Overview

Uses of tangent lines:
- Approximating a function near a fixed point
- Over/under estimate
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Uses of tangent lines:
- Approximating a function near a fixed point
- Over/under estimate
- Approximating the zeroes of a function
Approximate Value of $\sqrt{10}$
Approximate Value of $\sqrt{10}$

The equation of the tangent line is:

$$y = \frac{1}{3}x + 3.166$$

at the point $(10, 3.166)$. The graph shows the function $y = \sqrt{x}$ and the tangent line at $x = 10$. The approximate value of $\sqrt{10}$ is 3.166.
Approximate Value of $\sqrt{10}$

The equation of a tangent line at the point $(10, \sqrt{10})$ can be approximated as:

$$y = \sqrt{10} + \frac{1}{2\sqrt{10}}(x - 10)$$

Approximation:

$$\sqrt{10} \approx 3.166$$

Calculator:

$$\sqrt{10} = 3.162277...$$
Approximate Value of $\sqrt{10}$

The equation of a tangent line to the curve $y = \sqrt{x}$ at the point $(10, 3)$ is given by $y = \frac{1}{6}x + \frac{3}{2}$. This line is tangent to the curve at the point $(10, 3)$. The diagram illustrates this with the tangent line and the curve intersecting at $(10, 3)$. The approximate value of $\sqrt{10}$ is $3.166$, which is close to the actual value $3.162277...$. The $y$-intercept of the tangent line is at $(0, 1.5)$.
Approximate Value of $\sqrt{10}$

$\sqrt{10} \approx 3.166$

$\sqrt{10} \approx 3.162277$

$y = \frac{1}{6}x + \frac{3}{2}$

$y = \sqrt{x}$

(10, $\frac{19}{6}$)

Tangent line
Approximate Value of $\sqrt{10}$

Approximation: $\sqrt{10} \approx \frac{19}{6} = 3.166$
Approximate Value of $\sqrt{10}$

Approximation: $\sqrt{10} \approx \frac{19}{6} = 3.166$

Calculator: $\sqrt{10} = 3.162277...$
5.1 The equation of a tangent line

Approximate Value of $\sqrt{10}$

$y = x^2 - 10$

$\sqrt{10} \approx 3.162$
Approximate Value of $\sqrt{10}$

The equation of the tangent line at the point $(3, -1)$ is:

$$y = 6x - 19$$

The $x$-intercept is $\frac{19}{6}$. The graph shows the tangent line $y = x^2 - 10$.
Approximate Value of $\sqrt{10}$
Approximate Value of $\sqrt{10}$

The equation of a tangent line to the curve $y = x^2 - 10$ at the point $(3, -1)$ is given by $y = 6x - 19$. The tangent line is shown graphically in the diagram.
Approximate Value of $\sqrt{10}$

The equation of the tangent line is:

$$y = 6x - 19$$

The $x$-intercept of the tangent line is $\frac{19}{6}$. The point $(3, -1)$ is on the tangent line. The graph shows the tangent line $y = x^2 - 10$ and the point $(3, -1)$ on it.
Tangent Line Equation

Point-Slope Form - Review

A line with slope \( m \), passing through the point \((x_0, y_0)\), has equation

\[
(y - y_0) = m(x - x_0)
\]
Tangent Line Equation

Point-Slope Form - Review

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A line with slope \( m \), passing through the point \((x_0, y_0)\), has equation

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(y - y_0) = m(x - x_0)
\]

