x < 0 \)
(f) \( f'(x) \) negative when \( x > 0 \)
(g) \( f'(x) \) positive when \( x > 0 \)

I

II

III

IV
Each of the following functions have a critical point at $x = 0$. Match the derivatives with their graphs.

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   $f'(x)$ negative when $x > 0$

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   $f'(x)$ positive when $x > 0$

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(d) \( f'(x) \) positive when \( x < 0 \)
    \( f'(x) \) negative when \( x > 0 \)
Match the second derivatives with their graphs.

(a) $f''(x)$ negative

(b) $f''(x)$ positive

(c) $f''(x)$ negative when $x < 0$
   $f''(x)$ positive when $x > 0$

(d) $f''(x)$ positive when $x < 0$
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Match the second derivatives with their graphs.

(a) \( f''(x) \) negative
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Match the second derivatives with their graphs.

(a) \(f''(x)\) negative

(c) \(f''(x)\) negative when \(x < 0\)
\(f''(x)\) positive when \(x > 0\)

(b) \(f''(x)\) positive

(d) \(f''(x)\) positive when \(x < 0\)
\(f''(x)\) negative when \(x > 0\)
Suppose $x = a$ is a critical point of the continuous function $f(x)$. That is, $f'(a) = 0$ or $f'(x)$ does not exist at $a$.

(a) $f'(a)$ changes from neg to pos  
(b) $f'(a)$ changes from pos to neg  
(c) $f''(a) = 0$  
(d) $f''(a)$ changes from neg to pos  
(e) $f''(a)$ changes from pos to neg  
(f) $f''(a)$ positive and $f'(a) = 0$  
(g) $f''(a)$ negative and $f'(a) = 0$

I. inflection point  
II. local max  
III. local min  
IV. not a local extrema  
V. could be local max, local min, or neither
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(a) - III  (b) - II  (c) - V  (d),(e) - I,V  (f) - III  (g) - II
Tests

The information from the previous questions forms a method for finding local extrema.
Tests

The information from the previous questions forms a method for finding local extrema.

- First, find all critical points (where $f'(x) = 0$ or $f'(x)$ doesn’t exist). Local extrema can ONLY occur at critical points.
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  - Using the second derivative:
    - If $f''(x) > 0$, the CP is a local min.
    - If $f''(x) < 0$, the CP is a local max.
    - If $f''(x) = 0$, we need to look at the first derivative to decide.

  - Using the first derivative:
    - If $f'(x)$ does not change signs at the CP, then it’s neither a local max nor a local min–it’s just a momentary flat point.
    - If $f'(x)$ changes from increasing to decreasing, the CP is a local max.
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Chapter 6: Sketching the graph of a function using calculus tools
6.2 Special points on the graph of a function

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Chapter 6: Sketching the graph of a function using calculus tools

6.2 Special points on the graph of a function

\[ f(x) = \frac{1}{4}x^4 - 2x^3 + \frac{9}{2}x^2 + 1 \]

1. Find all critical points of \( f(x) \) (that is, find where \( f'(x) = 0 \))
2. Are they local maxima, local minima, or neither?

Challenge: sketch the function.
Chapter 6: Sketching the graph of a function using calculus tools

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To find CPs, we set \( f'(x) = 0 \).

\[ f'(x) = x^3 - 6x^2 + 9x = x(x^2 - 6x + 9) = x(x - 3)^2 \]

So the CPs are at \( x = 0 \) and \( x = 3 \)
Chapter 6: Sketching the graph of a function using calculus tools

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To classify them, note \( f''(x) = 3x^2 - 12x + 9 = 3(x - 1)(x - 3) \), so \( f''(0) > 0 \) and \( f''(3) = 0 \).

Then at \( x = 0 \) there is a LOCAL MIN, and we need more information for \( x = 3 \).
6.2 Special points on the graph of a function

\[ f(x) = \frac{1}{4}x^4 - 2x^3 + \frac{9}{2}x^2 + 1 \]

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Then at \( x = 0 \) there is a LOCAL MIN, and we need more information for \( x = 3 \). Since \( f'(x) \) is positive both just before and just after \( x = 3 \), there is neither a local max nor a local min there.

