Overview

Uses of tangent lines:

- Approximating a function near a fixed point
- Over/under estimate

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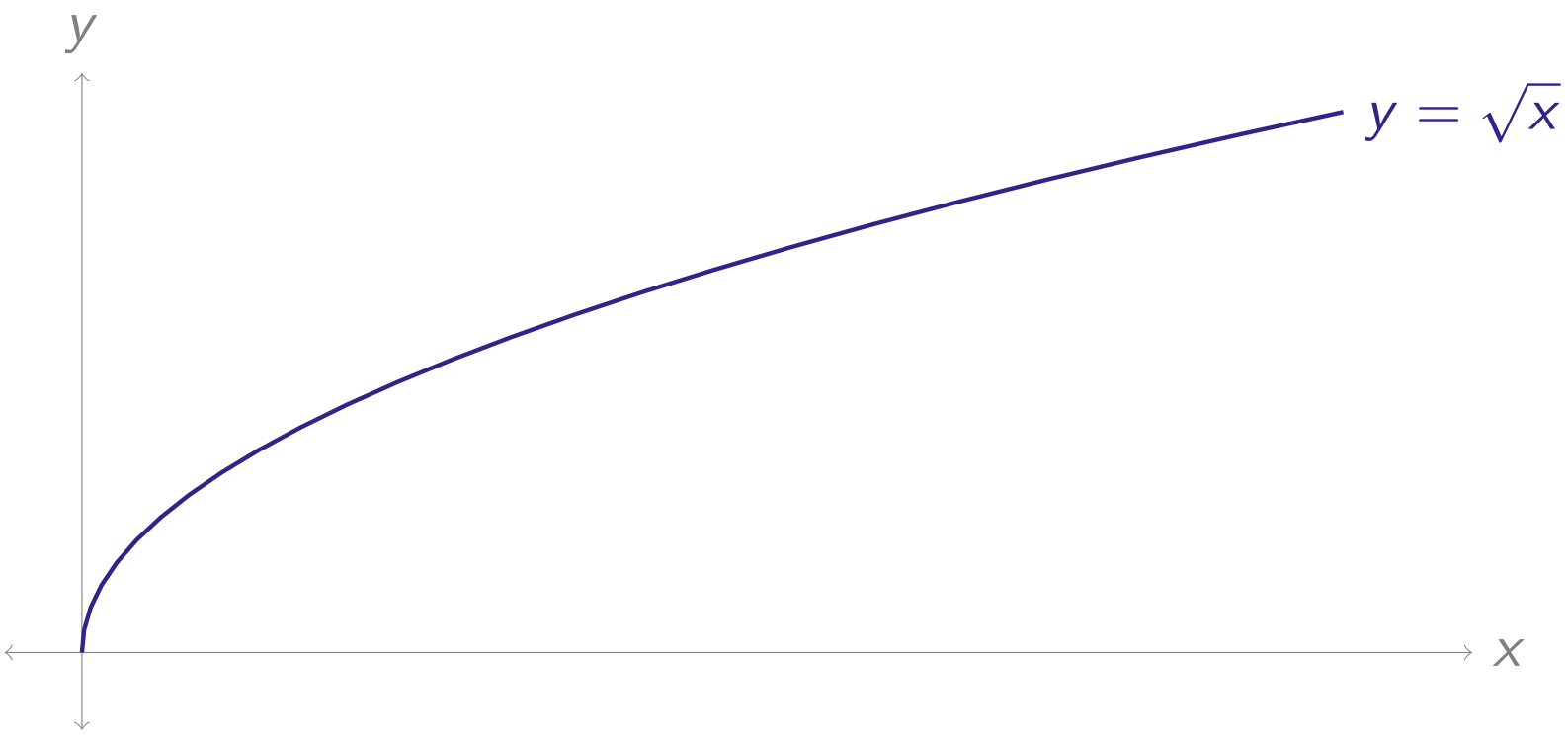
- Approximating a function near a fixed point
- Over/under estimate
- Approximating the zeroes of a function

Approximate Value of $\sqrt{10}$



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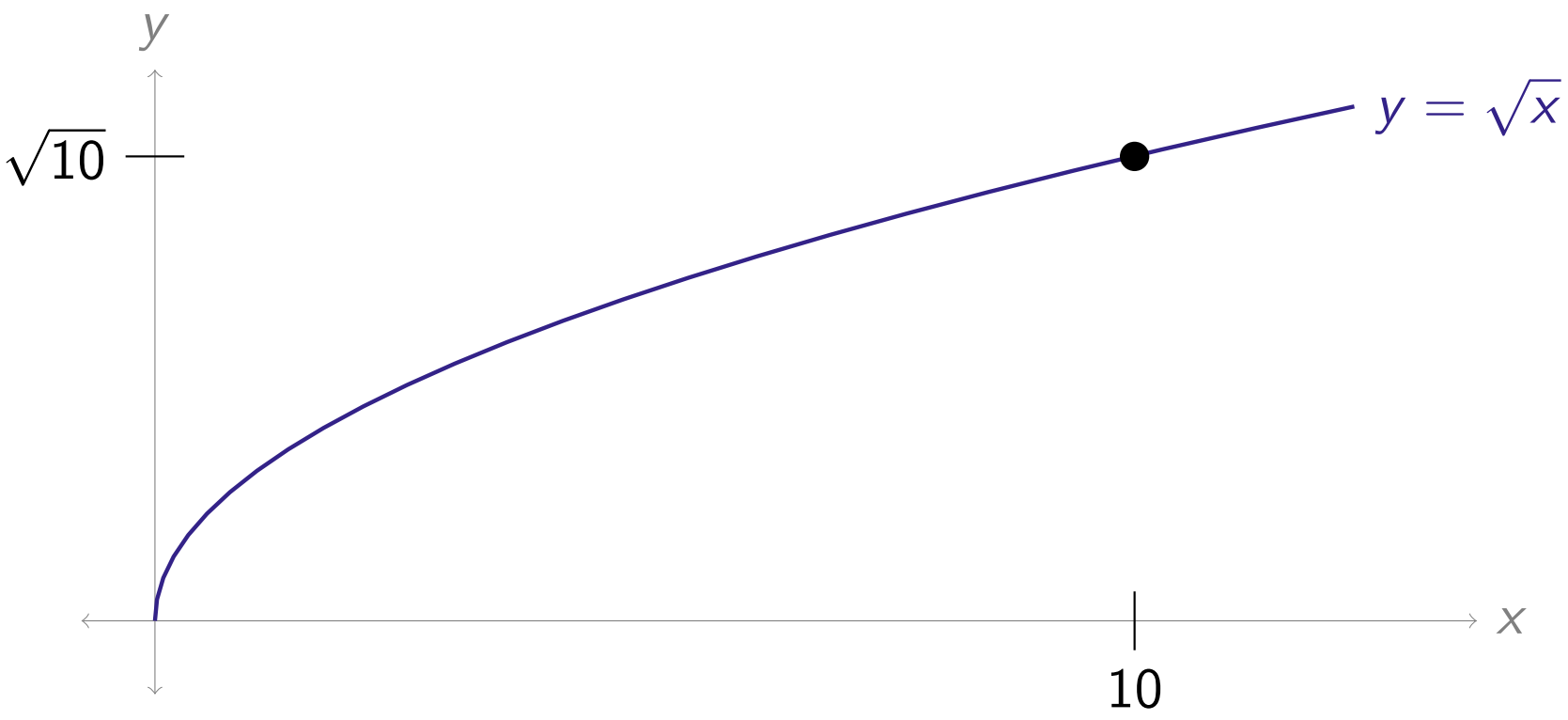

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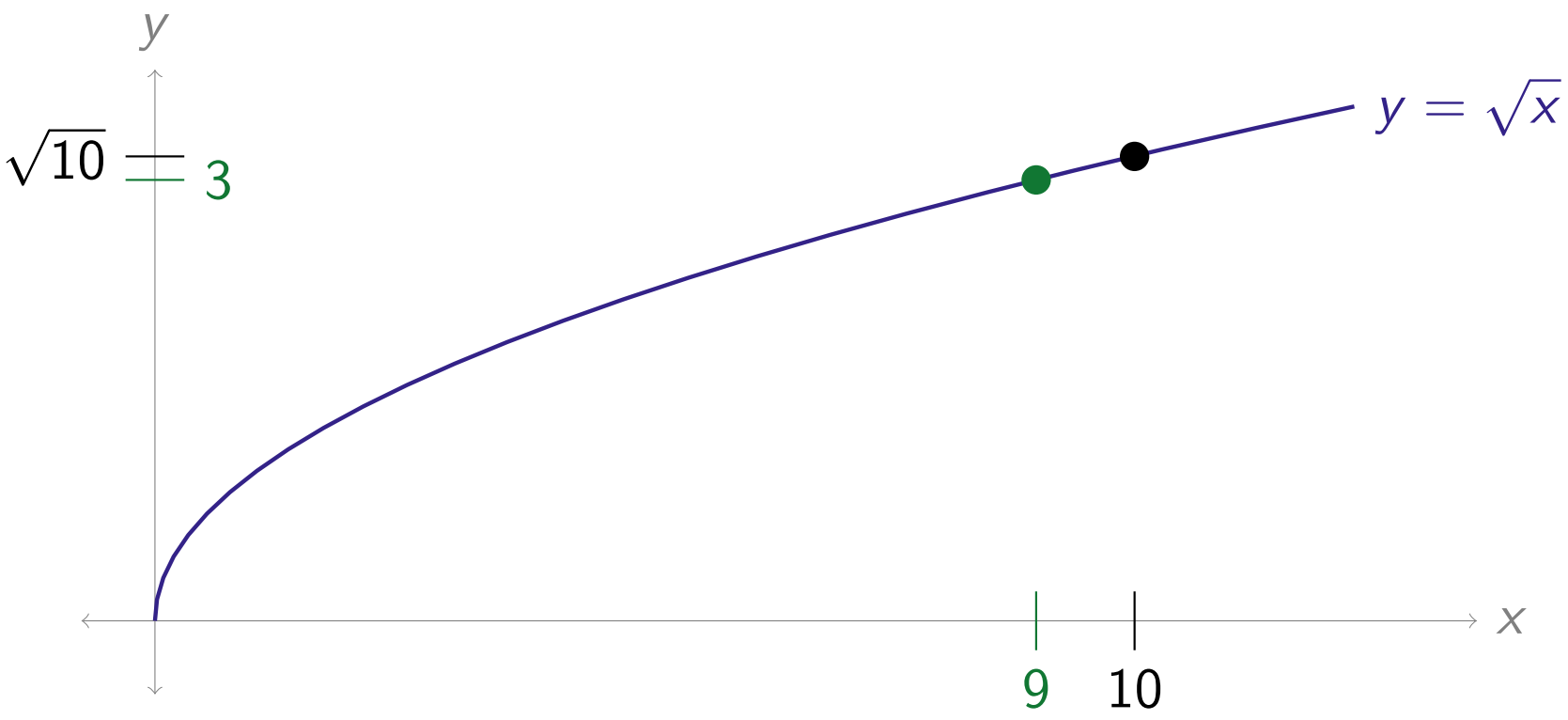

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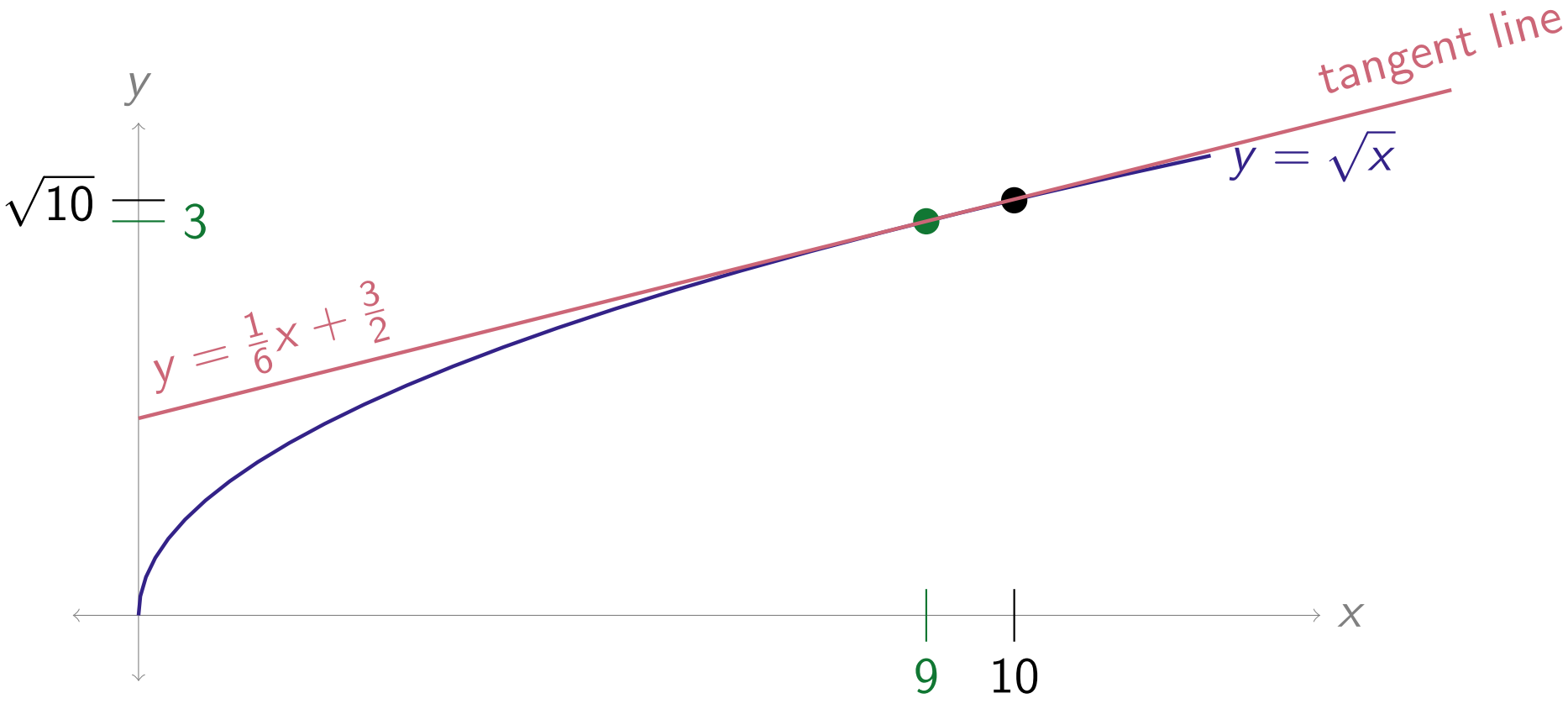

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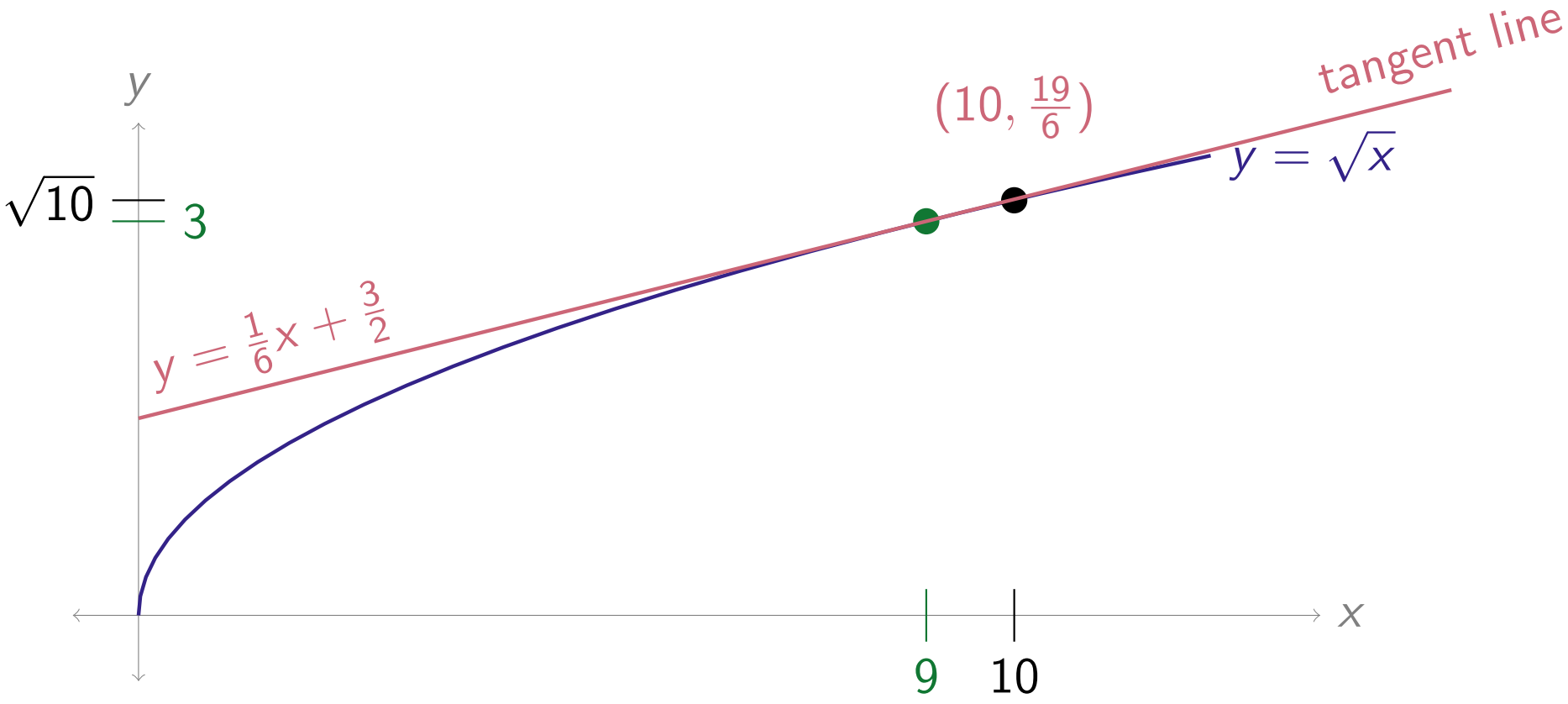
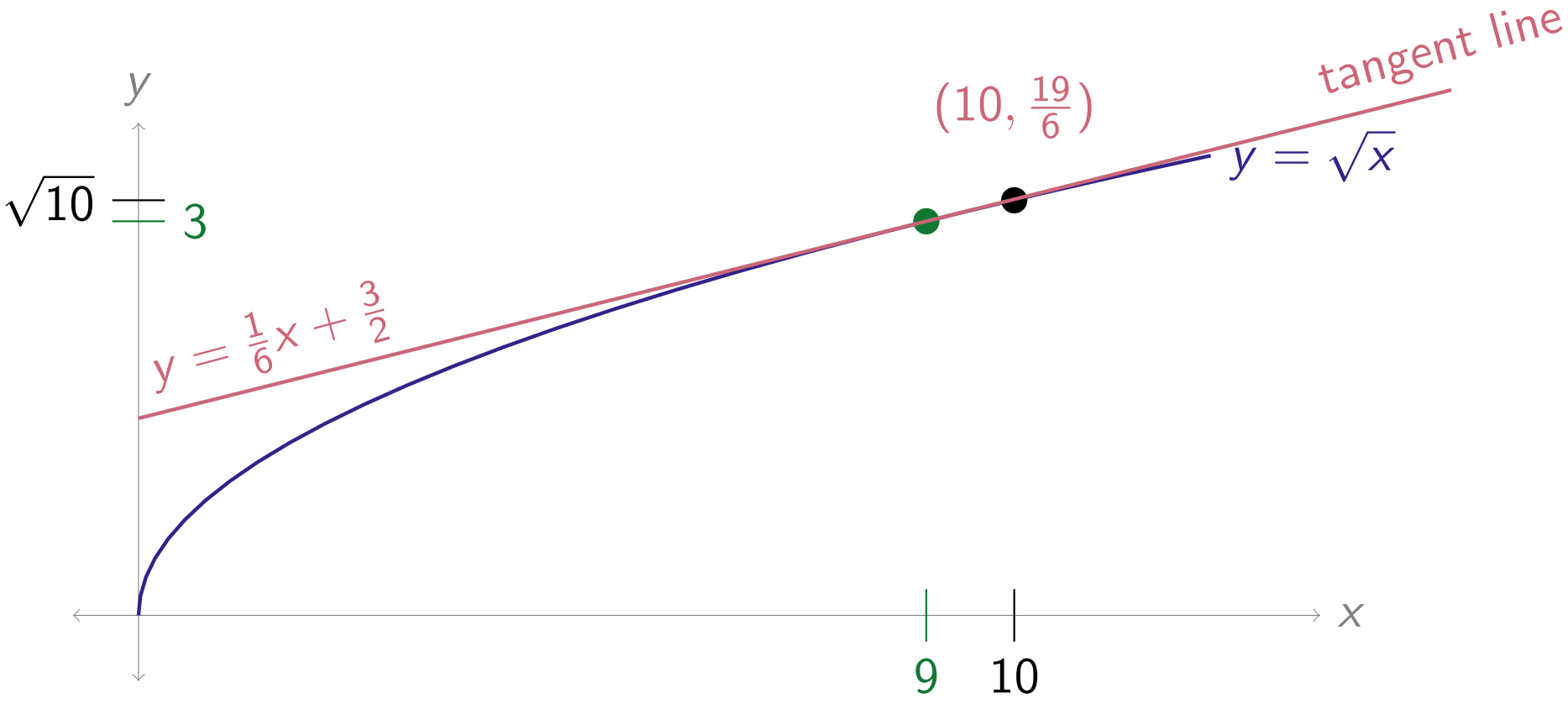