So, the tangent line to the curve \( y = f(x) \) at the point \( x_0 \) is

\[
y = f'(x_0)(x - x_0) + f(x_0)
\]
Tangent Line Equation

Point-Slope Form - Review

A line with slope $m$, passing through the point $(x_0, y_0)$, has equation

$$(y - y_0) = m(x - x_0)$$

So, the tangent line to the curve $y = f(x)$ at the point $x_0$ is

$$(y - f(x_0)) = f'(x_0)(x - x_0)$$
Tangent Line Equation

Point-Slope Form - Review

A line with slope $m$, passing through the point $(x_0, y_0)$, has equation

$$(y - y_0) = m(x - x_0)$$

So, the tangent line to the curve $y = f(x)$ at the point $x_0$ is

$$(y - f(x_0)) = f'(x_0)(x - x_0)$$

That is:

$$y = f'(x_0)(x - x_0) + f(x_0)$$
Tangent line $x$-intercept

**Generic tangent line**

So, the tangent line to the curve $y = f(x)$ at the point $x_0$ is

$$y = f'(x_0)(x - x_0) + f(x_0)$$

**Example 1:** Where does this line intersect the $x$-axis?
Tangent line $x$-intercept

**Generic tangent line**

So, the tangent line to the curve $y = f(x)$ at the point $x_0$ is

$$y = f'(x_0)(x - x_0) + f(x_0)$$

**Example 1: Where does this line intersect the $x$-axis?**

$$0 = f'(x_0)(x - x_0) + f(x_0)$$

$$-f(x_0) = f'(x_0)(x - x_0)$$

$$-\frac{f(x_0)}{f'(x_0)} = x - x_0$$

$$x = x_0 - \frac{f(x_0)}{f'(x_0)}$$
Tangent line $x$-intercept

**Generic tangent line**

So, the tangent line to the curve $y = f(x)$ at the point $x_0$ is

$$y = f'(x_0)(x - x_0) + f(x_0)$$

and its $x$-intercept is

$$x = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Example 2: Let $f(x) = x^3 + 5x - 5$. We want to find its root.

(a) What is $f(1)$?

(b) Where does the tangent line of $f(x)$ at $x = 1$ intersect the $x$-axis? Call that spot $x_1$.

(c) Is $f(x_1)$ closer to 0 than $f(1)$? (Use a calculator here.)
Tangent line $x$-intercept

**Generic tangent line**

So, the tangent line to the curve $y = f(x)$ at the point $x_0$ is

$$y = f'(x_0)(x - x_0) + f(x_0)$$

and its $x$-intercept is

$$x = x_0 - \frac{f(x_0)}{f'(x_0)}$$

**Example 2:** Let $f(x) = x^3 + 5x - 5$.

We want to find its root.

(a) What is $f(1)$?

(b) Where does the tangent line of $f(x)$ at $x = 1$ intersect the $x$-axis? Call that spot $x_1$.

(c) Is $f(x_1)$ closer to 0 than $f(1)$? (Use a calculator here.)
Solution 2:

(a) \( f(1) = 1^3 + 5(1) - 5 = 1 \). Note that 1 is not very close to 0, and we want to find the value of \( x \) that makes \( f(x) = 0 \).

(b) As we saw before, the spot where the tangent line intersects the \( x \)-axis is \( x = 1 - \frac{f(1)}{f'(1)} \). Note \( f'(x) = 3x^2 + 5 \), so \( f'(1) = 8 \). Then the tangent line has \( x \)-intercept \( x = 1 - \frac{1}{8} = \frac{7}{8} \).

(c) \( f(1) = 1 \), while \( f(7/8) \approx 0.04 \). So, if we’re looking for an \( x \)-value that gives us \( f(x) = 0 \), \( x = 7/8 \) is a much better approximation than what we started with, \( x = 1 \).
Tangent Line Approximation

\[ y = f(x_0)(x - x_0) + f(x_0) \]

Linear Approximation

For \( x \) near \( x_0 \),

\[ f(x) \approx f'(x_0)(x - x_0) + f(x_0) \]
Tangent Line Approximation

Tangent Line Approximation

\[ y = f(x_0)(x - x_0) + f(x_0) \]

Linear Approximation

For \( x \) near \( x_0 \),

\[ f(x) \approx f'(x_0)(x - x_0) + f(x_0) \]
Tangent Line Approximation

\[ y = f(x_0) + f'(x_0)(x - x_0) \]