Since \( f''(x) \) changes signs at \( x = 3 \), there is an inflection point there.
Sketch.

\[ f(x) = \frac{1}{4}x^4 - 2x^3 + \frac{9}{2}x^2 + 1 \]
6.2 Special points on the graph of a function

Sketch.

\[ f(x) = \frac{1}{4}x^4 - 2x^3 + \frac{9}{2}x^2 + 1 \]

local min at \((0, 1)\) also global min
inflection points at \((3, 31/4)\) and \((1, 15/4)\)
Chapter 6: Sketching the graph of a function using calculus tools

6.2 Special points on the graph of a function

Sketch.

\[ f(x) = \frac{1}{4}x^4 - 2x^3 + \frac{9}{2}x^2 + 1 \]

local min at \((0, 1)\)

Also, global min at \((3, \frac{31}{4})\) and \((1, \frac{15}{4})\).
Chapter 6: Sketching the graph of a function using calculus tools  

6.2 Special points on the graph of a function

\[
f(x) = \frac{1}{4}x^4 - 2x^3 + \frac{9}{2}x^2 + 1
\]

Sketch.

local min at \((0, 1)\)  
also **global** min
Sketch the graph of the function using calculus tools.

The function is given by:

\[ f(x) = \frac{1}{4}x^4 - 2x^3 + \frac{9}{2}x^2 + 1 \]

Sketch.

- Local min at \((0, 1)\)
- Also global min
- Inflection points at \((3, \frac{31}{4})\) and \((1, \frac{15}{4})\)
Sketch.

$$f(x) = \frac{1}{4}x^4 - 2x^3 + \frac{9}{2}x^2 + 1$$

local min at $(0, 1)$
also **global** min

inflection points at $\left(3, \frac{31}{4}\right)$ and $\left(1, \frac{11}{4}\right)$
Chapter 6: Sketching the graph of a function using calculus tools

6.2 Special points on the graph of a function

\[ f(x) = \frac{1}{4}x^4 - 2x^3 + \frac{9}{2}x^2 + 1 \]

Sketch.
Global extrema over a closed interval

Where do global extrema occur?
Global extrema over a closed interval

Where do global extrema occur?

- Global max at endpoint
- Global min at critical point
Global extrema over a closed interval

Where do global extrema occur?

1. **Global max at endpoint**
2. **Global min at critical point**
Global extrema over a closed interval

Where do global extrema occur?

- Global max at endpoint
- Global min at critical point
- Global max at critical point
Global extrema over a closed interval

Where do global extrema occur?

- **global max at endpoint**
- **global min at critical point**
- **global max at critical point**
- **global mins at endpoints**
Global extrema over a closed interval

Where do global extrema occur? endpoints or critical points

global max at endpoint

global min at critical point

global max at critical point

global mins at endpoints
Cell Division
Chapter 6: Sketching the graph of a function using calculus tools

6.3 Sketching the graph of a function

Cell Division

A cell of age $t$ hours has a probability\(^1\) of dividing given by the function

$$P(t) = \frac{at}{t^3 + 16},$$

where $a$ is the constant $\frac{9\sqrt{3}}{2\pi}$.

From time $t = 0$ to time $t = 10$, when is the cell likeliest to divide? When is it least likely to divide?

\(^1\)Actually, we’re giving a probability density function... it’s a little more complicated than what we said. Still, a higher value of $P(t)$ means a cell age that is more reproduce-y, so we just want to find the max and min of $P(t)$ when $0 \leq t \leq 10$. 
Cell Division

We're looking for the absolute maximum and minimum of $P(t)$ on the interval $[0, 10]$. This can occur at the endpoints (that is, $t = 0$ and $t = 10$) or at critical points.

To find the critical points, we take the first derivative.

$$P'(t) = \frac{(t^3 + 16)(a) - at(3t^2)}{(t^3 + 16)^2} = a\frac{16 - 2t^3}{(t^3 + 16)^2}$$

We find this is undefined only at negative numbers—not in our interval, so we don't care. It's zero when $16 - 2t^3 = 0$, i.e. when $t = 2$. So, our sole critical point is at $t = 2$.

Now, we compare $P(0)$, $P(2)$, and $P(10)$ to see which one is largest and which smallest.