Approximate Value of $\sqrt{10}$

Approximation: $\sqrt{10} \approx \frac{19}{6} = 3.1\overline{66}$



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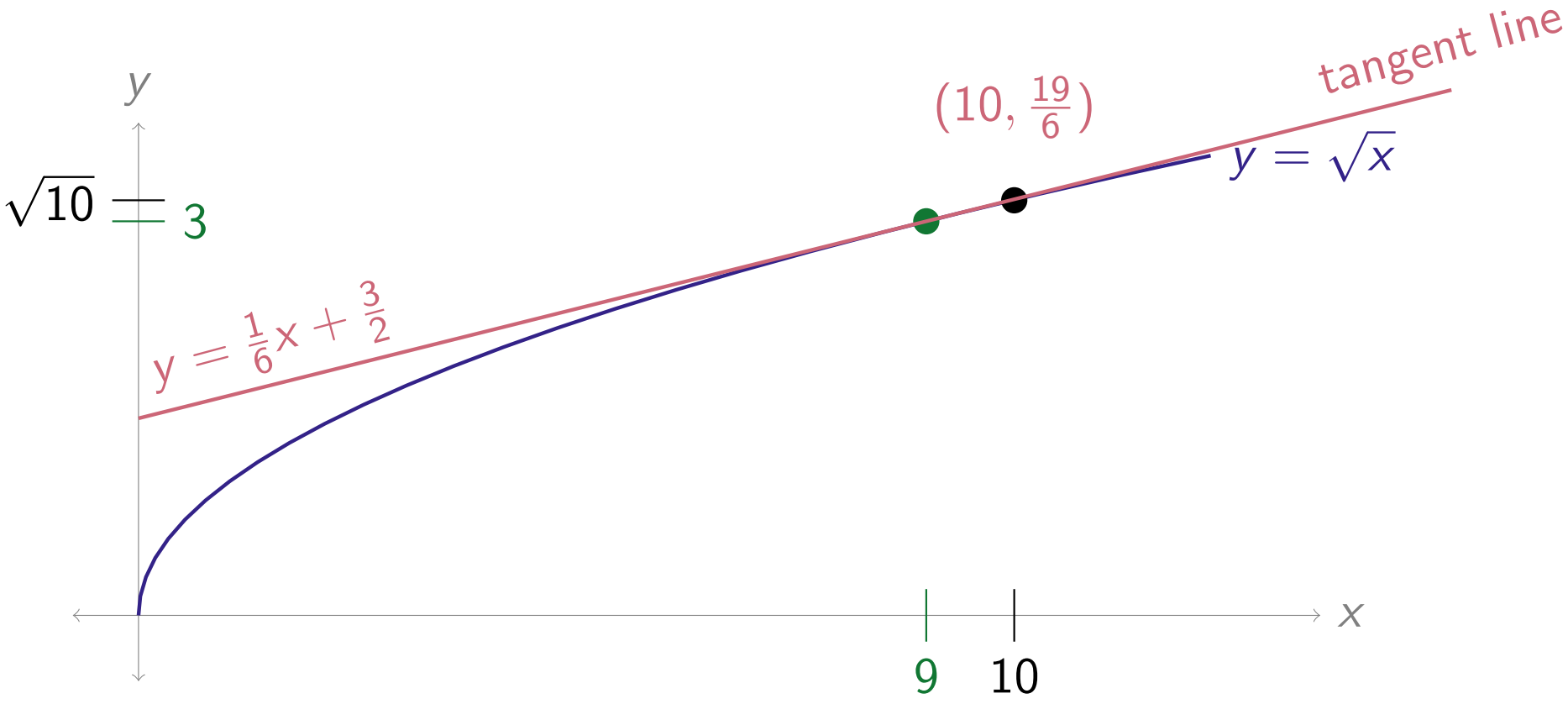
Calculator: $\sqrt{10} = 3.162277...$



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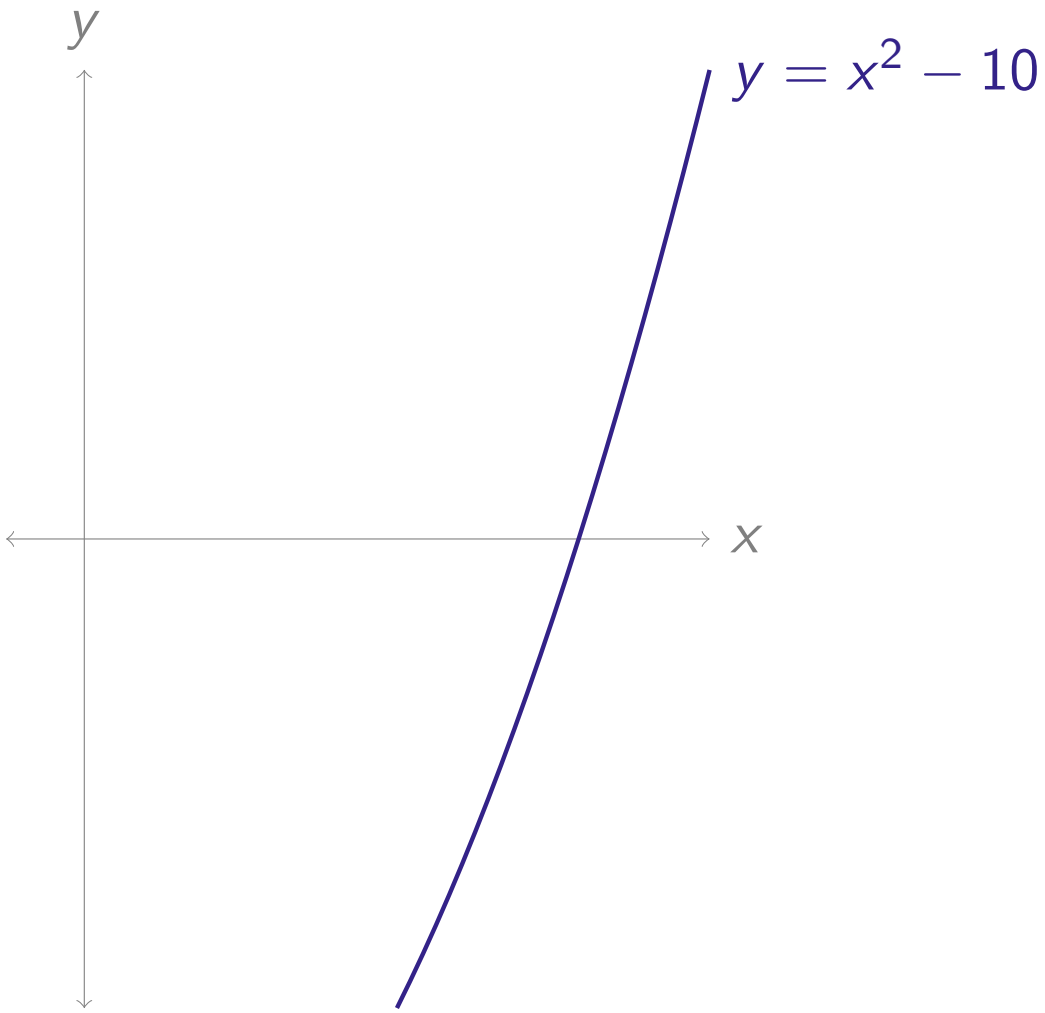

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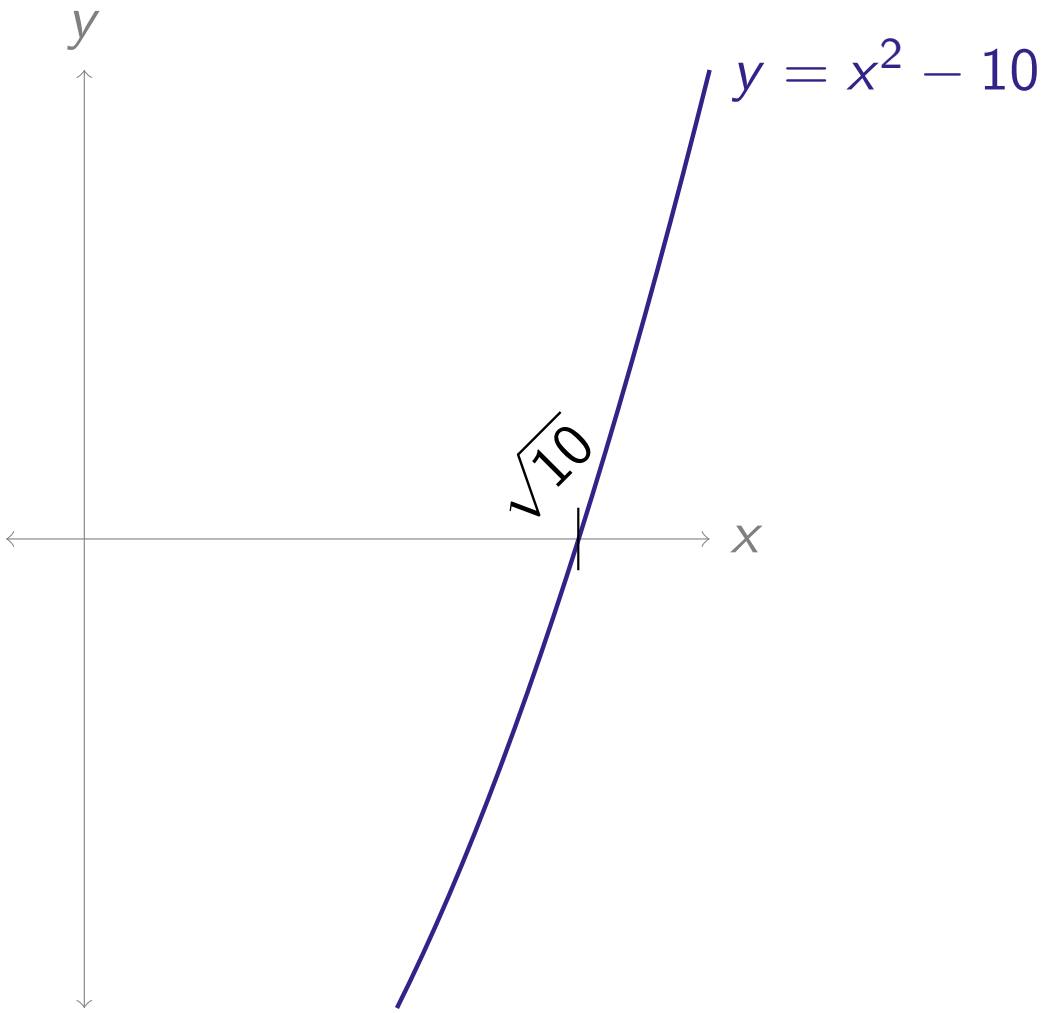

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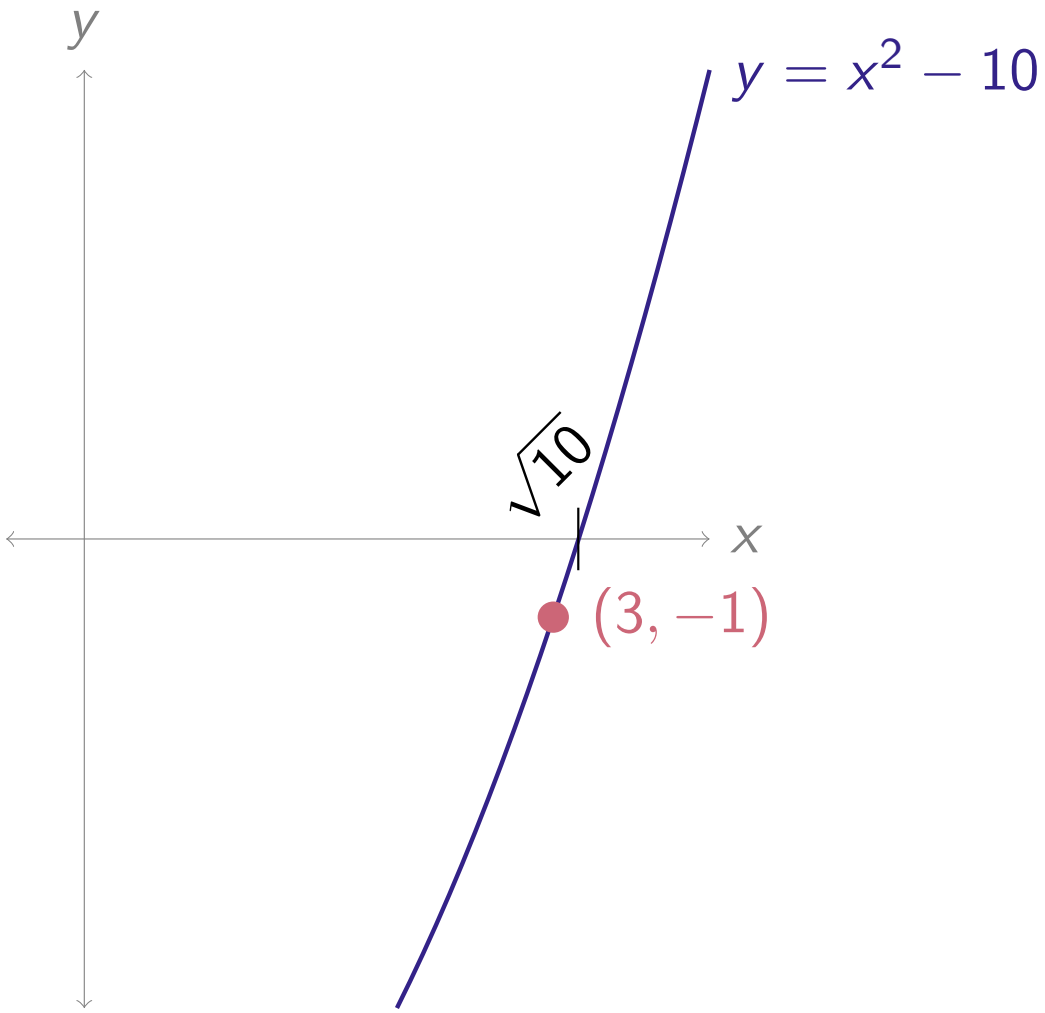

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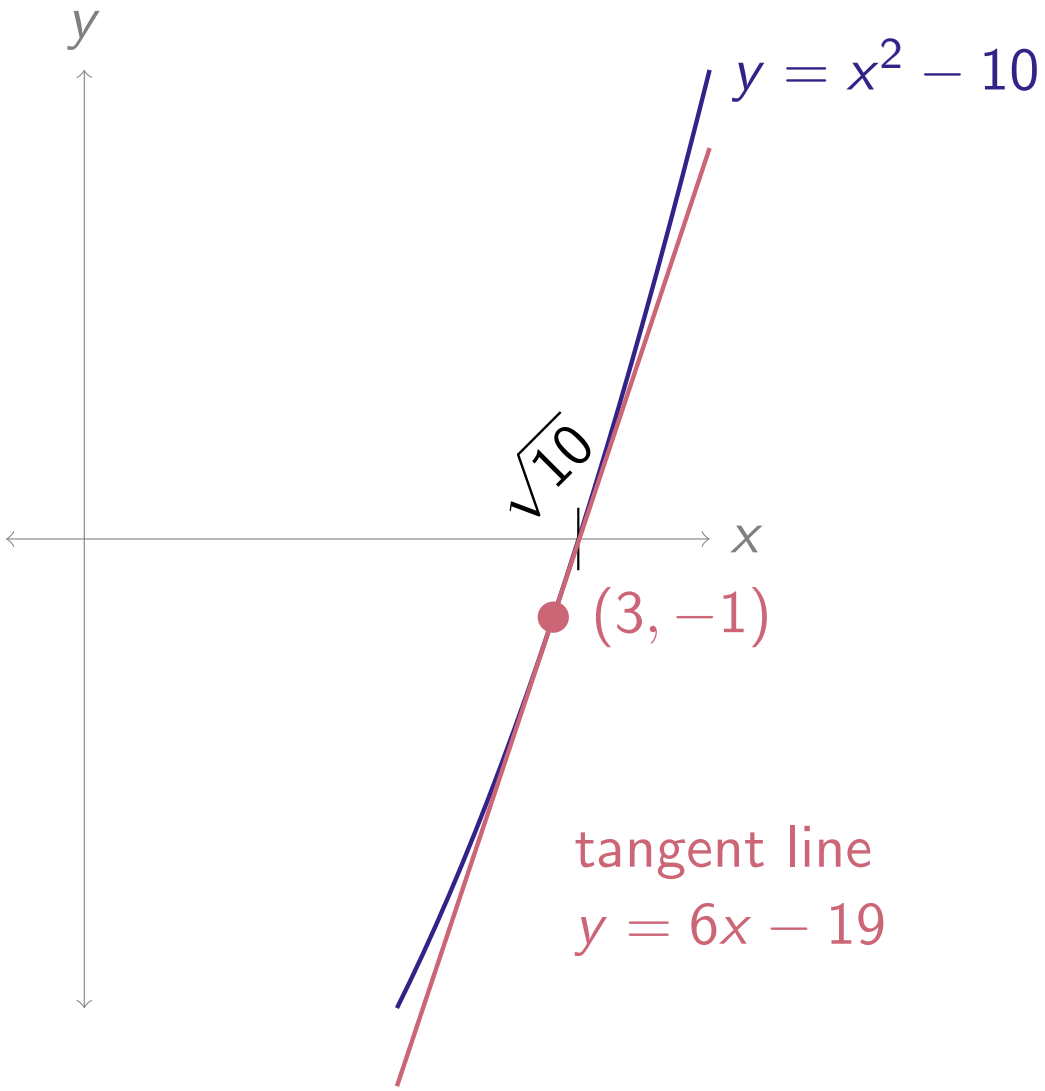

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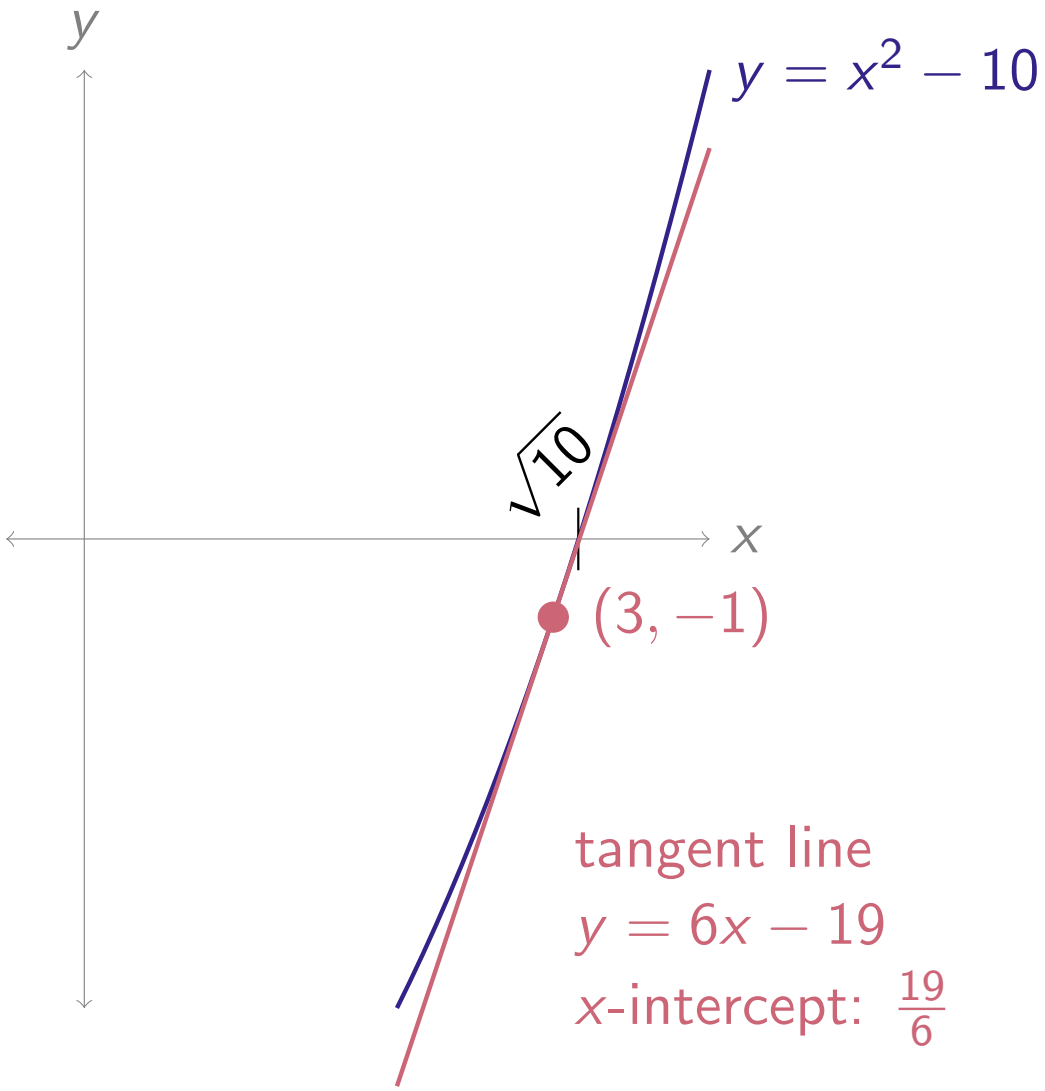