Linear Approximation

For \( x \) near \( x_0 \),

\[ f(x) \approx f'(x_0)(x - x_0) + f(x_0) \]
Chapter 5: Tangent lines, linear approximation, Newton’s method

5.3 Approximating a function by its tangent line

**Tangent Line Approximation**

For $x$ near $x_0$, $f(x) \approx f'(x_0)(x - x_0) + f(x_0)$.
Tangent Line Approximation

For $x$ near $x_0$,
\[ f(x) \approx f'(x_0)(x - x_0) + f(x_0). \]
Linear Approximation

For $x$ near $x_0$, \( f(x) \approx f'(x_0)(x - x_0) + f(x_0) \).

Example 3:

(a) Using a linear approximation of \( f(x) = \sqrt[3]{x} \), approximate the value of \( \sqrt[3]{28} \).

(b) Using a linear approximation, approximate the values of \( 99^{100} \) and \( 101^{100} \).
Linear Approximation

For $x$ near $x_0$, $f(x) \approx f'(x_0)(x - x_0) + f(x_0)$.

Example 3:

(a) Using a linear approximation of $f(x) = \sqrt[3]{x}$, approximate the value of $\sqrt[3]{28}$.

(b) Using a linear approximation, approximate the values of $99^{100}$ and $101^{100}$. 
Solution 3:

(a) If \( f(x) = x^{1/3} \), then \( f'(x) = \frac{1}{3} x^{-2/3} = \frac{1}{3 \sqrt[3]{x^2}} \).

We choose \( x_0 = 27 \), because this is close to 28, and is easy to work with. Then: \( f(x_0) = \sqrt[3]{27} = 3 \) and \( f'(x_0) = \frac{1}{3 \sqrt[3]{27^2}} = \frac{1}{3 \cdot 3^2} = \frac{1}{27} \).

So, our linear approximation is:

\[
f(x) \approx f'(x_0)(x - x_0) + f(x_0) = \frac{1}{27}(x - 27) + 3
\]

Therefore,

\[
f(28) \approx \frac{1}{27}(28 - 27) + 3 = \frac{1}{27} + 3 = \frac{82}{27}
\]
Solution 3:

(a) We can check:

\[
\left(3 + \frac{1}{27}\right)^3 = 3^3 + 3 \cdot 3^2 \cdot \frac{1}{27} + 3 \cdot 3 \cdot \frac{1}{27^2} + \frac{1}{27^3}
\]

\[
= 27 + 1 + \frac{1}{81} + \frac{1}{27^3}
\]

\[
\approx 28
\]

So, \(3^{\sqrt{28}}\) should be fairly close to our approximation, \(3 + \frac{1}{27}\).
Solution 3:

(b) Let’s approximate the function $f(x) = x^{100}$, with $x_0 = 100$.

$f'(x) = 100x^{99}$, so:

$f(100) = f'(100) = 100^{100} = 10^{200}$.

Then our linear approximation is

$$f(x) \approx f'(100)(x - 100) + f(100)$$

$$f(x) \approx 10^{200}(x - 100) + 10^{200}$$

$$f(x) \approx 10^{200}(x - 99)$$

This leads to:

$$f(99) \approx 10^{200}(0) = 0$$

$$f(101) \approx 2(10^{200})$$
Solution 3:

(b) How good are these approximations?

- \( f(101) = 101^{100} \) (exact value);
- \( f(101) \approx 2(100^{100}) \) (linear approximation);
- \( f(101) \approx 100^{100} \) (using 100 \( \approx \) 101)

\[
\frac{\text{approx}}{\text{exact}} = \frac{2(100^{100})}{101^{100}} = 2 \left( \frac{100}{101} \right)^{100} \approx 0.74
\]

So, our linear approximation was about 74% of the way there.

\[
\frac{\text{approx}}{\text{exact}} = \frac{100^{100}}{101^{100}} = \left( \frac{100}{101} \right)^{100} \approx 0.37
\]

So, our other approximation was only about a third of the actual value.
(b) How good are these approximations?