- $P(0) = 0$: this must be smallest, since probabilities can't be negative
- $P(2) = \frac{2a}{2^3 + 16} = \frac{a}{12}$
- $P(10) = \frac{10a}{10^3 + 16} = \frac{10a}{1016} = \frac{a}{101.6}$

Since $P(2) > P(10) > P(0)$, the cell is likeliest to divide when $t = 2$, and least likely to divide when $t = 10$. 
Sketching the graph of a function using calculus tools

6.3 Sketching the graph of a function

Sketch

\[ f(x) = \frac{(x - 1)^2}{x^3} \]

Include the following, if they exist:

- Roots (zeroes)
- Discontinuities
- Asymptotes
- Local and global extrema (maxes and mins)
- Inflection points and concavity
Sketch

\[ f(x) = \frac{(x-1)^2}{x^3} \]

\[ f'(x) = \frac{-(x-1)(x-3)}{x^4} \]

\[ f''(x) = \frac{2(x^2-6x+6)}{x^5} \]
Sketch

\[ f(x) = \frac{(x-1)^2}{x^3} \]

- \( f(1) = 0 \)
- blow-up discontinuity at \( x = 0 \)
- horizontal asymptote \( y = 0 \) both sides

\[ f'(x) = \frac{-(x-1)(x-3)}{x^4} \]

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\[ f'(x) = \frac{-(x-1)(x-3)}{x^4} \]

- CPs at \( x = 1, 3 \)
- \( f(x) \) increasing on (1, 3), decreasing elsewhere (except \( x = 0 \))
- So, local min at \( x = 1 \), local max at \( x = 3 \)

\[ f''(x) = \frac{2(x^2-6x+6)}{x^5} \]
Chapter 6: Sketching the graph of a function using calculus tools

6.3 Sketching the graph of a function

Sketch

\[ f(x) = \frac{(x-1)^2}{x^3} \]
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- CPs at \( x = 1, 3 \)
- \( f(x) \) increasing on \((1, 3)\), decreasing elsewhere (except \( x = 0 \))
- So, local min at \( x = 1 \), local max at \( x = 3 \)

\[ f''(x) = \frac{2(x^2-6x+6)}{x^5} \]
- \( f''(x) = 0 \) when \( x = 3 \pm \sqrt{3} \) (quadratic formula)
- concave down on \((3 - \sqrt{3}, 3 + \sqrt{3})\) and \((-\infty, 0)\);
- concave up on \((0, 3 - \sqrt{3})\) and \((3 + \sqrt{3}, \infty)\)
- So, inflection points at \( 3 \pm \sqrt{3} \)
Sketch

\[ f(x) = \frac{(x - 1)^2}{x^3} \]

- Local min at \((1, 0)\)
- Local max at \((3, \frac{4}{27})\)
- No global extrema
- Inflection points at \(x = 3 \pm \sqrt{3}\)
- \(\lim_{x \to \pm\infty} f(x) = 0\)
Sketch

\[ f(x) = \frac{(x - 1)^2}{x^3} \]

local min at \((1, 0)\); local max at \((3, \frac{4}{27})\)
no global extrema

inflection points at \(x = 3 \pm \sqrt{3}\)

\[ \lim_{x \to \pm \infty} f(x) = 0 \]
Sketch:

\[ f(x) = \frac{10x}{x^3 + 16} \]
Sketch:

\[ f(x) = \frac{10x}{x^3 + 16} \]

- \[ f'(x) = \frac{20(8-x^3)}{(x^3+16)^2} \]
- \[ f''(x) = \frac{60x^2(x^3-32)}{(x^3+16)^3} \]
Chapter 6: Sketching the graph of a function using calculus tools

6.3 Sketching the graph of a function

Sketch:

\[ f(x) = \frac{10x}{x^3 + 16} \]

- Roots: \( f(0) = 0 \)
- When \( x \approx 0 \), \( f(x) \approx \frac{5}{8}x \) (straight line)
- Vertical asymptote: \( x = -3\sqrt{16} \)
- Horizontal asymptote: \( y = 0 \)
- CP: \((-2, 5/6)\)
- Increasing: \((-\infty, -2) \cup (-2, 2)\); Decreasing: \((2, \infty)\)
- Inflection point: \( x = 2^{5/3} \)
- Concave up: \((-\infty, -2) \cup (2^{5/3}, \infty)\); Concave down: \((-2, 2^{5/3})\)
- Local max: \( x = 2, y = 5/6 \)
- No global extrema, no local min
Sketch:

\[ f(x) = \frac{10x}{x^3 + 16} \]