Tangent Line Equation

Point-Slope Form - Review

A line with slope m, passing through the point (x_0, y_0), has equation

$$(y - y_0) = m(x - x_0)$$

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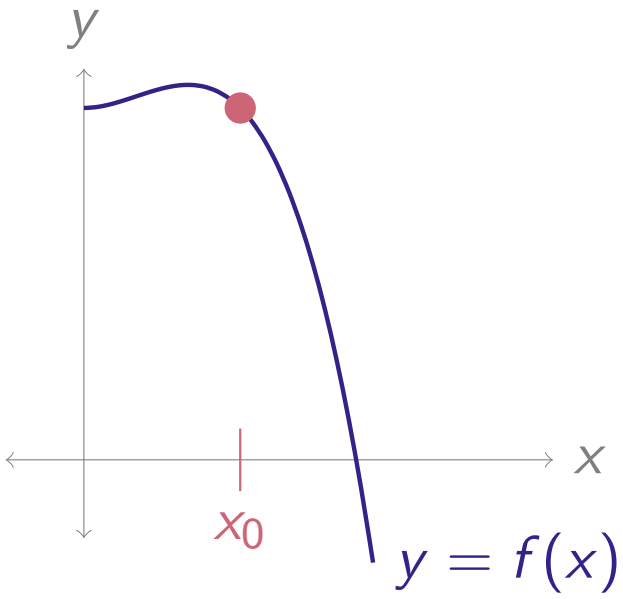

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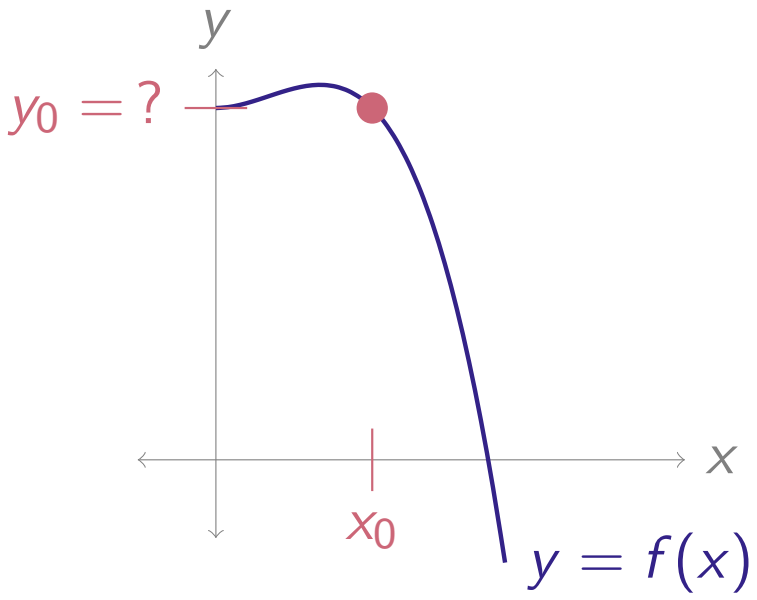

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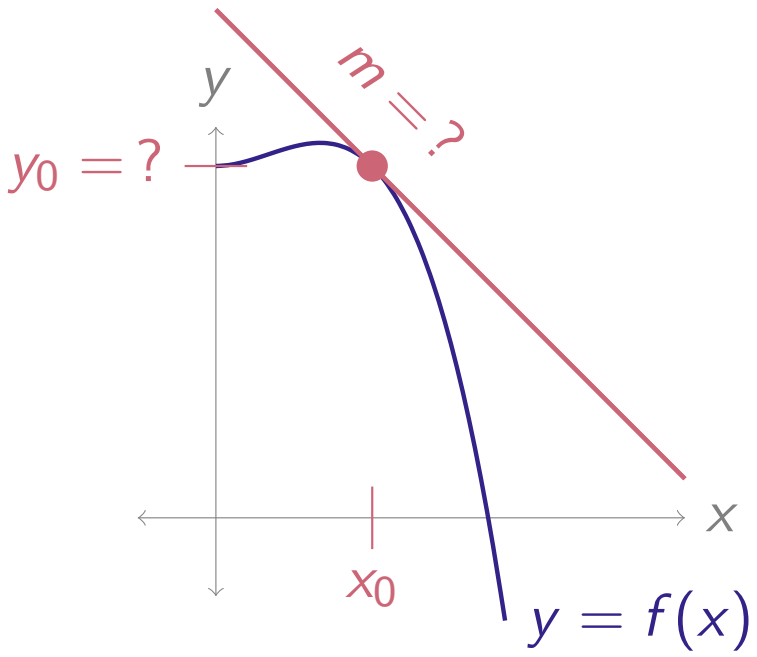

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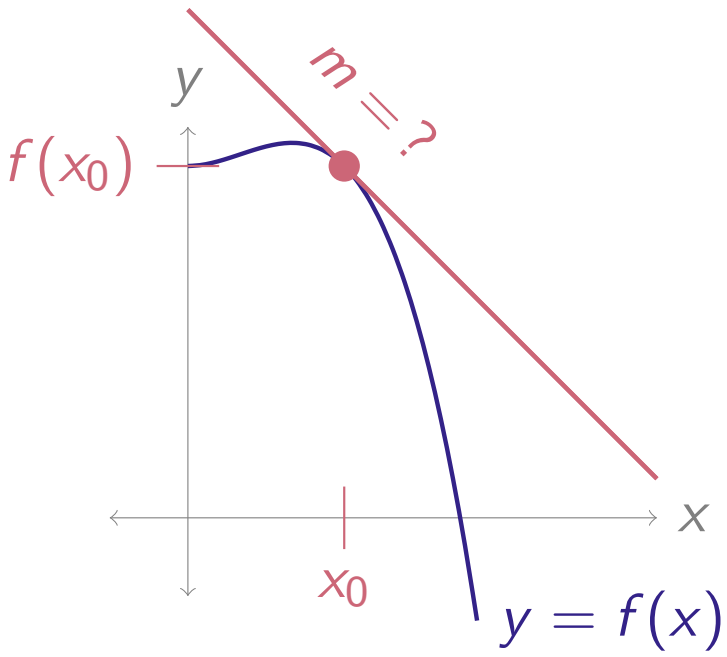

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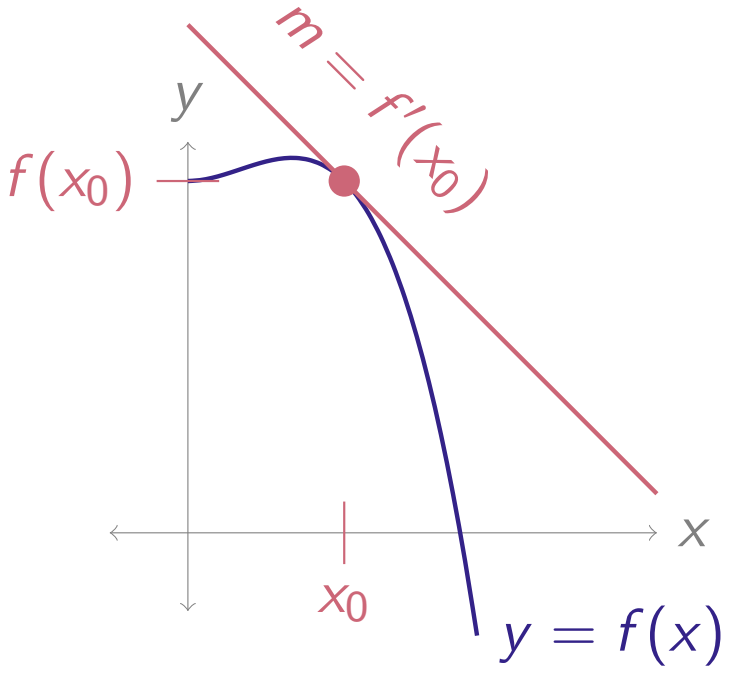

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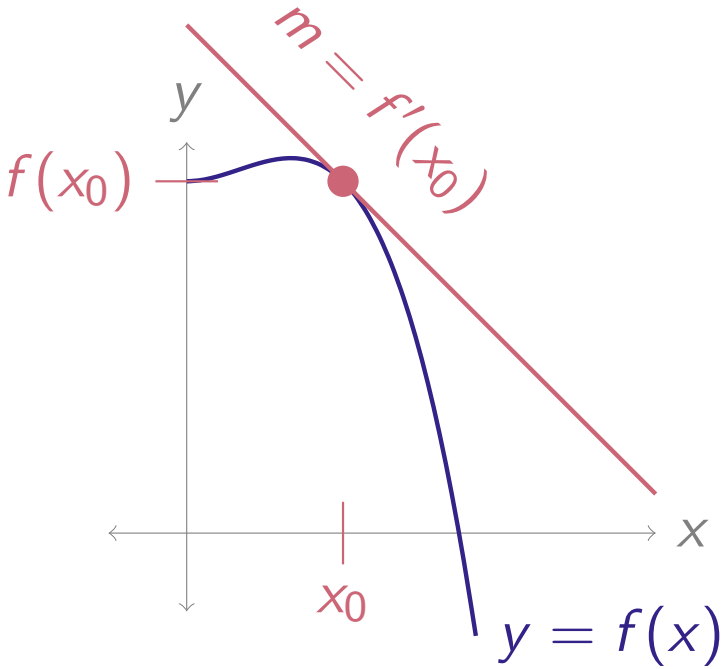

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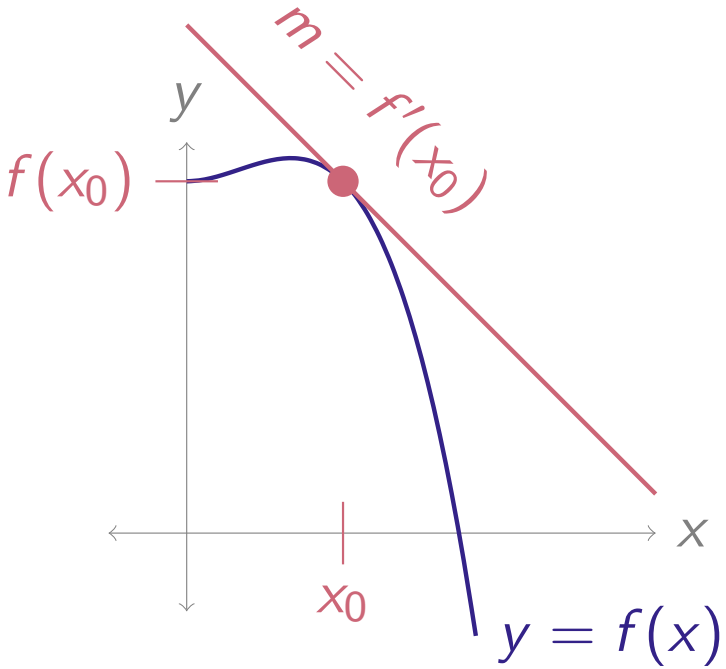

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$$(y - f(x_0)) = f'(x_0)(x - x_0)$$

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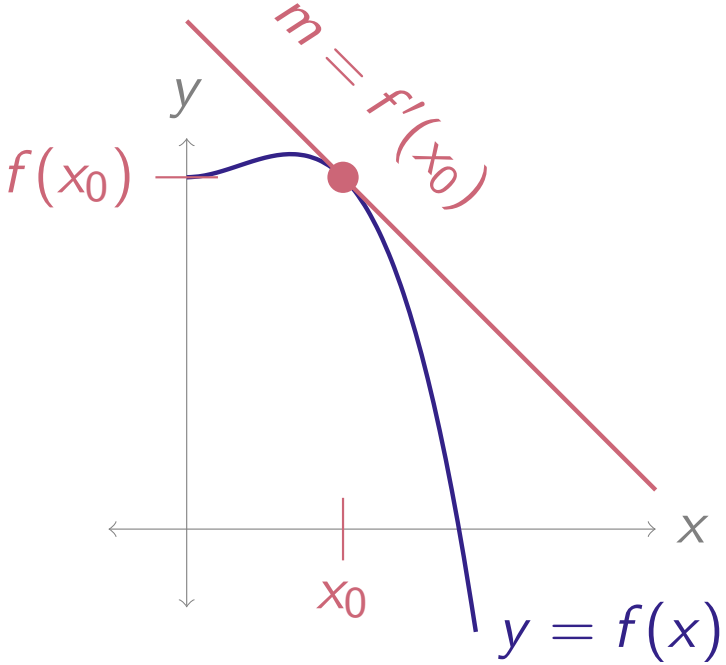

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$$(y - f(x_0)) = f'(x_0)(x - x_0)$$

That is:

$$y = f'(x_0)(x - x_0) + f(x_0)$$

Tangent line x-intercept

Generic tangent line

So, the tangent line to the curve $y = f(x)$ at the point x_0 is

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Example 1: Where does this line intersect the x-axis?

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Example 1: Where does this line intersect the x-axis?

$$\begin{aligned} 0 &= f'(x_0)(x - x_0) + f(x_0) \\ -f(x_0) &= f'(x_0)(x - x_0) \\ -\frac{f(x_0)}{f'(x_0)} &= x - x_0 \\ x &= x_0 - \frac{f(x_0)}{f'(x_0)} \end{aligned}$$

Tangent line x-intercept

Generic tangent line

So, the tangent line to the curve $y = f(x)$ at the point x_0 is

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and its x-intercept is

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Example 2: Let $f(x) = x^3 + 5x - 5$.

We want to find its root.

(a) What is $f(1)$?

(b) Where does the tangent line of $f(x)$ at $x = 1$ intersect the x-axis? Call that spot x_1.

(c) Is $f(x_1)$ closer to 0 than $f(1)$? (Use a calculator here.)

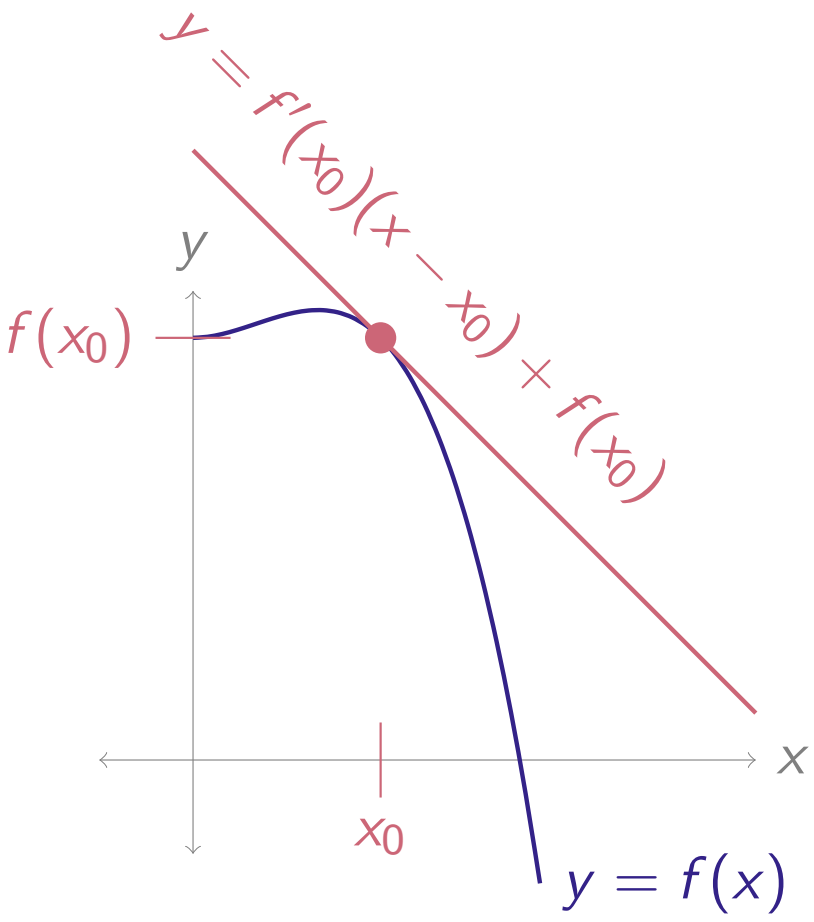
Solution 2:

(a) $f(1) = 1^3 + 5(1) - 5 = 1$. Note that 1 is not very close to 0, and we want to find the value of x that makes $f(x) = 0$.

(b) As we saw before, the spot where the tangent line intersects the x-axis is $x = 1 - \frac{f(1)}{f'(1)}$. Note $f'(x) = 3x^2 + 5$, so $f'(1) = 8$. Then the tangent line has x-intercept $x = 1 - \frac{1}{8} = \frac{7}{8}$.

(c) $f(1) = 1$, while $f(7/8) \approx 0.04$. So, if we're looking for an x-value that gives us $f(x) = 0$, $x = 7/8$ is a much better approximation than what we started with, $x = 1$.

Tangent Line Approximation



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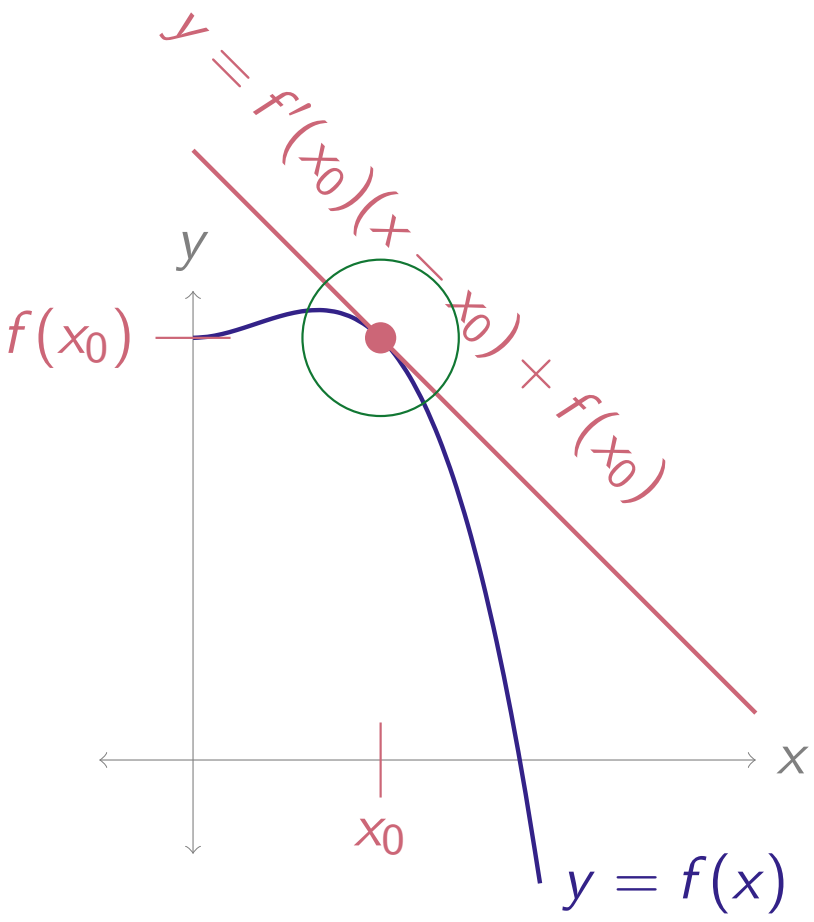

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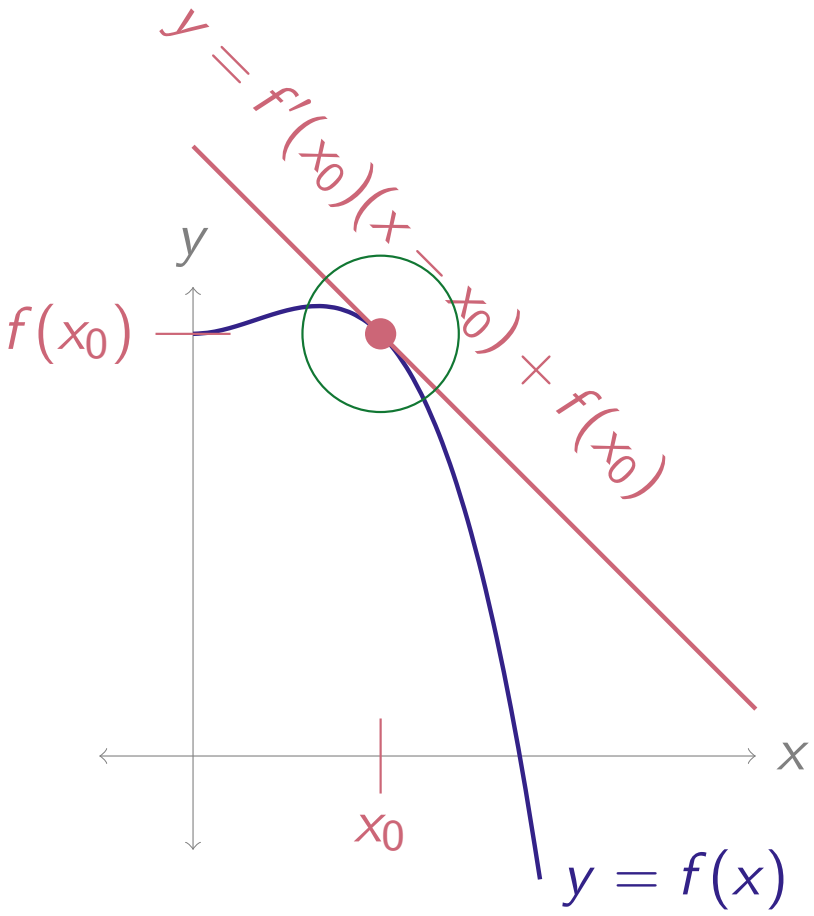