- $f(99) = 99^{100}$ (exact value);
- $f(99) \approx 0$ (linear approximation);
- $f(99) \approx 100^{100}$ (using $100 \approx 99$)
Linear Approximation

For $x$ near $x_0$, $f(x) \approx f'(x_0)(x - x_0) + f(x_0)$.

(a) Using a linear approximation of $f(x) = \sqrt[3]{x}$, approximate the value of $\sqrt[3]{28}$.

(b) Using a linear approximation, approximate the values of $99^{100}$ and $101^{100}$.

Example 4: Were these overestimates or underestimates?
Solution 4:

The tangent line is higher than the function, so the linear approximation is an overestimate. That is, the approximation is higher than the actual value.
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\[ y = \sqrt[3]{x} \]
Solution 4:

The tangent line is higher than the function, so the linear approximation is an overestimate. That is, the approximation is higher than the actual value.

Graph showing the tangent line and the function $y = \sqrt[3]{x}$ at $x = 27$. The tangent line is higher than the function at this point.
Solution 4:

The tangent line is **higher** than the function, so the linear approximation is an **overestimate**. That is, the approximation is higher than the actual value.
Solution 4:

The tangent line is lower than the function, so the linear approximation is an underestimate. That is, the approximation is smaller than the actual value.
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The tangent line is lower than the function, so the linear approximation is an underestimate. That is, the approximation is smaller than the actual value.
Solution 4:

The tangent line is **lower** than the function, so the linear approximation is an **underestimate**. That is, the approximation is smaller than the actual value.
Example 5:

Let \( f(x) = \frac{5\sqrt{x}}{2 + \sqrt{x}} \).

Using the methods of this chapter, find a rational number approximating \( f(8.75) \), and decide whether your approximate value is an overestimation or an underestimation.
Example 5:

Let \( f(x) = \frac{5\sqrt{x}}{2 + \sqrt{x}} \).

Using the methods of this chapter, find a rational number approximating \( f(8.75) \), and decide whether your approximate value is an overestimation or an underestimation.

\[ f(8.75) \approx 3 - \frac{1}{60}; \text{ overestimation} \]
5.3 Approximating a function by its tangent line

\[ f(9) = \frac{5(3)}{2 + 3} = 3 \]

\[ f'(x) = \frac{(2 + \sqrt{x}) \left( \frac{5}{2\sqrt{x}} \right) - 5\sqrt{x} \left( \frac{1}{2\sqrt{x}} \right)}{(2 + \sqrt{x})^2} \]  
\hspace{1cm} \text{(quotient rule)}

\[ = \frac{\frac{5}{\sqrt{x}} + \frac{5}{2} - \frac{5}{2}}{(2 + \sqrt{x})^2} \]
\[ = \frac{5}{\sqrt{x}(2 + \sqrt{x})^2} \]

\[ f'(9) = \frac{5}{3(2 + 3)^2} = \frac{1}{15} \]  
\hspace{1cm} \text{(slope of tan line)}

\[ f(x) \approx \frac{1}{15}(x - 9) + 3 \]  
\hspace{1cm} \text{(linear approx)}

\[ f(8.75) \approx \frac{1}{15}(8.75 - 9) + 3 \]
\[ = \frac{1}{15} \left( -\frac{1}{4} \right) + 3 = 3 - \frac{1}{60} \]  
\hspace{1cm} \text{(answer)}
Our function $f(x)$ generally has the form of a Hill function (even though its exponent, $n = \frac{1}{2}$, is not an integer).

The tangent lines seem to be above the function, suggesting our approximation is an overestimation—that is, our guess is higher than the actual value.
Newton’s Method: Intro

Suppose you take the tangent line at $x = x_0$ to a differentiable function $f(x)$. Where does that tangent line hit the $x$-axis?