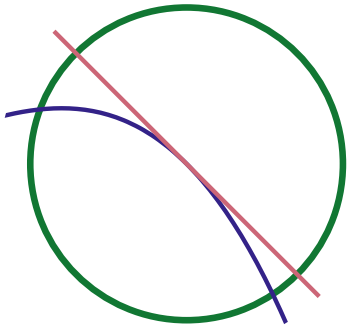

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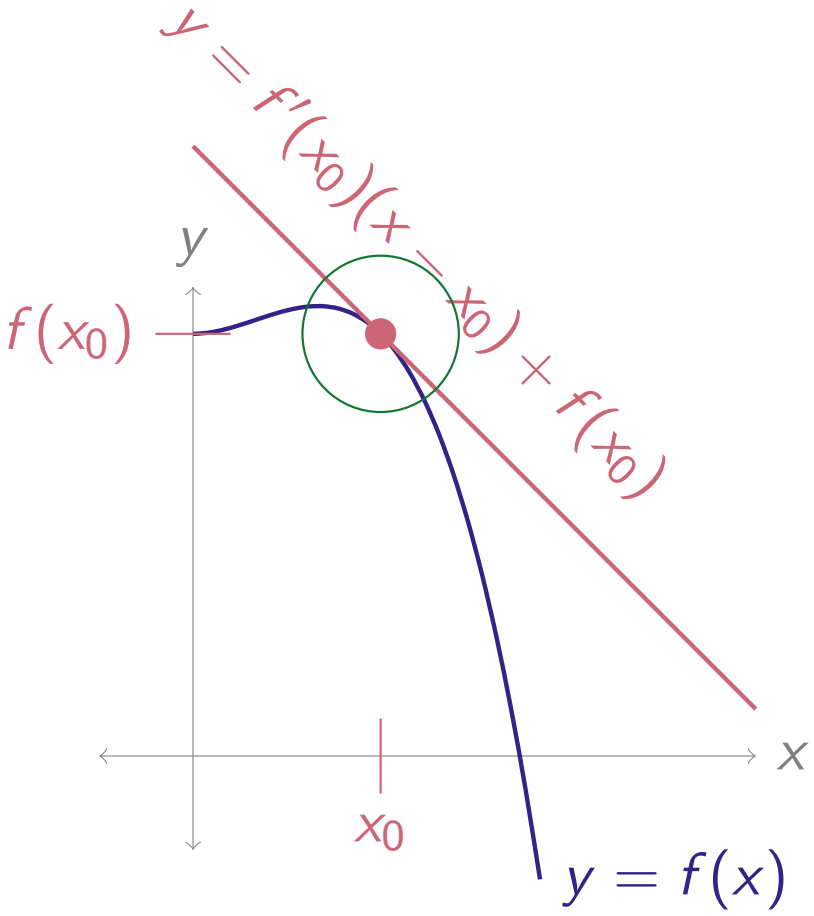
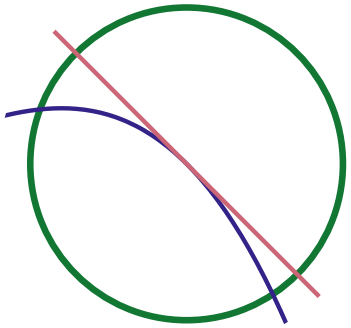
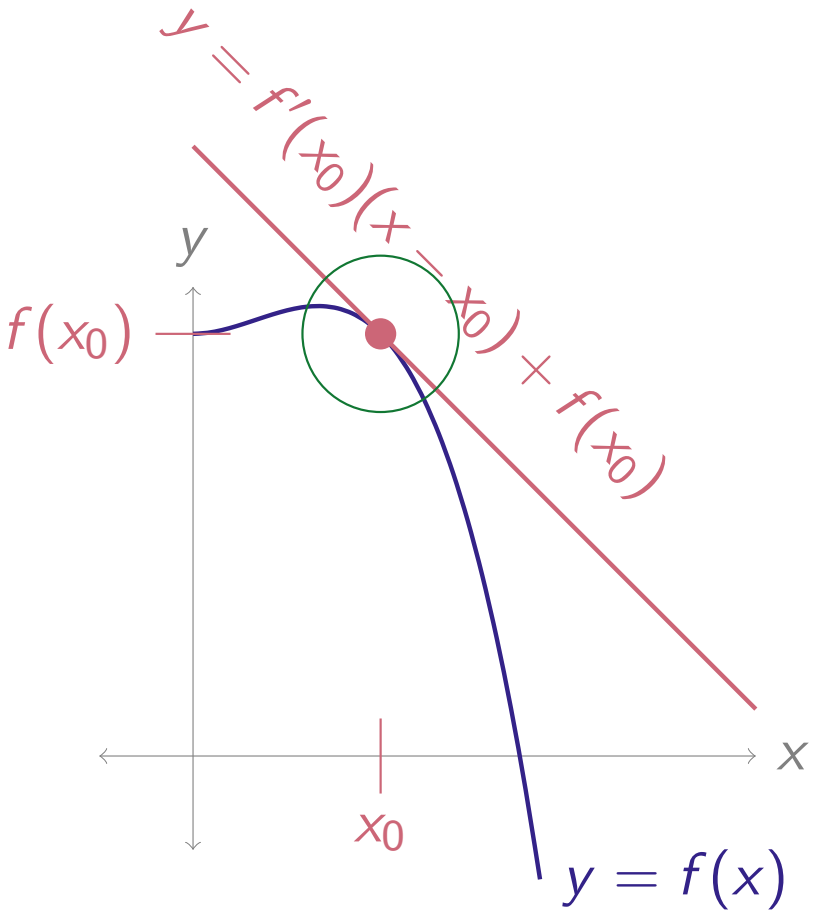
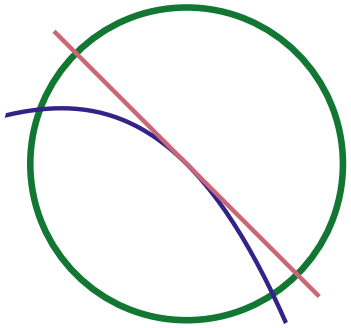




$$f(x) \approx f'(x_0)(x - x_0) + f(x_0)$$

Tangent Line Approximation





$$f(x) \approx f'(x_0)(x - x_0) + f(x_0)$$

Linear Approximation

For x near x_0,
$f(x) \approx f'(x_0)(x - x_0) + f(x_0)$.

Linear Approximation

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Example 3:

(a) Using a linear approximation of $f(x) = \sqrt[3]{x}$, approximate the value of $\sqrt[3]{28}$.

(b) Using a linear approximation, approximate the values of 99^{100} and 101^{100}.

Linear Approximation

For x near x_0, $f(x) \approx f'(x_0)(x - x_0) + f(x_0)$.

Example 3:

(a) Using a linear approximation of $f(x) = \sqrt[3]{x}$, approximate the value of $\sqrt[3]{28}$.

(b) Using a linear approximation, approximate the values of 99^{100} and 101^{100}.

Solution 3:

(a) If $f(x) = x^{1/3}$, then $f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x}^2}$.

We choose $x_0 = 27$, because this is close to 28, and is easy to work with.
Then: $f(x_0) = \sqrt[3]{27} = 3$ and $f'(x_0) = \frac{1}{3\sqrt[3]{27}^2} = \frac{1}{3 \cdot 3^2} = \frac{1}{27}$

So, our linear approximation is:

$$\begin{aligned} f(x) &\approx f'(x_0)(x - x_0) + f(x_0) \\ &= \frac{1}{27}(x - 27) + 3 \end{aligned}$$

Therefore,

$$\begin{aligned} f(28) &\approx \frac{1}{27}(28 - 27) + 3 \\ &= \frac{1}{27} + 3 = \frac{82}{27} \end{aligned}$$

Solution 3:

(a)
We can check:

$$\begin{aligned}\left(3+\frac{1}{27}\right)^3 &= 3^3 + 3\cdot 3^2\cdot\frac{1}{27} + 3\cdot 3\cdot\frac{1}{27^2} + \frac{1}{27^3}\\ &= 27 + 1 + \frac{1}{81} + \frac{1}{27^3}\\ &\approx 28\end{aligned}$$

So, $\sqrt[3]{28}$ should be fairly close to our approximation, $3+\frac{1}{27}$.

Solution 3:

(b) Let's approximate the function $f(x) = x^{100}$, with $x_0 = 100$.
$f'(x) = 100x^{99}$, so:
$f(100) = f'(100) = 100^{100} = 10^{200}$.
Then our linear approximation is

$$f(x) \approx f'(100)(x - 100) + f(100)$$
$$f(x) \approx 10^{200}(x - 100) + 10^{200}$$
$$f(x) \approx 10^{200}(x - 99)$$

This leads to:

$$f(99) \approx 10^{200}(0) = 0$$
$$f(101) \approx 2(10^{200})$$

Solution 3:

(b) How good are these approximations?

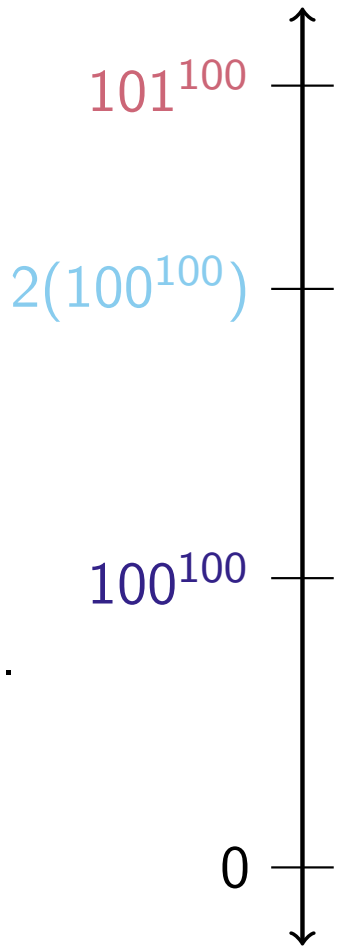
- $f(101) = 101^{100}$ (exact value);
- $f(101) \approx 2(100^{100})$ (linear approximation);
- $f(101) \approx 100^{100}$ (using $100 \approx 101$)

$$\frac{\text{approx}}{\text{exact}} = \frac{2(100^{100})}{101^{100}} = 2\left(\frac{100}{101}\right)^{100} \approx 0.74$$

So, our linear approximation was about 74% of the way there.

$$\frac{\text{approx}}{\text{exact}} = \frac{100^{100}}{101^{100}} = \left(\frac{100}{101}\right)^{100} \approx 0.37$$

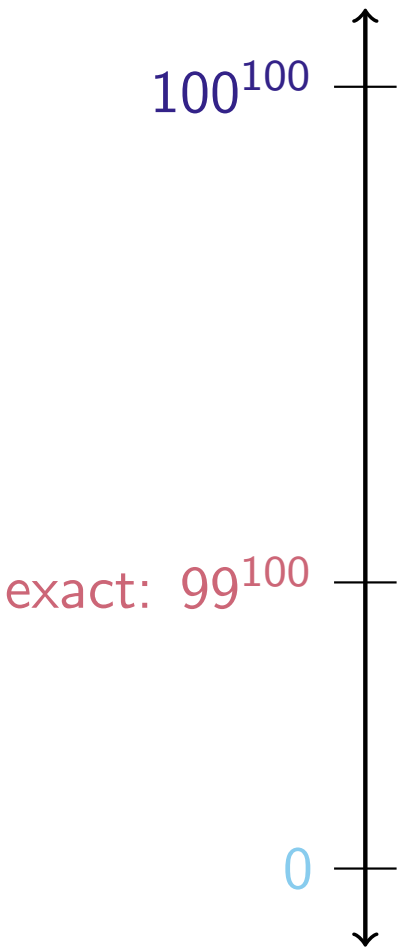
So, our other approximation was only about a third of the actual value.



Solution 3:

(b) How good are these approximations?

- $f(99) = 99^{100}$ (exact value);
- $f(99) \approx 0$ (linear approximation);
- $f(99) \approx 100^{100}$ (using $100 \approx 99$)



Linear Approximation

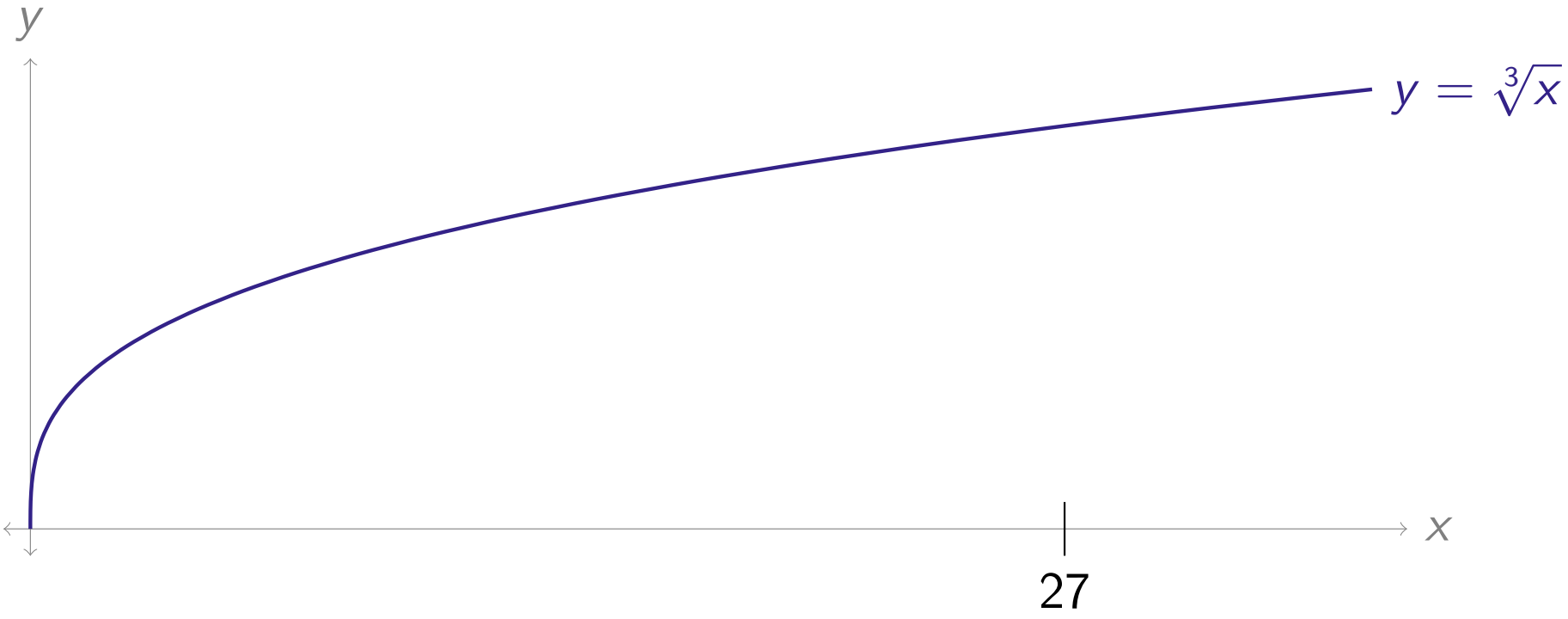
For x near x_0, $f(x) \approx f'(x_0)(x - x_0) + f(x_0)$.

(a) Using a linear approximation of $f(x) = \sqrt[3]{x}$, approximate the value of $\sqrt[3]{28}$.

(b) Using a linear approximation, approximate the values of 99^{100} and 101^{100}.

Example 4: Were these overestimates or underestimates?

Solution 4:



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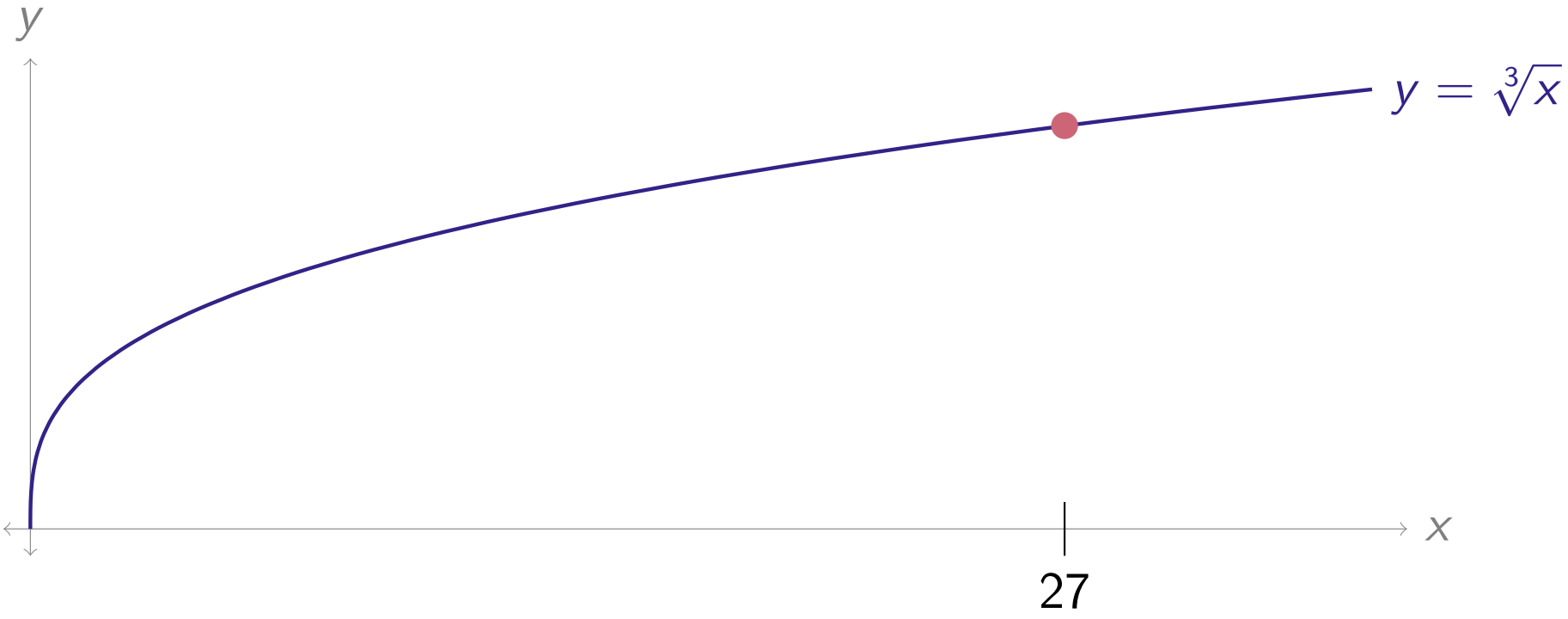

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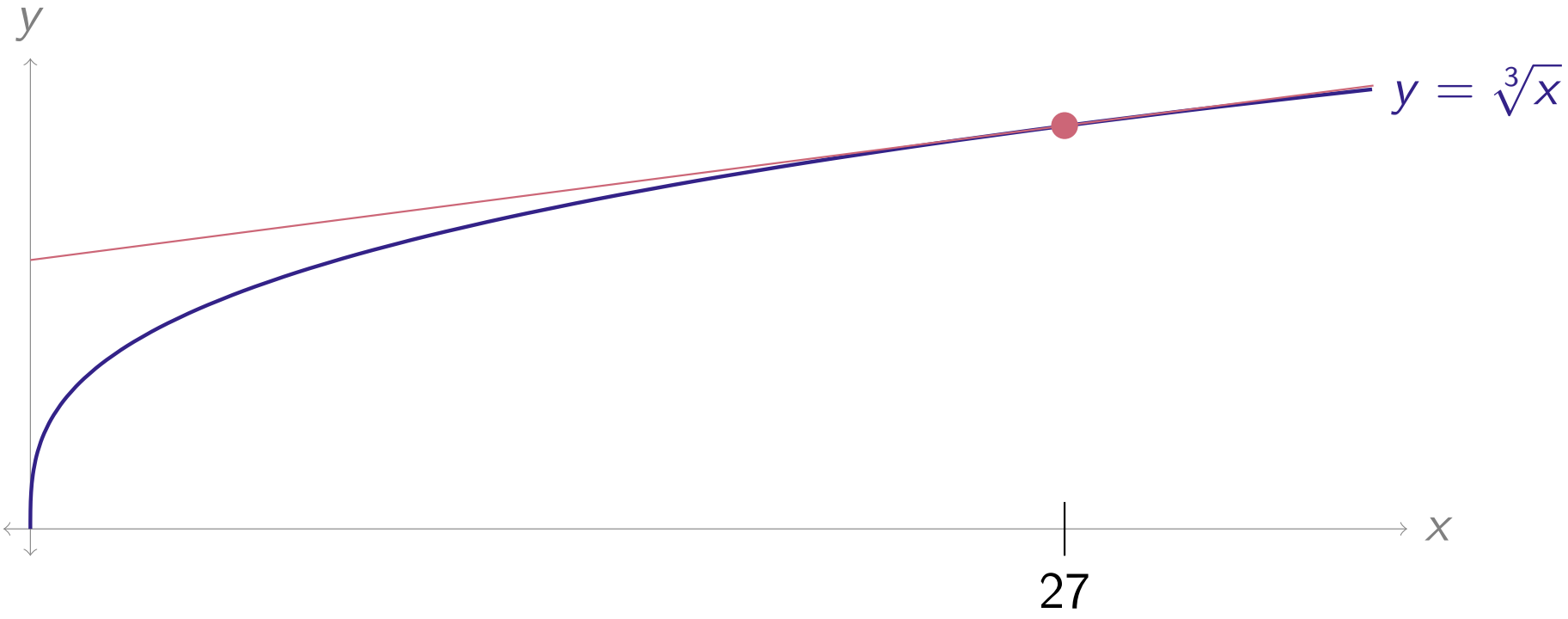