A. $x_0 - \frac{f(x_0)}{f'(x_0)}$  
B. $x_0 - \frac{f'(x_0)}{f(x_0)}$  
C. $x_0 + \frac{f(x_0)}{f'(x_0)}$  
D. $x_0 + \frac{f'(x_0)}{f(x_0)}$
Newton’s Method: Intro

Suppose you take the tangent line at \( x = x_0 \) to a differentiable function \( f(x) \). Where does that tangent line hit the \( x \)-axis?

\[
x_0 - \frac{f(x_0)}{f'(x_0)}
\]

A. \( x_0 - \frac{f(x_0)}{f'(x_0)} \)  
B. \( x_0 - \frac{f'(x_0)}{f(x_0)} \)  
C. \( x_0 + \frac{f(x_0)}{f'(x_0)} \)  
D. \( x_0 + \frac{f'(x_0)}{f(x_0)} \)
Newton’s Method: Finding Roots

Newton’s Method
Given an approximation $x_k$ for the root of the equation $f(x) = 0$, we can improve the accuracy of that approximation with another iteration using

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$
Newton’s Method: Finding Roots

Newton’s Method

Given an approximation \( x_k \) for the root of the equation \( f(x) = 0 \), we can improve the accuracy of that approximation with another iteration using

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Newton’s Method: Finding Roots

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Given an approximation $x_k$ for the root of the equation $f(x) = 0$, we can improve the accuracy of that approximation with another iteration using

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$
Example 6:

Sketch the locations of $x_1$ and $x_2$ that arise from Newton’s method on the function shown below if $x_0 = 0$. 

\[ y \]

\[ x \]
Newton’s Method: Finding Roots

**Example 6:**

Sketch the locations of $x_1$ and $x_2$ that arise from Newton’s method on the function shown below if $x_0 = 0$. 

![Graph of function with tangent lines and points labeled $x_1$ and $x_2$.]
Newton’s Method: Finding Roots

Example 6:

Sketch the locations of $x_1$ and $x_2$ that arise from Newton’s method on the function shown below if $x_0 = 0$. 
Example 6:

Sketch the locations of $x_1$ and $x_2$ that arise from Newton’s method on the function shown below if $x_0 = 0$. 

![Graph showing a function and two points labeled $x_1$ and $x_2$.]
Newton’s Method: Finding Roots

Example 6:

Sketch the locations of $x_1$ and $x_2$ that arise from Newton’s method on the function shown below if $x_0 = 0$. 
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![Graph showing Newton's method application](image)
Example 6:

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Newton’s Method: Finding Roots

Example 6:

Sketch the locations of $x_1$ and $x_2$ that arise from Newton’s method on the function shown below if $x_0 = 0$. 

![Graph showing the locations of $x_1$, $x_2$, and $x_3$.]
Newton’s Method: Finding Roots

Newton’s Method

Given an approximation \( x_k \) for the root of the equation \( f(x) = 0 \), we can improve the accuracy of that approximation with another iteration using

\[
x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}
\]

Example 7:

Use three iterations of Newton’s Method to approximate a root of

\[
f(x) = x^3 - 2x + 1.
\]

with \( x_0 = 0 \).
Solution 7:

Use three iterations of Newton’s Method to approximate a root of $f(x) = x^3 - 2x + 1$ with $x_0 = 0$.

\[ f'(x) = 3x^2 - 2 \]

\[ x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{f(0)}{f'(0)} = 0 - \frac{1}{-2} = \frac{1}{2} \]

\[ x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{1}{2} - \frac{f(1/2)}{f'(1/2)} = \frac{1}{2} - \frac{1/8}{-5/4} = \frac{3}{5} = 0.6 \]

\[ x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \frac{3}{5} - \frac{f(3/5)}{f'(3/5)} = \frac{3}{5} - \frac{2/125}{-23/25} = \frac{71}{115} \approx 0.617 \]