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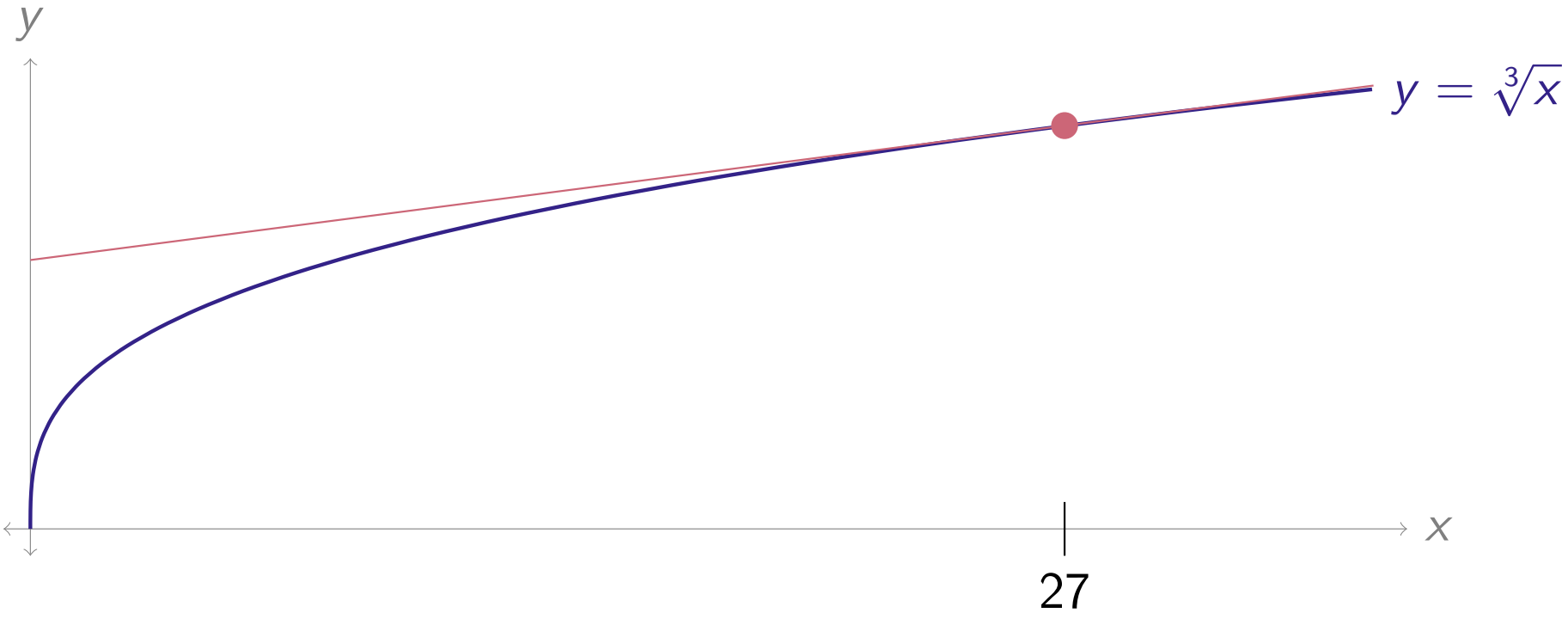
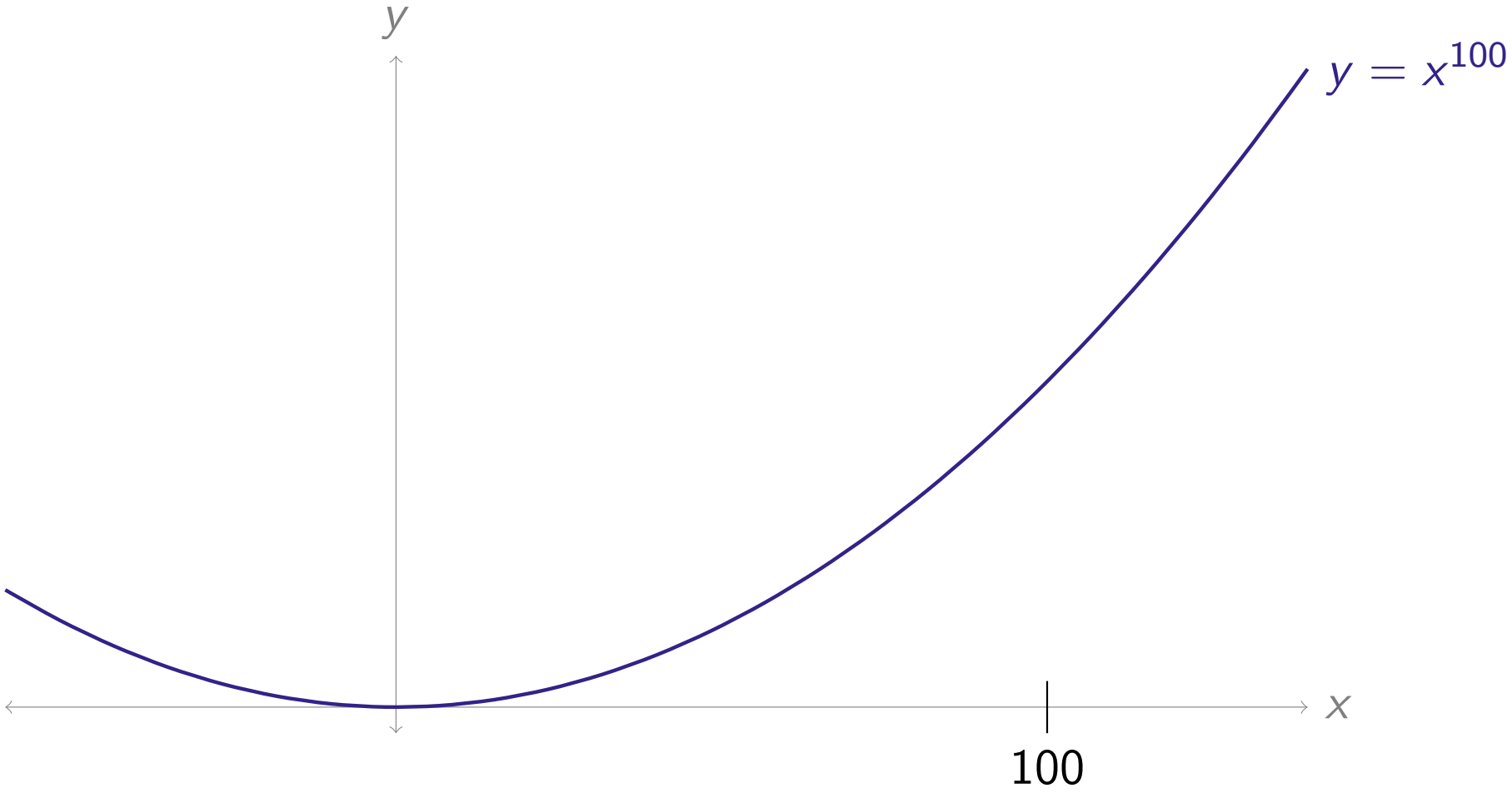


The tangent line is **higher** than the function, so the linear approximation is an **overestimate**. That is, the approximation is higher than the actual value.

Solution 4:



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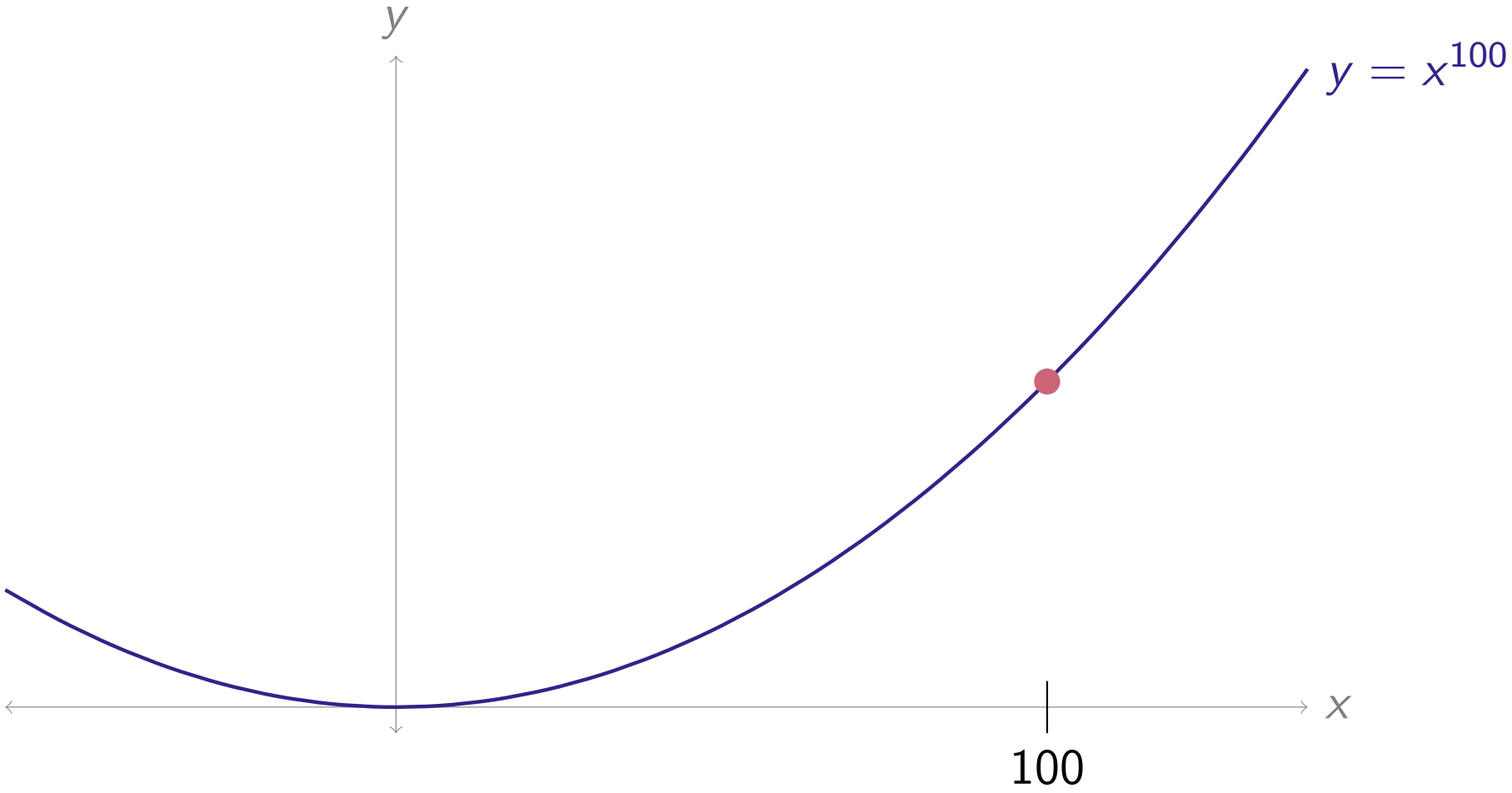

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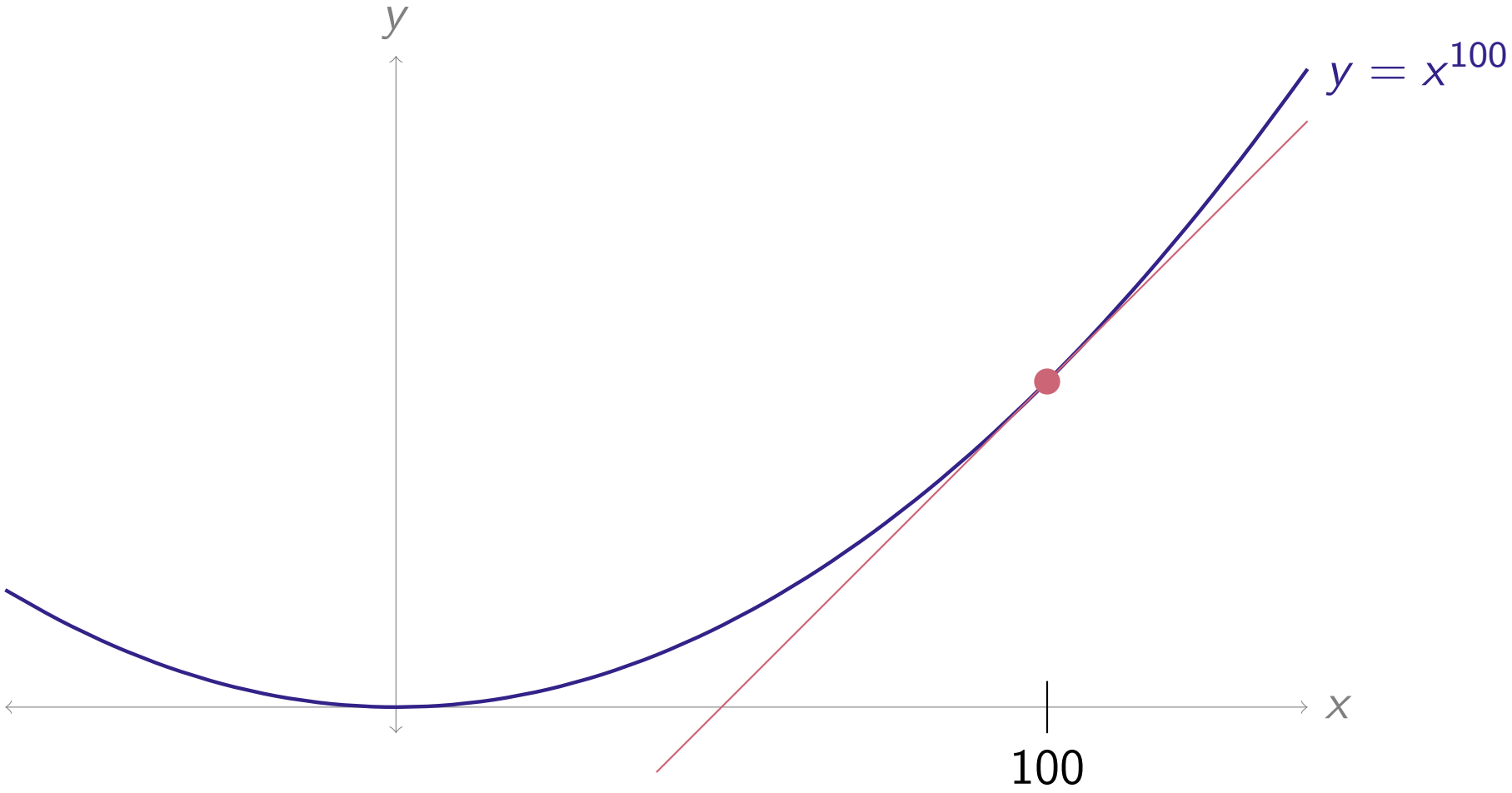

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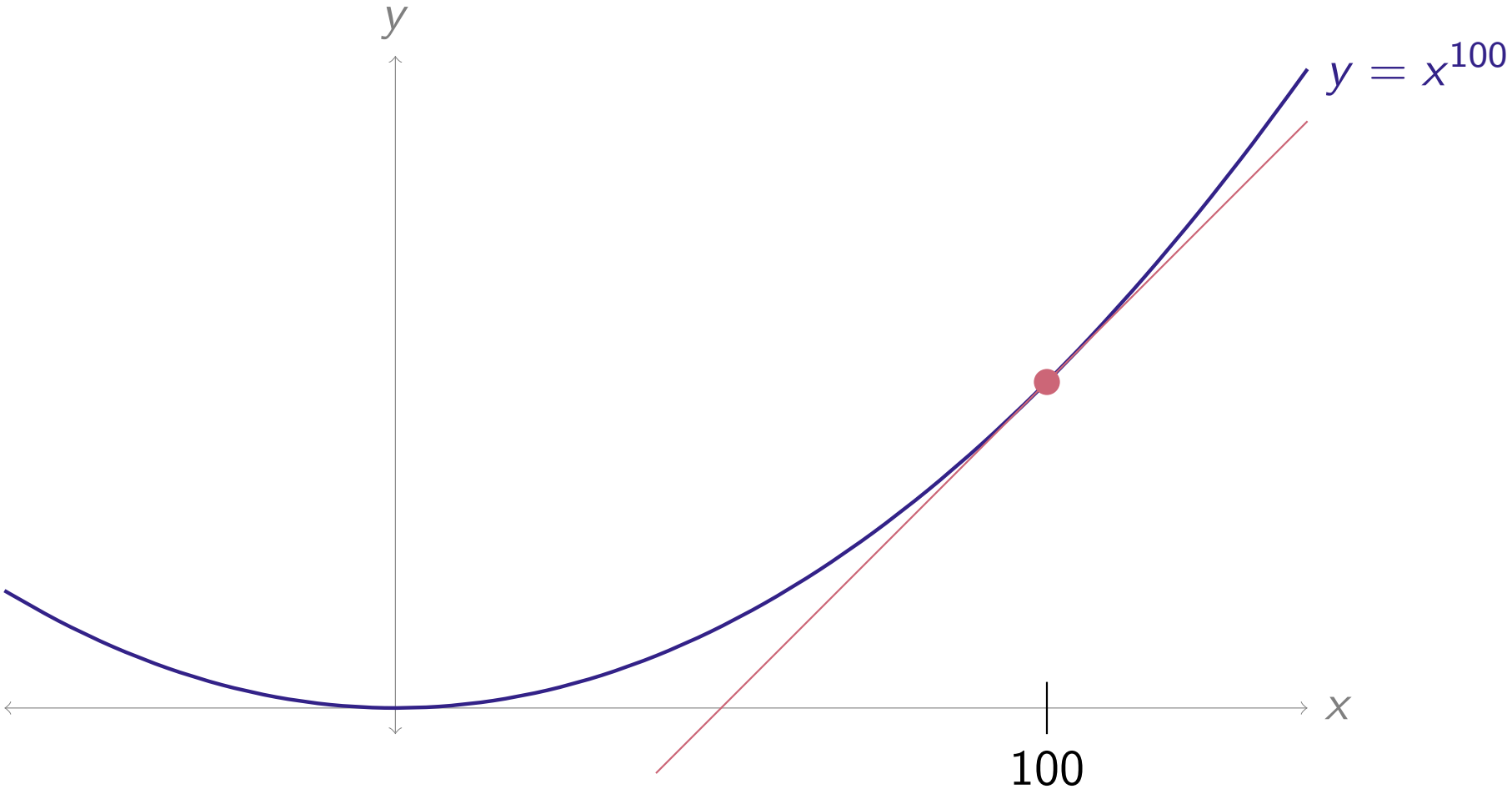


The tangent line is **lower** than the function, so the linear approximation is an **underestimate**. That is, the approximation is smaller than the actual value.

Linear Approximations

Example 5:

Let $f(x) = \dfrac{5\sqrt{x}}{2+\sqrt{x}}$.
Using the methods of this chapter, find a rational number approximating $f(8.75)$, and decide whether your approximate value is an overestimation or an underestimation.

Linear Approximations

Example 5:

Let $f(x) = \dfrac{5\sqrt{x}}{2+\sqrt{x}}$.
Using the methods of this chapter, find a rational number approximating $f(8.75)$, and decide whether your approximate value is an overestimation or an underestimation.
$f(8.75) \approx 3 - \frac{1}{60}$; overestimation

$$f(9) = \frac{5(3)}{2+3} = 3$$

$$f'(x) = \frac{(2+\sqrt{x})\left(\frac{5}{2\sqrt{x}}\right) - 5\sqrt{x}\left(\frac{1}{2\sqrt{x}}\right)}{(2+\sqrt{x})^2} \qquad \text{(quotient rule)}$$

$$= \frac{\frac{5}{\sqrt{x}} + \frac{5}{2} - \frac{5}{2}}{(2+\sqrt{x})^2}$$

$$= \frac{5}{\sqrt{x}(2+\sqrt{x})^2}$$

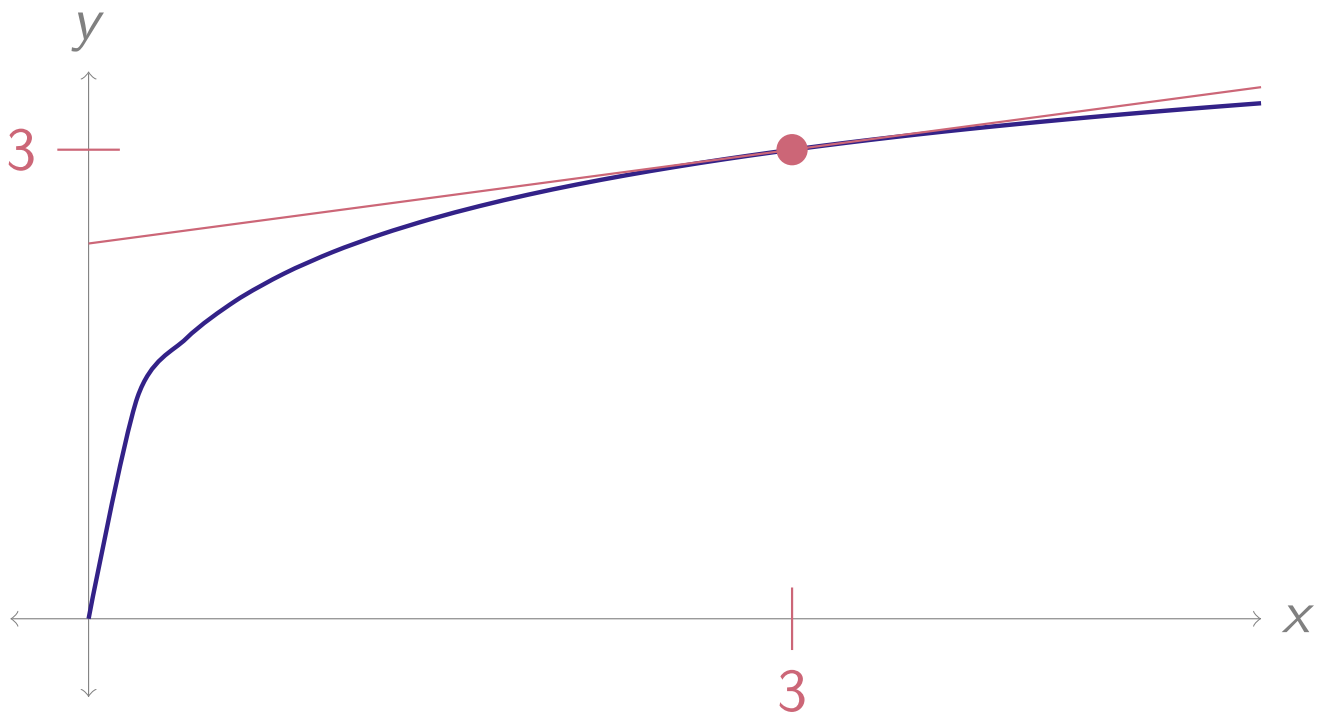
$$f'(9) = \frac{5}{3(2+3)^2} = \frac{1}{15} \qquad \text{(slope of tan line)}$$

$$f(x) \approx \frac{1}{15}(x-9) + 3 \qquad \text{(linear approx)}$$

$$f(8.75) \approx \frac{1}{15}(8.75-9) + 3$$

$$= \frac{1}{15}\left(-\frac{1}{4}\right) + 3 = 3 - \frac{1}{60} \qquad \text{(answer)}$$

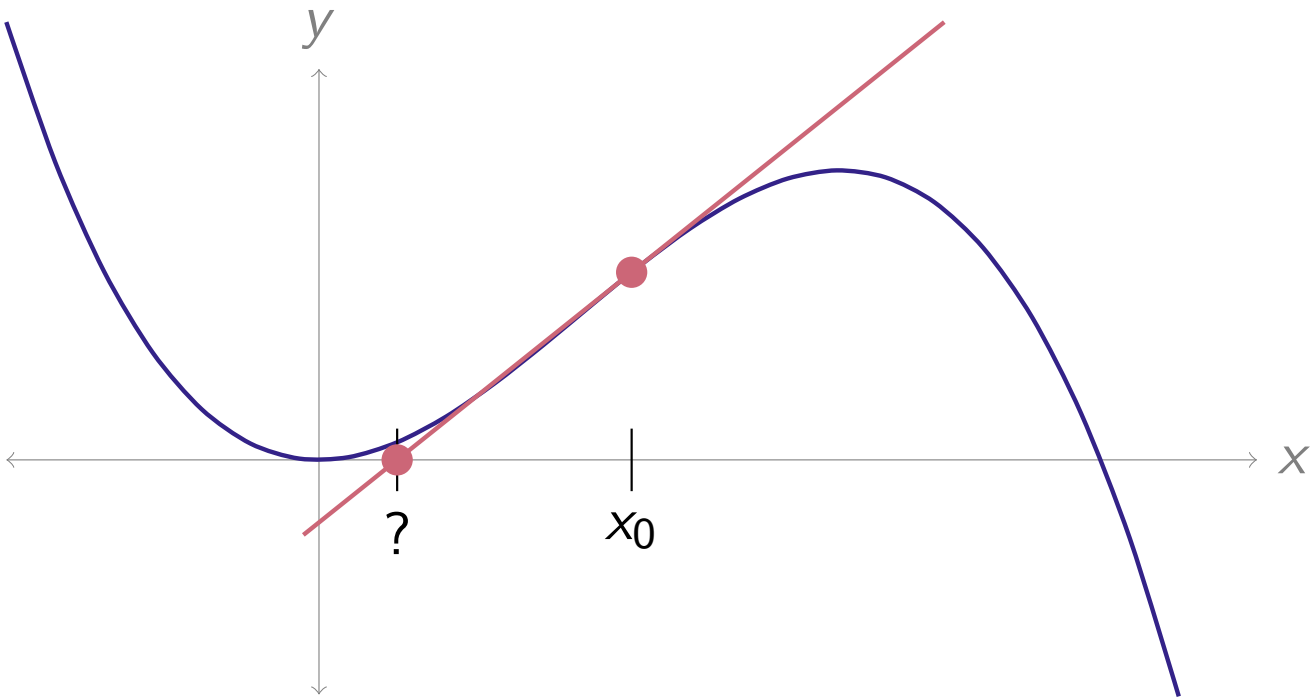
Our functon $f(x)$ generally has the form of a Hill function (even though its exponent, $n = \frac{1}{2}$, is not an integer).



The tangent lines seem to be above the function, suggesting our approximation is an **overestimation**–that is, our guess is higher than the actual value.

Newton's Method: Intro

Suppose you take the tangent line at $x = x_0$ to a differentiable function $f(x)$. Where does that tangent line hit the x-axis?



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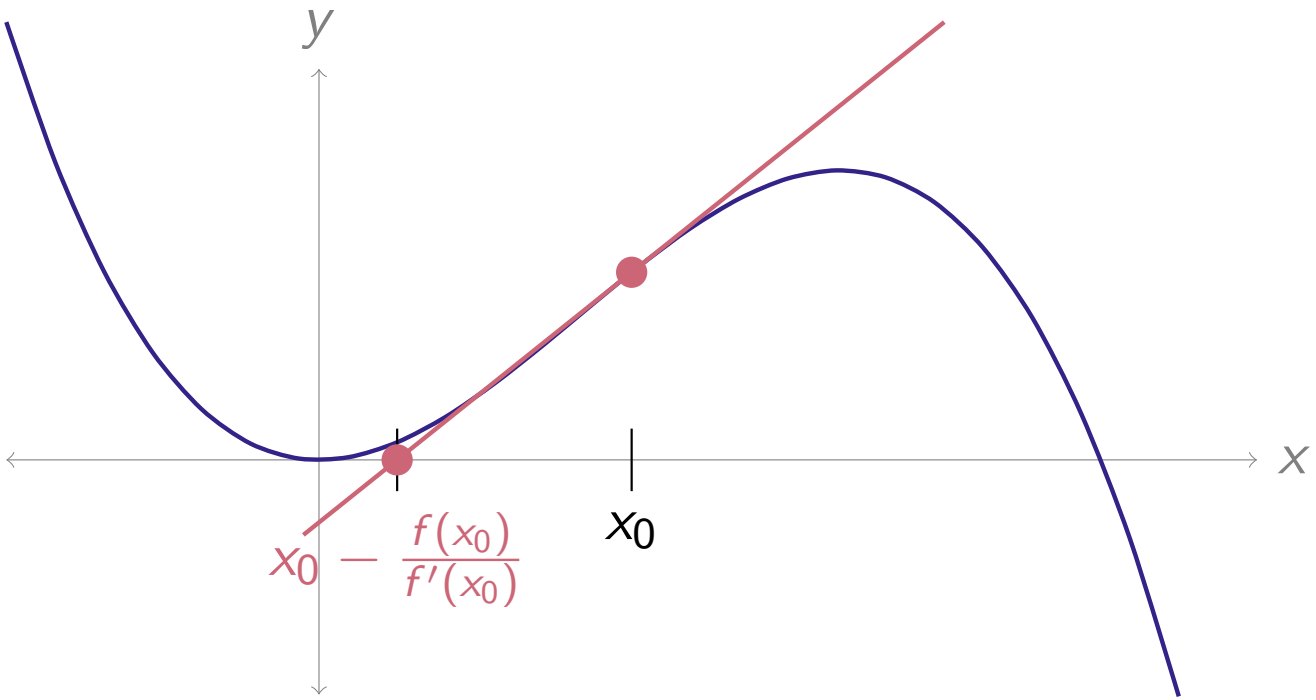

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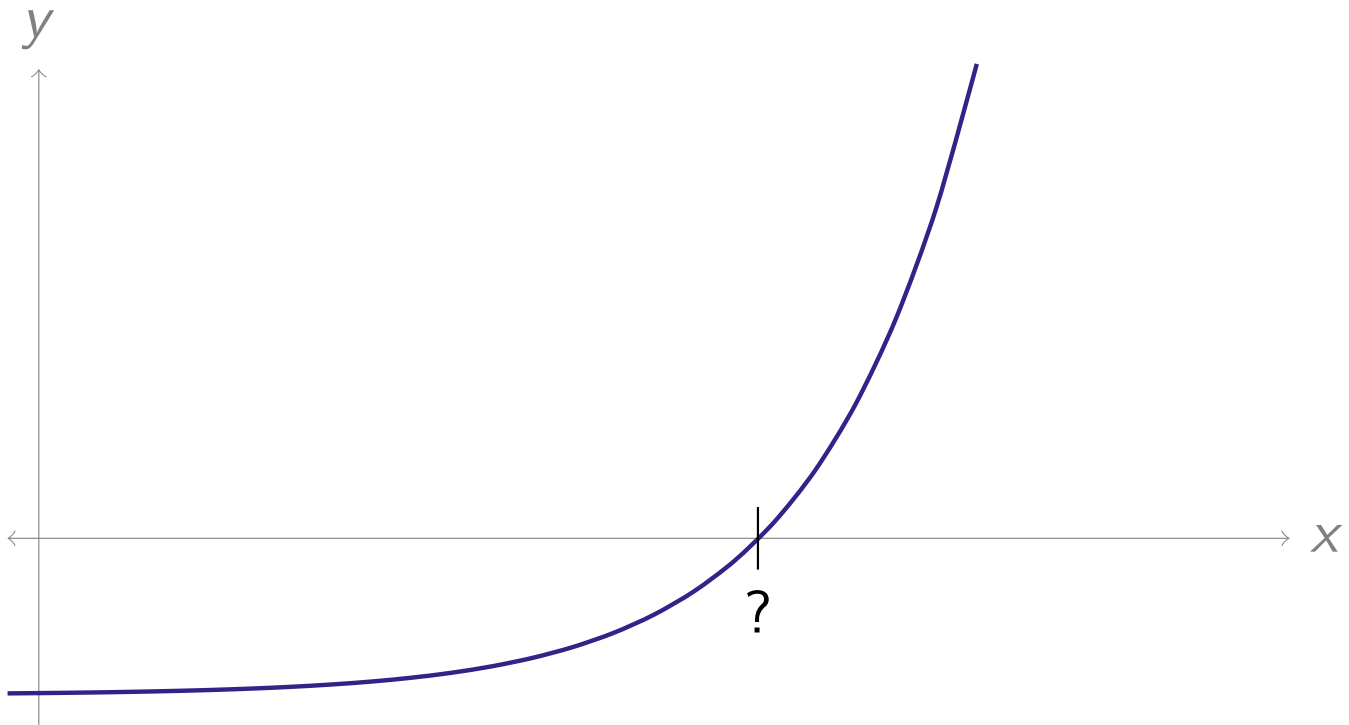
Newton's Method: Finding Roots

Newton's Method

Given an approximation x_k for the root of the equation $f(x) = 0$, we can improve the accuracy of that approximation with another iteration using

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$



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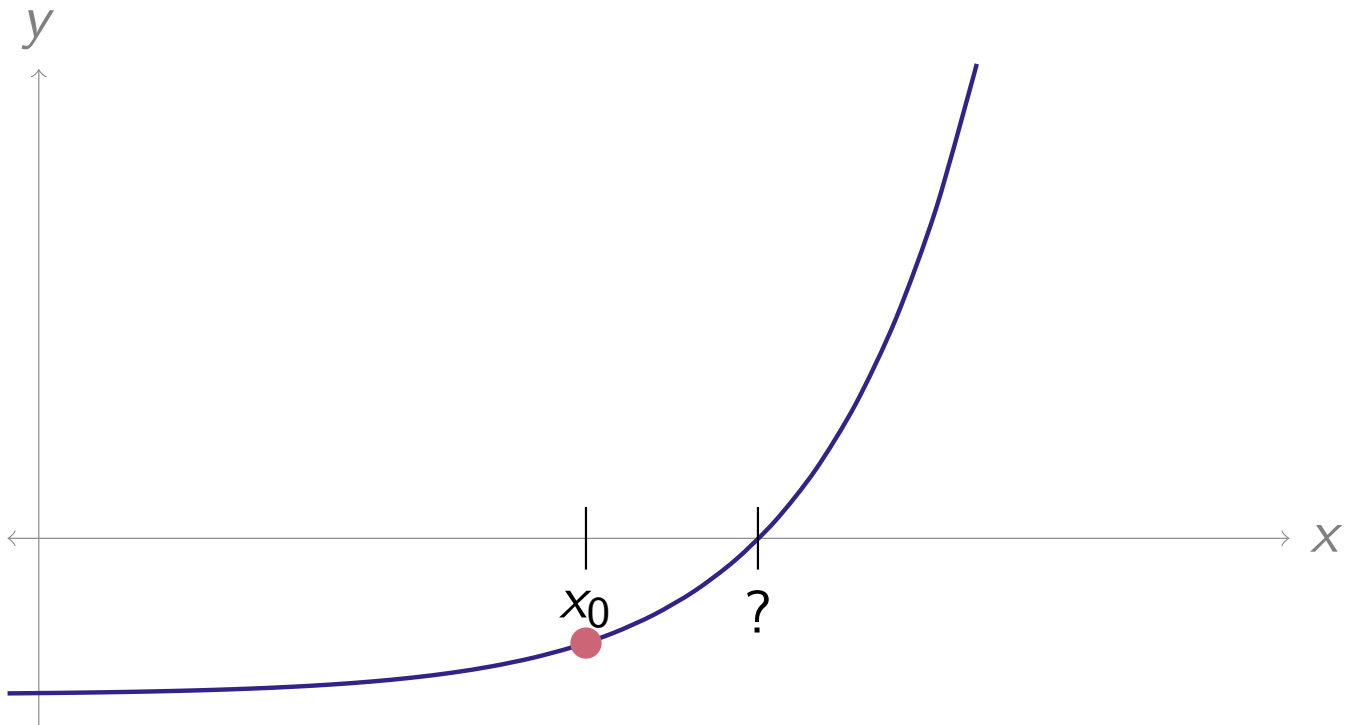

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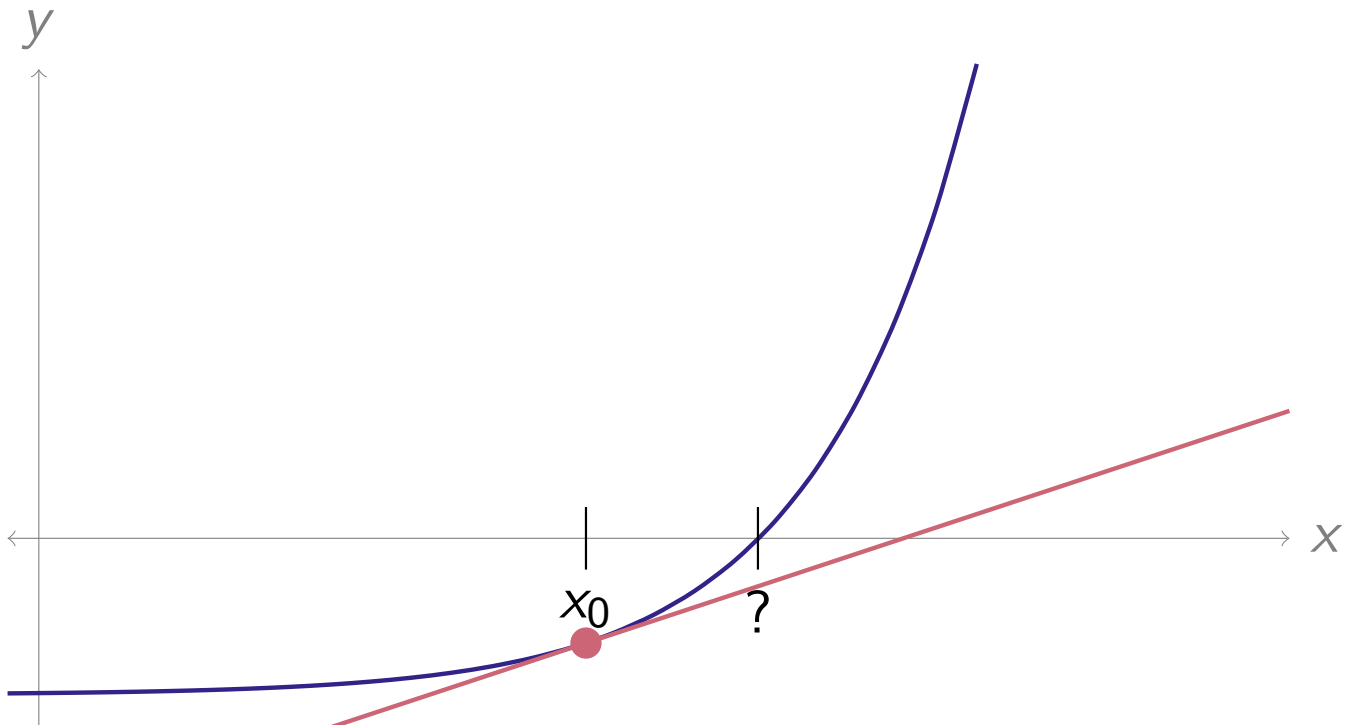

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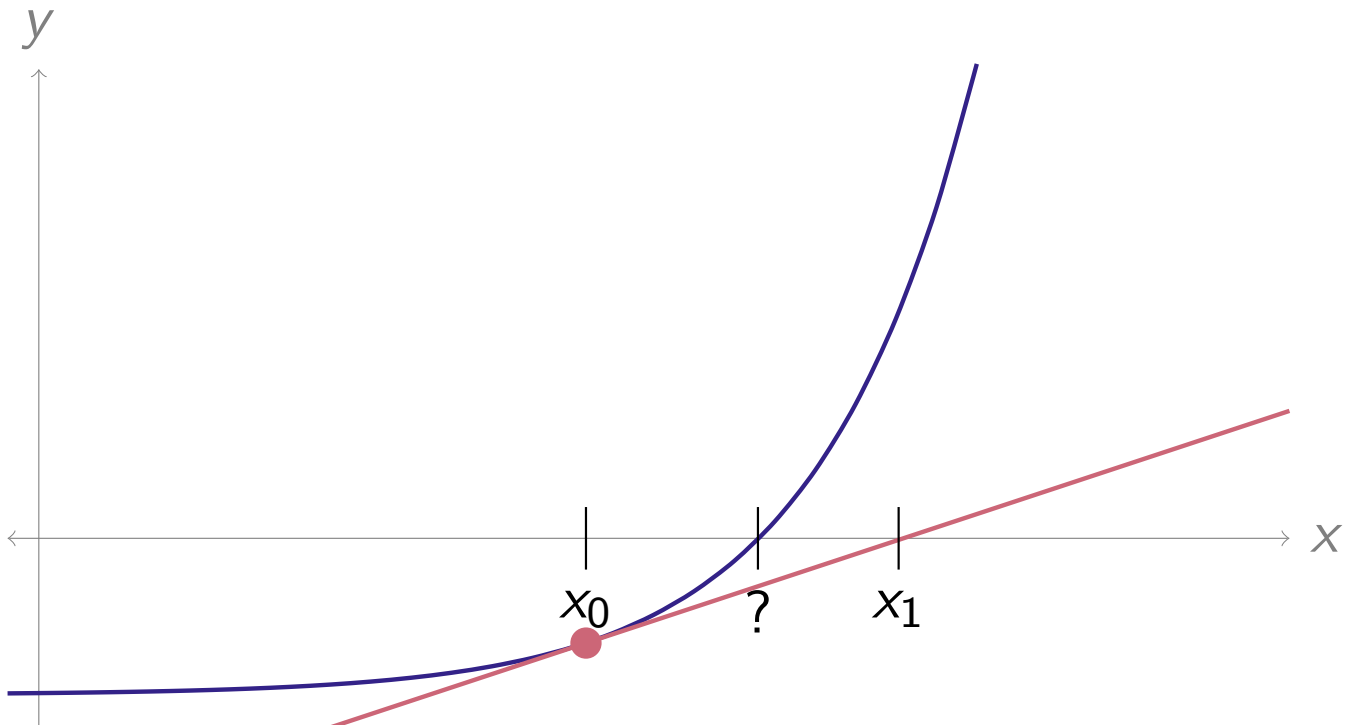

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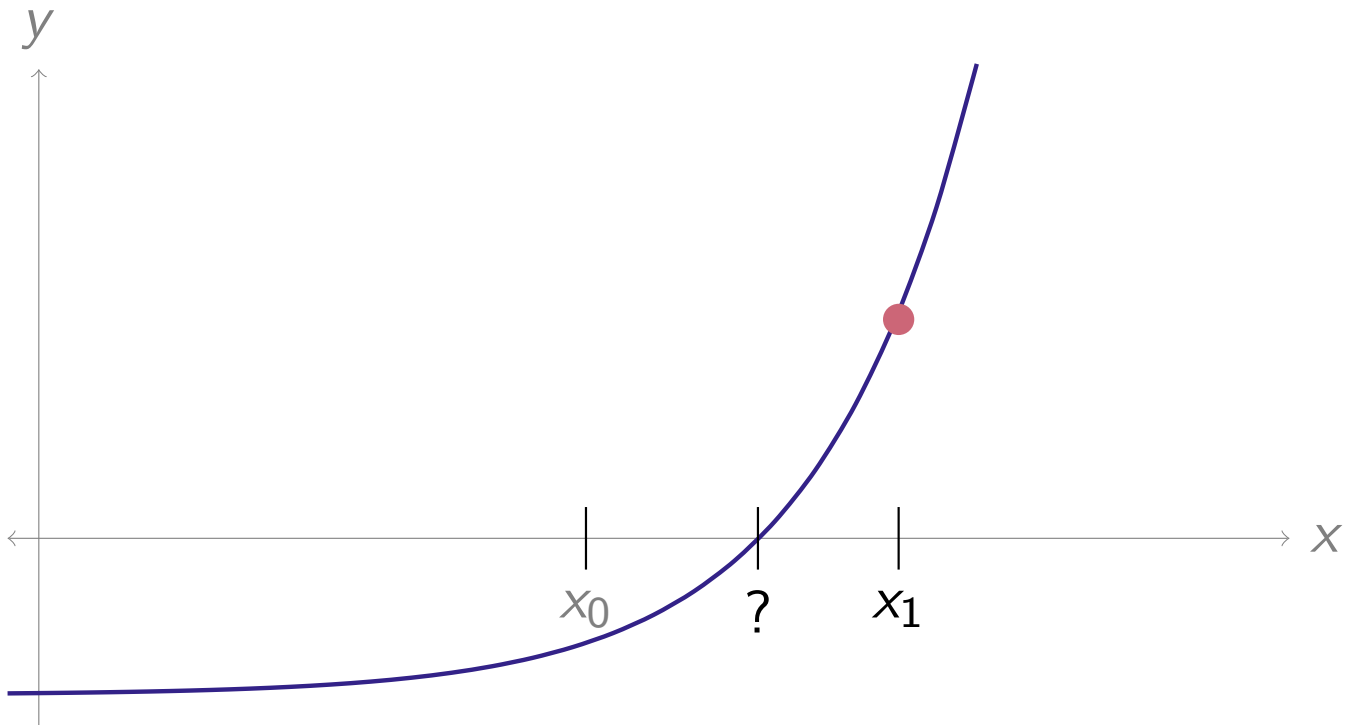