$f(x) = (x - 1)(x^2 + x - 1)$; one root: $x = \frac{-1 + \sqrt{5}}{2} \approx 0.618$
Newton’s Method: Finding Roots

Newton’s Method

\[ x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \]

Example 8:

Use Newton’s method to approximate a root of

\[ f(x) = x^5 + x - 1. \]
Solution 8:

Use Newton’s method to approximate a root of

\[ f(x) = x^5 + x - 1. \]

Choosing \( x_0 \), and when to terminate, is up to you. We’ll choose \( x_0 = 0 \), but we didn’t have to.

\[
f'(x) = 5x^4 + 1
\]

\[
x_1 = 0 - \frac{f(0)}{f'(0)} = 0 - \frac{-1}{1} = 1
\]

\[
x_2 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1}{6} = \frac{5}{6} = 0.833
\]

\[
x_3 = \frac{5}{6} - \frac{f(5/6)}{f'(5/6)} = \frac{5}{6} - \frac{(5/6)^5 + (5/6) - 1}{5(5/6)^4 + 1} \approx 0.764
\]
Example 9:

(a) Use Newton's Method to approximate $\sqrt{50}$.

(b) Use a linear approximation to approximate $\sqrt{50}$. 
Newton’s Method: Finding Roots

Example 9:

(a) Use Newton’s Method to approximate $\sqrt{50}$.

(b) Use a linear approximation to approximate $\sqrt{50}$.

(a) Let $f(x) = x^2 - 50$, so $f’(x) = 2x$, and let $x_0 = 7$.

\[
x_1 = 7 - \frac{-1}{14} = 7 + \frac{1}{14} = \frac{99}{14}
\]

\[
x_2 = \frac{99}{14} - \frac{(99/14)^2 - 50}{99/7} = \frac{19601}{2772}
\]

Remark: $(7 + \frac{1}{14})^2 \approx 50.005$
Newton’s Method: Finding Roots

Example 9:

(a) Use Newton’s Method to approximate \( \sqrt{50} \).

(b) Use a linear approximation to approximate \( \sqrt{50} \).

(b)
Let \( g(x) = \sqrt{x} \), so \( g'(x) = \frac{1}{2\sqrt{x}} \), and use the point \((49, 7)\) as an “anchor”:

\[
g(x) \approx g'(49)(x - 49) + g(49) = \frac{1}{14}(x - 49) + 7
\]

\[
g(50) \approx \frac{1}{14}(50 - 49) + 7 = 7 + \frac{1}{14}
\]

Remark: \((7 + \frac{1}{14})^2 \approx 50.005\) \(\left(\frac{19601}{2772}\right)^2 \approx 50.0000001\)
Ladybugs and Aphids

Size of aphid population: $x$

Ladybug predation rate: $P(x) = 30x^3 + x^3$

Aphid population growth rate: $G(x) = 0.5P(x) = G(x)$
Ladybugs and Aphids

Size of aphid population: $x$

Ladybug predation rate: $P(x) = 30x^3 + x^3$

Aphid population growth rate: $G(x) = 0.5P(x)$
Ladybugs and Aphids

Size of aphid population: $x$

Ladybug predation rate: $P(x) = 30x^3 + x^3$

Aphid population growth rate: $G(x) = 0$.

$P(x) = G(x)$
Ladybugs and Aphids

Ladybug predation rate:

\[ P(x) = 30x^3 + x^3 \]

Aphid population growth rate:

\[ G(x) = 0.5x \cdot P(x) = G(x) \]
Ladybugs and Aphids

Size of aphid population: \( x \)

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Aphid population growth rate: \( G(x) = 0.5x \)
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\frac{30x^3}{20^3 + x^3} = 0.5x
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\( P(x) = G(x) \)
Ladybugs and Aphids

Size of aphid population: \( x \)

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Aphid population growth rate: \( G(x) = 0.5x \)

\[
\frac{30x^3}{20^3 + x^3} = 0.5x \\
\frac{60x^2}{20^3 + x^3} = 1
\]
Ladybugs and Aphids