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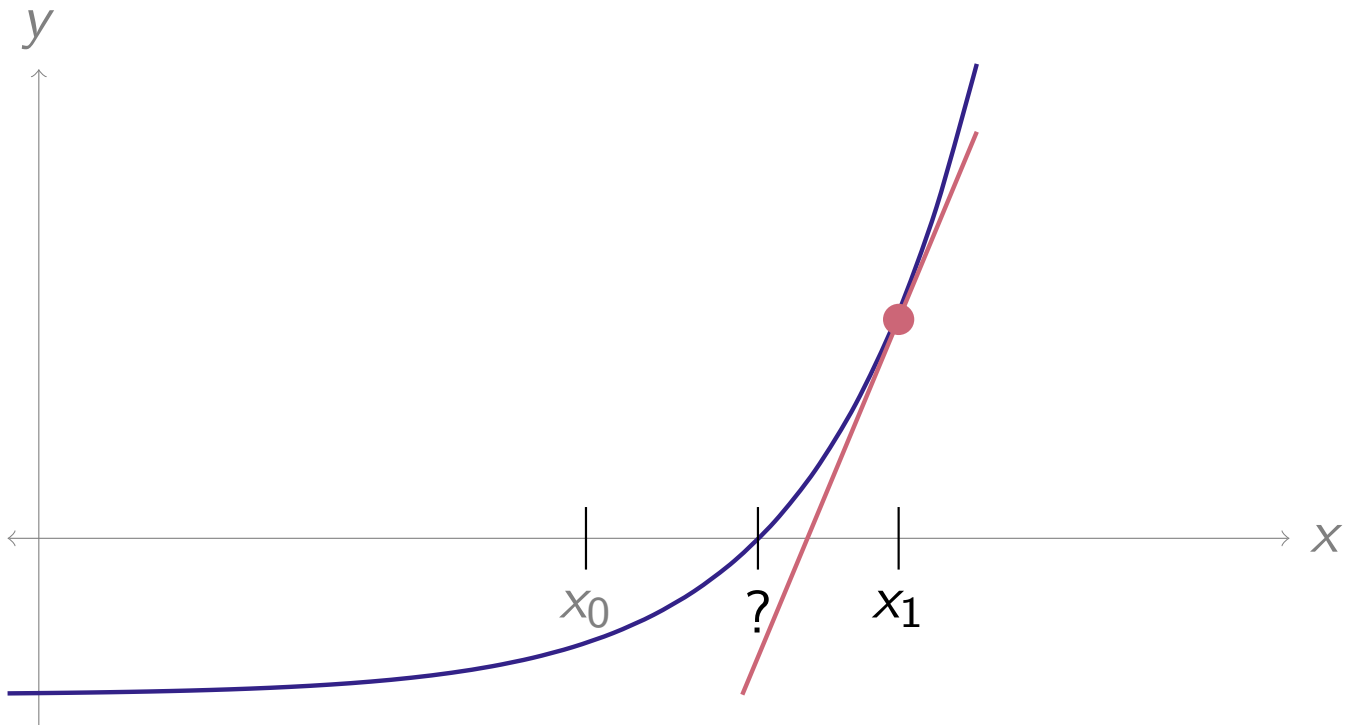

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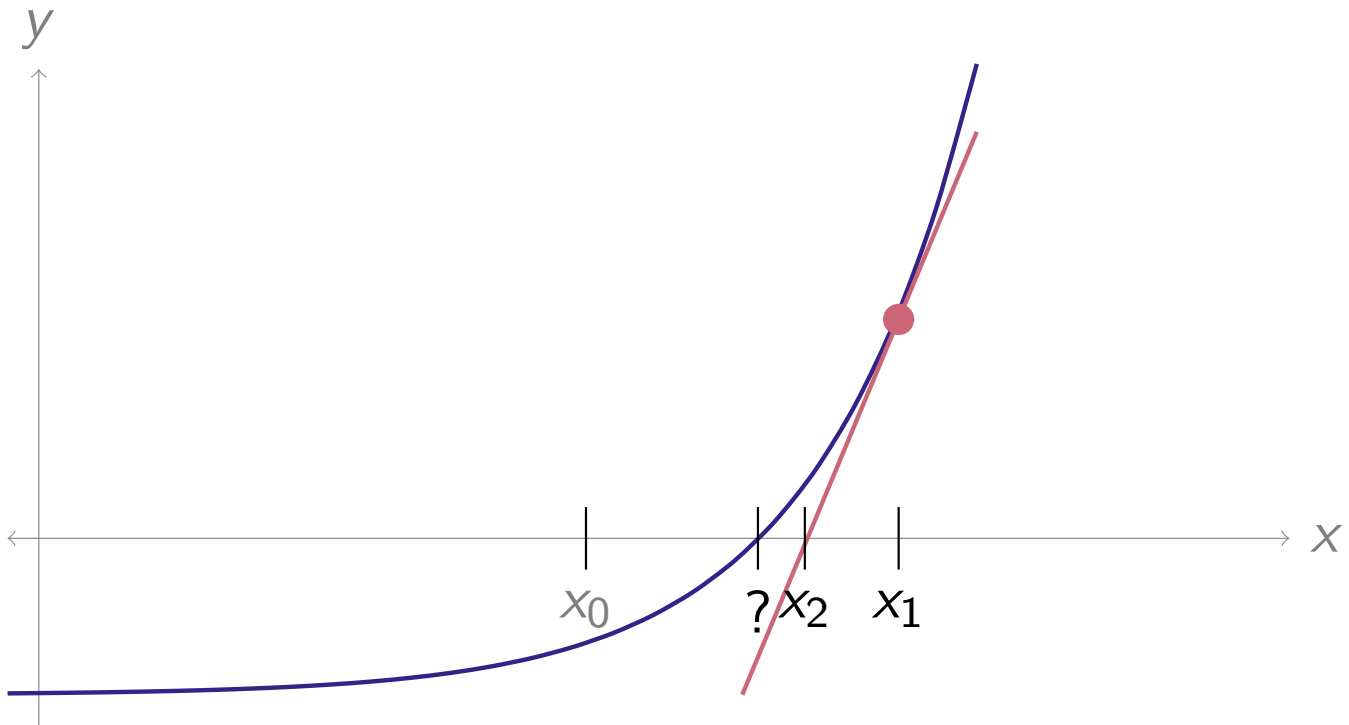

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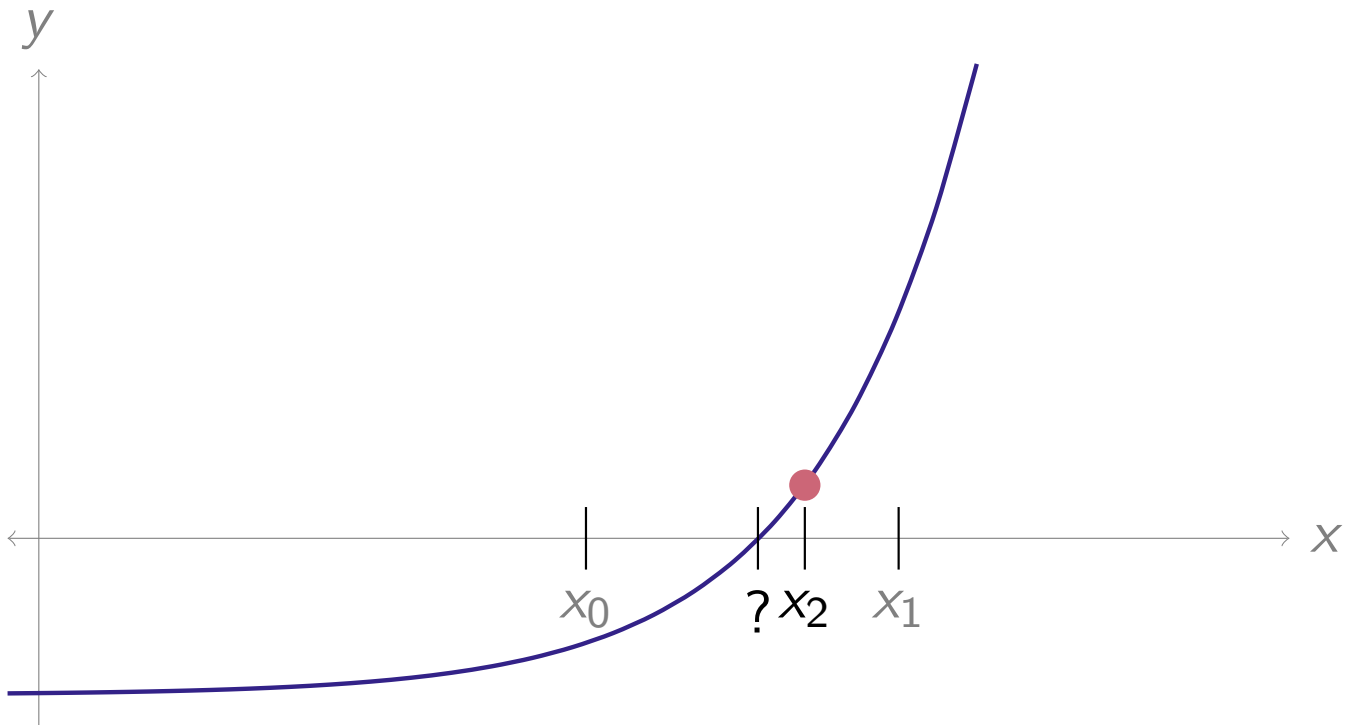

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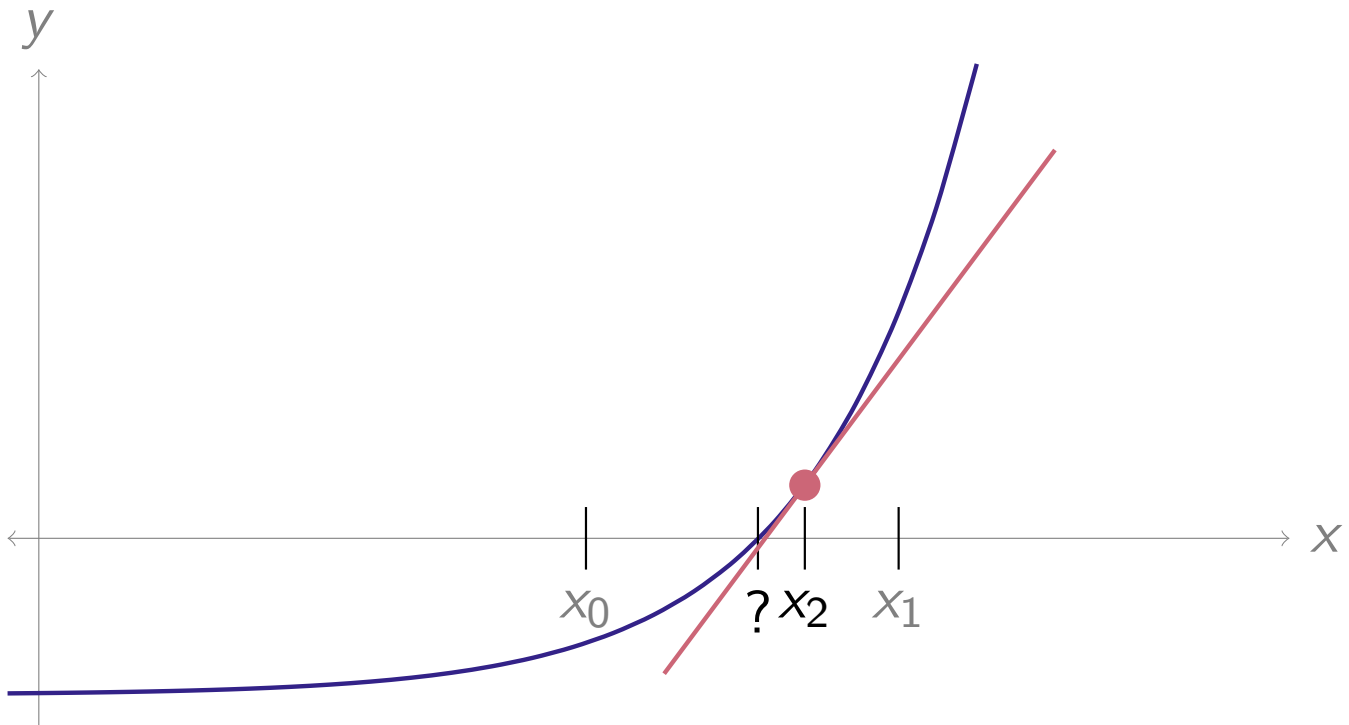

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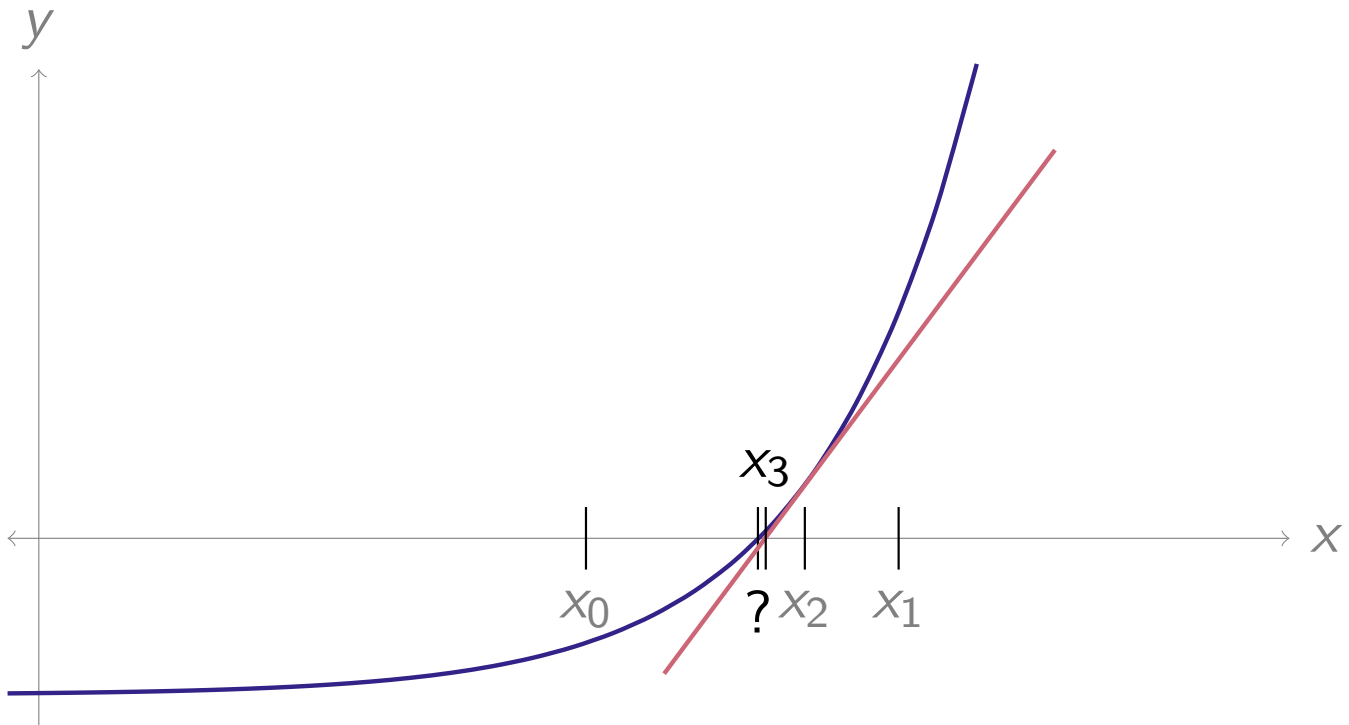

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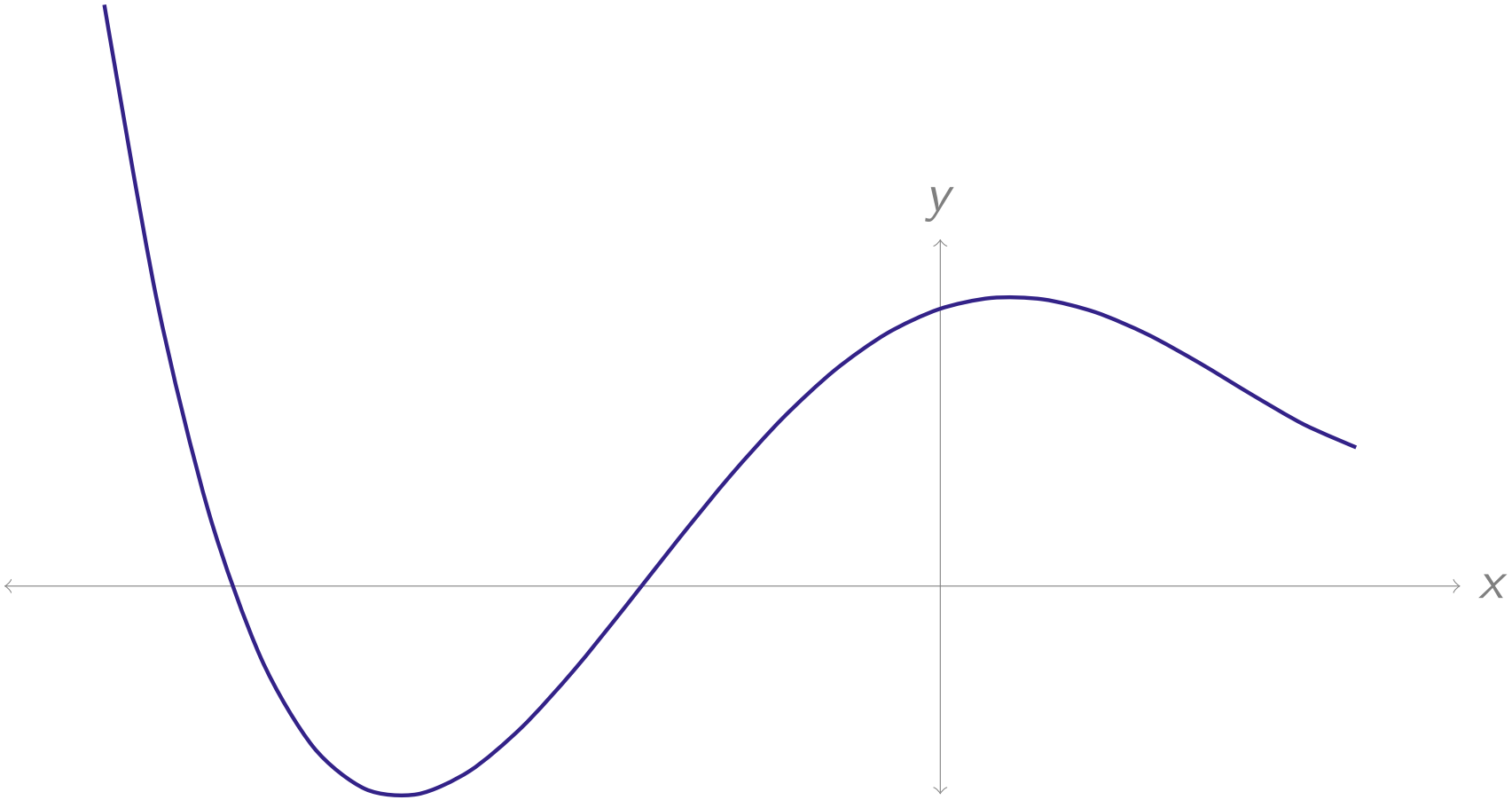
Example 6:

Sketch the locations of x_1 and x_2 that arise from Newton's method on the function shown below if $x_0 = 0$.