Size of aphid population: \( x \)

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\frac{30x^3}{20^3 + x^3} = 0.5x
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\frac{60x^2}{20^3 + x^3} = 1
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60x^2 = 20^3 + x^3
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\( P(x) = G(x) \)
Ladybugs and Aphids

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\[
60x^2 = 20^3 + x^3
\]

\[
x^3 - 60x^2 + 20^3 = 0
\]

Hard to solve explicitly—use Newton's Method
Ladybugs and Aphids

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60x^2 = 20^3 + x^3
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x^3 - 60x^2 + 20^3 = 0
\]

Hard to solve explicitly— use Newton’s Method
Ladybugs and Aphids

\[ x^3 - 60x^2 + 20^3 = 0 \]

\[ f(x) = x^3 - 60x^2 + 20^3 \]

\[ x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \]
Ladybugs and Aphids

\[ x^3 - 60x^2 + 20^3 = 0 \]

\[ f(x) = x^3 - 60x^2 + 20^3 \]
\[ f'(x) = 3x^2 - 120x \]
\[ x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \]

Spreadsheet (link)
Anthill problem
Chapter 5: Tangent lines, linear approximation, Newton’s method

5.1 again

Anthill problem

\[ y = 4 - x^2 \]

How long is your ladder?

Photo: Guido Gerring
Anthill problem
Anthill problem
Chapter 5: Tangent lines, linear approximation, Newton’s method

5.1 again

Anthill problem

Photo: Guido Gerring
Anthill problem
Chapter 5: Tangent lines, linear approximation, Newton’s method

5.1 again

Anthill problem

How long is your ladder?

Photo: Guido Gerring
Anthill problem
Anthill problem
Anthill problem

\[ y = 4 - x^2 \]
Anthill problem

\[ y = 4 - x^2 \]
Anthill problem

How long is your ladder?

\[ y = 4 - x^2 \]
Anthill Problem

- The ladder is a tangent line to the anthill.
- The ladder intersects the hill at point \((p, 4 - p^2)\) for some constant \(p\).
- Since \(\frac{d}{dx}\{4 - x^2\} = -2x\), the slope of the tangent line is \(-2p\).
- Since the tangent line passes through the point \((0, 5)\), it has equation
  \[(y - 5) = -2px\]

- Also, the tangent line passes through the point \((p, 4 - p^2)\). So:
  \[
  \begin{align*}
  (4 - p^2) - 5 &= -2p(p) \\
  &\underline{y} \quad \underline{x}
  \end{align*}
  \]
  \[p^2 = 1 \quad p = -1\]

- Then the equation of the tangent line is
  \[y - 5 = 2x\]

- The tangent line has \(x\)-intercept \(x = -\frac{2}{5}\). So, the ladder is the hypotenuse of a triangle with side lengths 5 and \(-\frac{2}{5}\). By the Pythagorean Theorem, the ladder has length \(\sqrt{5^2 + \left(\frac{2}{5}\right)^2} = \frac{5\sqrt{5}}{2}\).
Example 10:

\[ f(x) = x^3 - 4x \]

1. Sketch \( f(x) \), using the methods we learned earlier.
2. Give an equation of the tangent line to \( y = f(x) \) when \( x = -1 \).
3. Give all points where the tangent line intersects the curve.
Linear Approximation

Solution 10:

\[ y = x^3 - 4x \text{ at } (-1, 3) \]

\[ y' = 3x^2 - 4 \]

\[ m = -1 \]

\[ y = 2 - x \]
Solution 10:

Given the function $y = x^3 - 4x$, we have:

- $y' = 3x^2 - 4$
- $m = y'(-1, 3) = 3(-1)^2 - 4 = -1$

Thus, the equation of the tangent line at $(-1, 3)$ is:

$y - 3 = -1(x + 1)$

or

$y = -x + 2$
Solution 10:

The point $(-1, 3)$ is on the curve $y = x^3 - 4x$. The derivative of the function at $x = -1$ is $y' = 3x^2 - 4$, which gives $y'(-1) = -1$. Thus, the slope of the tangent line at $(-1, 3)$ is $-1$. The linear approximation at $(-1, 3)$ is given by the equation:

$$y = (x + 1)(-1) + 3 = -x + 2.$$
Solution 10:

Given the function $y = x^3 - 4x$ and its derivative $y' = 3x^2 - 4$, we want to find the linear approximation at the point $(-1, 3)$.

The linear approximation $m$ at $x = a$ is given by:

$$m = y'(a) = 3(-1)^2 - 4 = 3 - 4 = -1$$

The equation of the tangent line at $(-1, 3)$ is:

$$y - 3 = -1(x + 1)$$

Simplifying, we get:

$$y = -x - 1 + 3$$

$$y = -x + 2$$
Solution 10:

Given $y = x^3 - 4x$, we find the derivative $y' = 3x^2 - 4$. At the point $(-1, 3)$, the slope $m = -1$. The linear approximation $y = 2 - x$ is shown on the graph.
Linear Approximation

To find where the curves $y = x^3 - 4x$ and $y = 2 - x$ meet, we start by setting them equal to each other.

$$x^3 - 4x = 2 - x$$

$$x^3 - 3x - 2 = 0$$

We know that one place these curves meet is $x = -1$. So, $x = -1$ is a solution to the above equation; hence $(x + 1)$ is a factor.

$$x + 1 \overline{) x^3 - 3x - 2}$$

$$\phantom{x + 1) x^3} - x^2 - 2$$

$$\phantom{x + 1) x^3} - x^2 - 3x$$

$$\phantom{x + 1) x^3} x^2 + x$$

$$\phantom{x + 1) x^3} - 2x - 2$$

$$\phantom{x + 1) x^3} 2x + 2$$

$$\phantom{x + 1) x^3} 0$$

Then,

$$x^3 - 3x - 2 = (x + 1)(x^2 - x - 2)$$

$$= (x + 1)(x - 2)(x + 1)$$

$$= (x + 1)^2(x - 2)$$

The curves meet at $x = -1$ and $x = 2$. 
Example 11:

Use Newton’s Method to approximate $\sqrt{10}$. 

Choose $f(x)$ such that $f(\sqrt{10}) = 0$.

Choose $x_0$ (preferably close to $\sqrt{10}$).

Spreadsheet link:

$f(x) = x^2 - 10$

$f'(x) = 2x$

$x_0 = 3$

$x_1 = 3 - \frac{1}{3} = 19/6 = 3.1666666667$

$x_2 = 19/6 - 1/36 = 721/228 \approx 3.16228$
Extra Practice

**Example 11:**

Use Newton’s Method to approximate $\sqrt{10}$.

- Choose $f(x)$ such that $f(\sqrt{10}) = 0$
- Choose $x_0$ (preferably close to $\sqrt{10}$)
Extra Practice

Example 11:

Use Newton’s Method to approximate \( \sqrt{10} \).

- Choose \( f(x) \) such that \( f(\sqrt{10}) = 0 \)
- Choose \( x_0 \) (preferably close to \( \sqrt{10} \))

Spreadsheet link
Extra Practice

Example 11:

Use Newton’s Method to approximate $\sqrt{10}$.

- Choose $f(x)$ such that $f(\sqrt{10}) = 0$
- Choose $x_0$ (preferably close to $\sqrt{10}$)

Spreadsheet link

$$f(x) = x^2 - 10 \quad f'(x) = 2x$$

$x_0 = 3$

$$x_1 = 3 - \frac{-1}{6} = \frac{19}{6} = 3.166$$

$$x_2 = \frac{19}{6} - \frac{1/36}{38/6} = \frac{721}{228} \approx 3.16228$$