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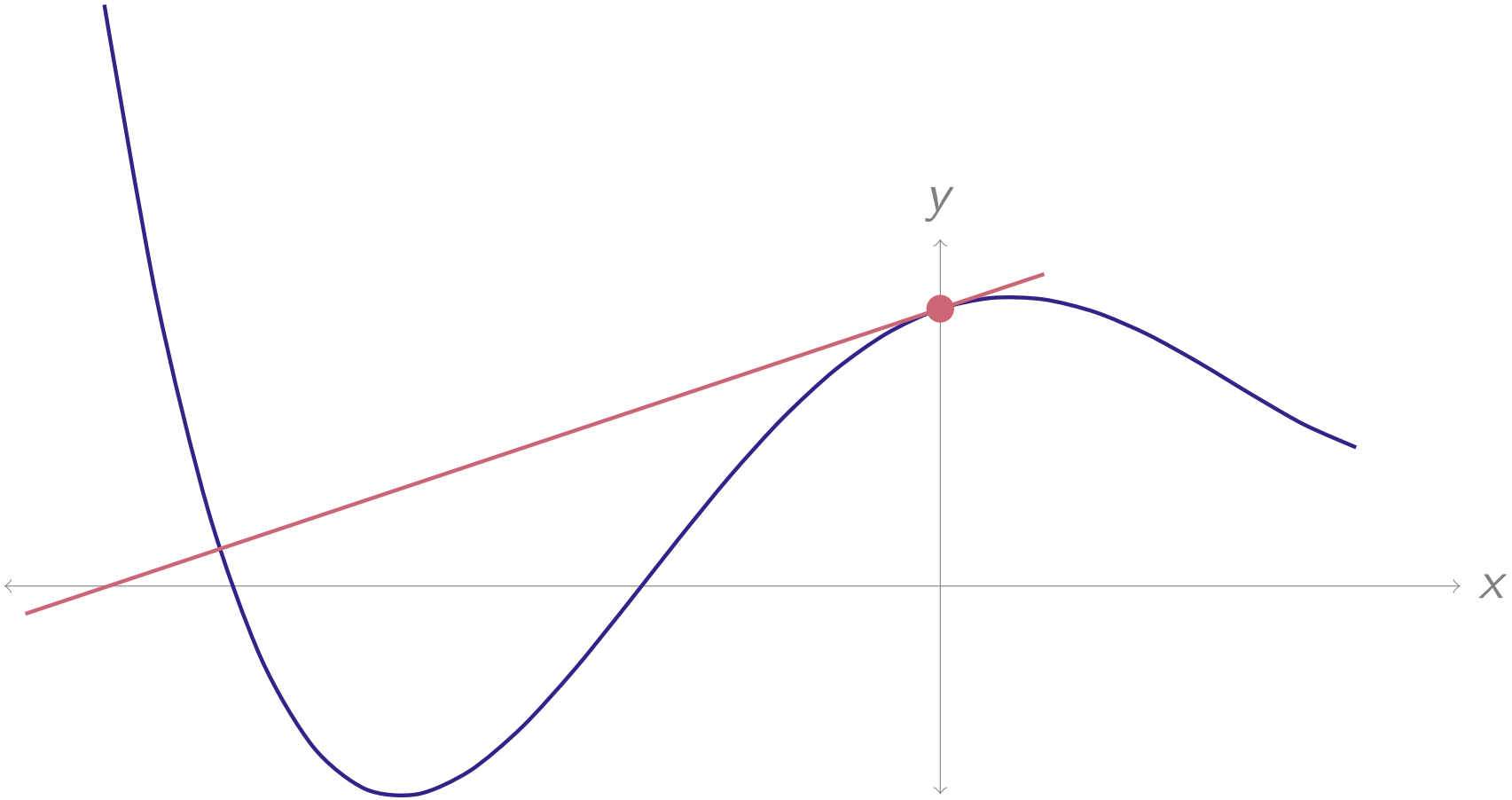

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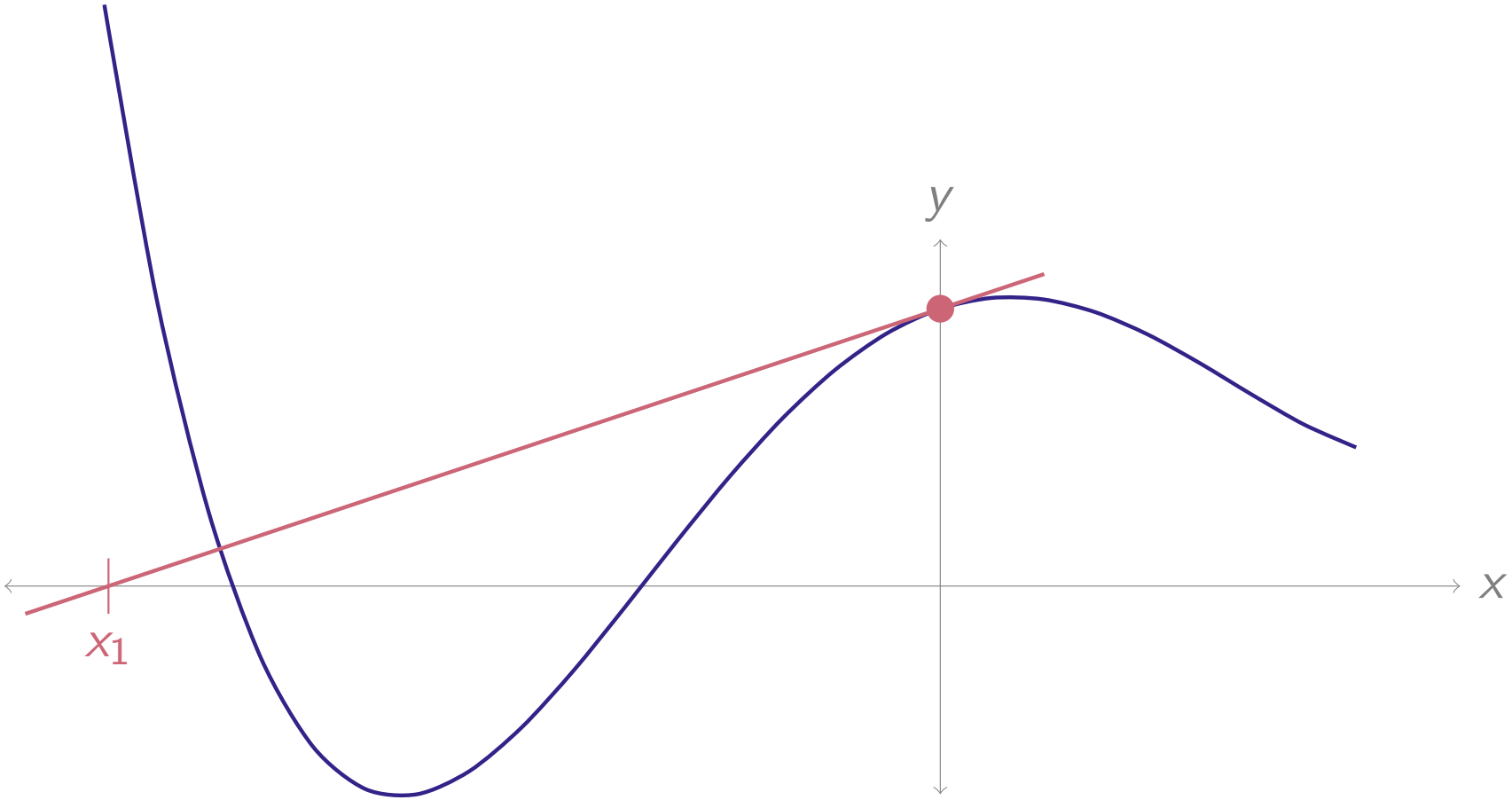

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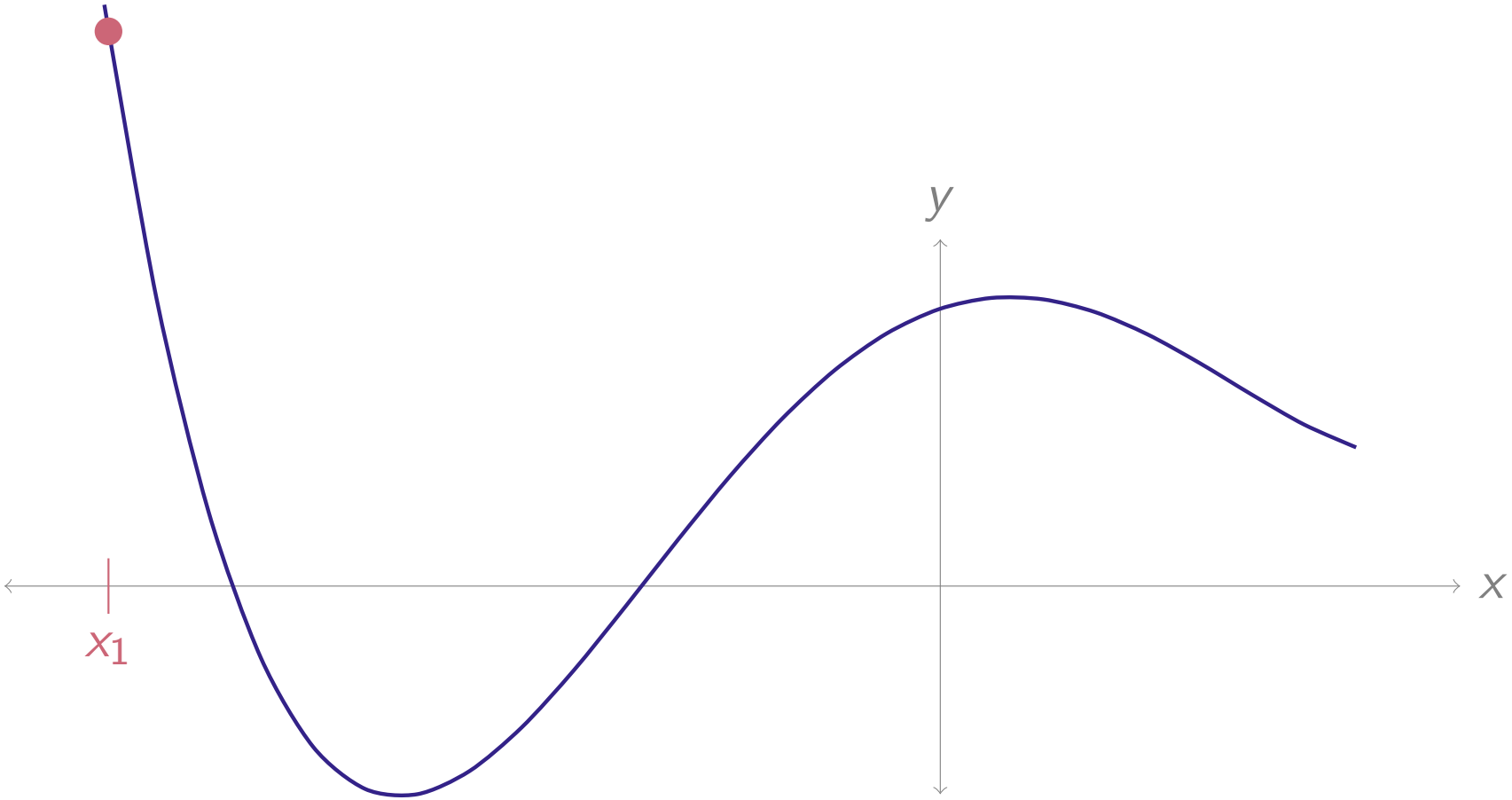

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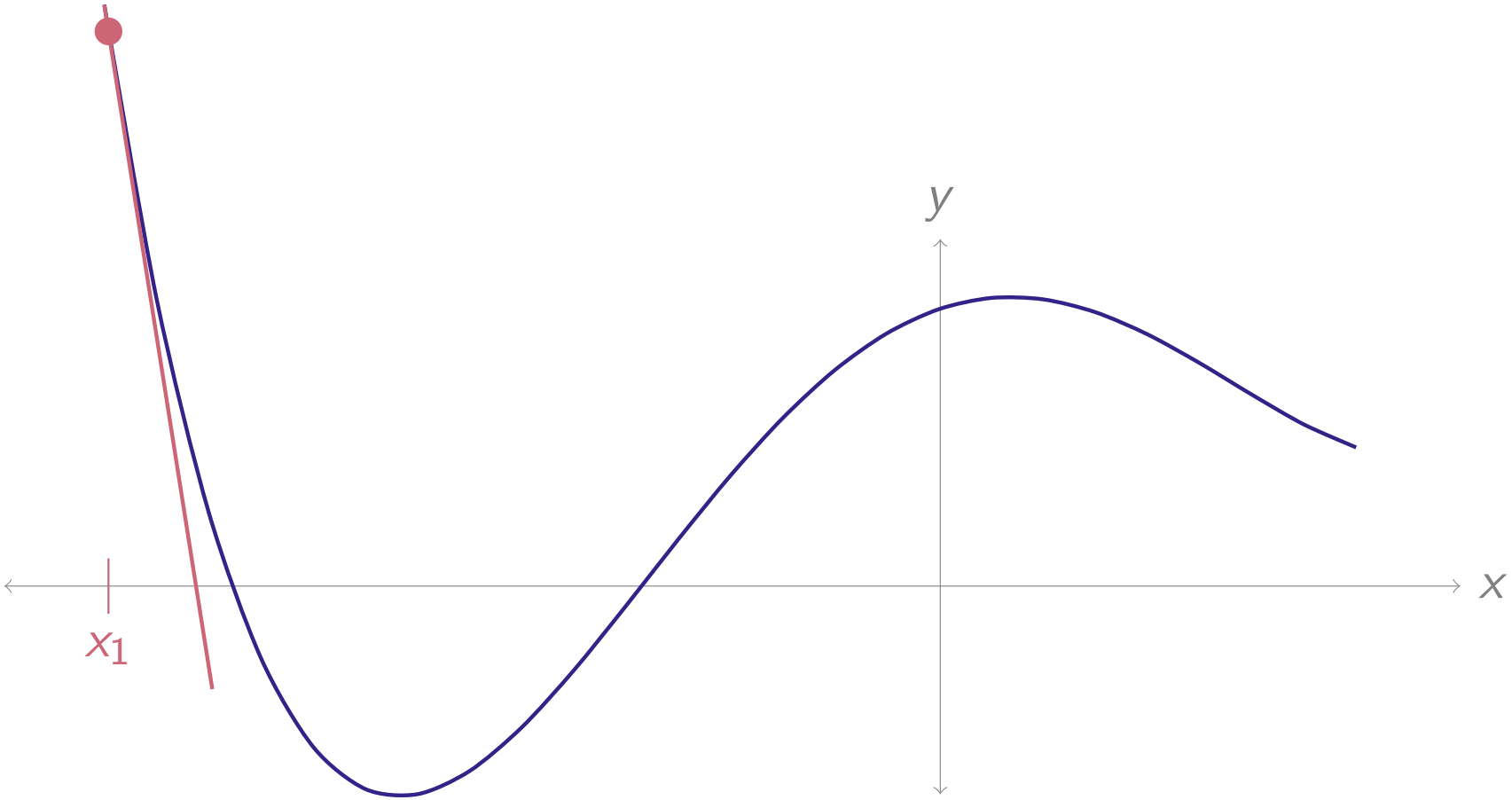

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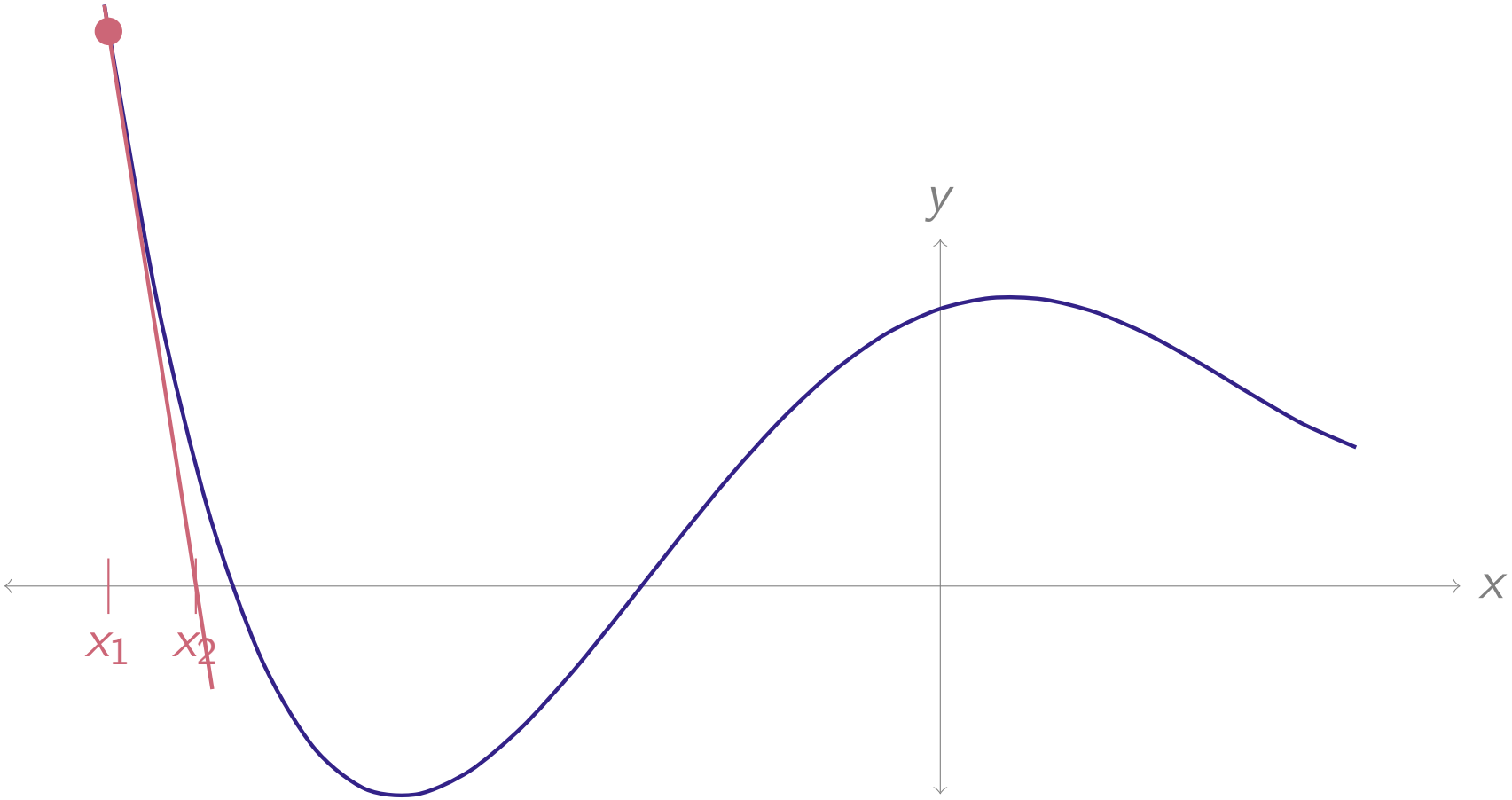

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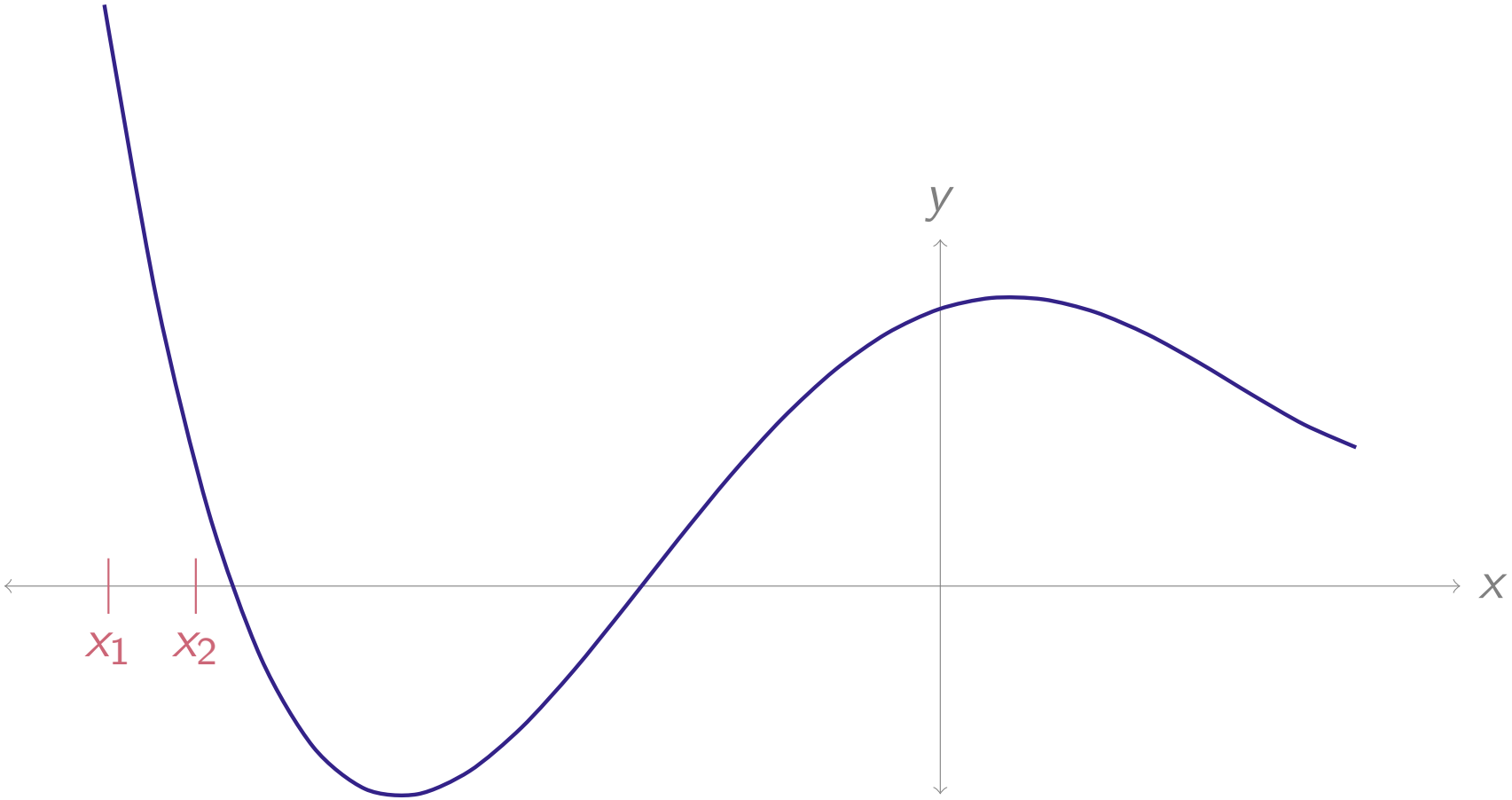

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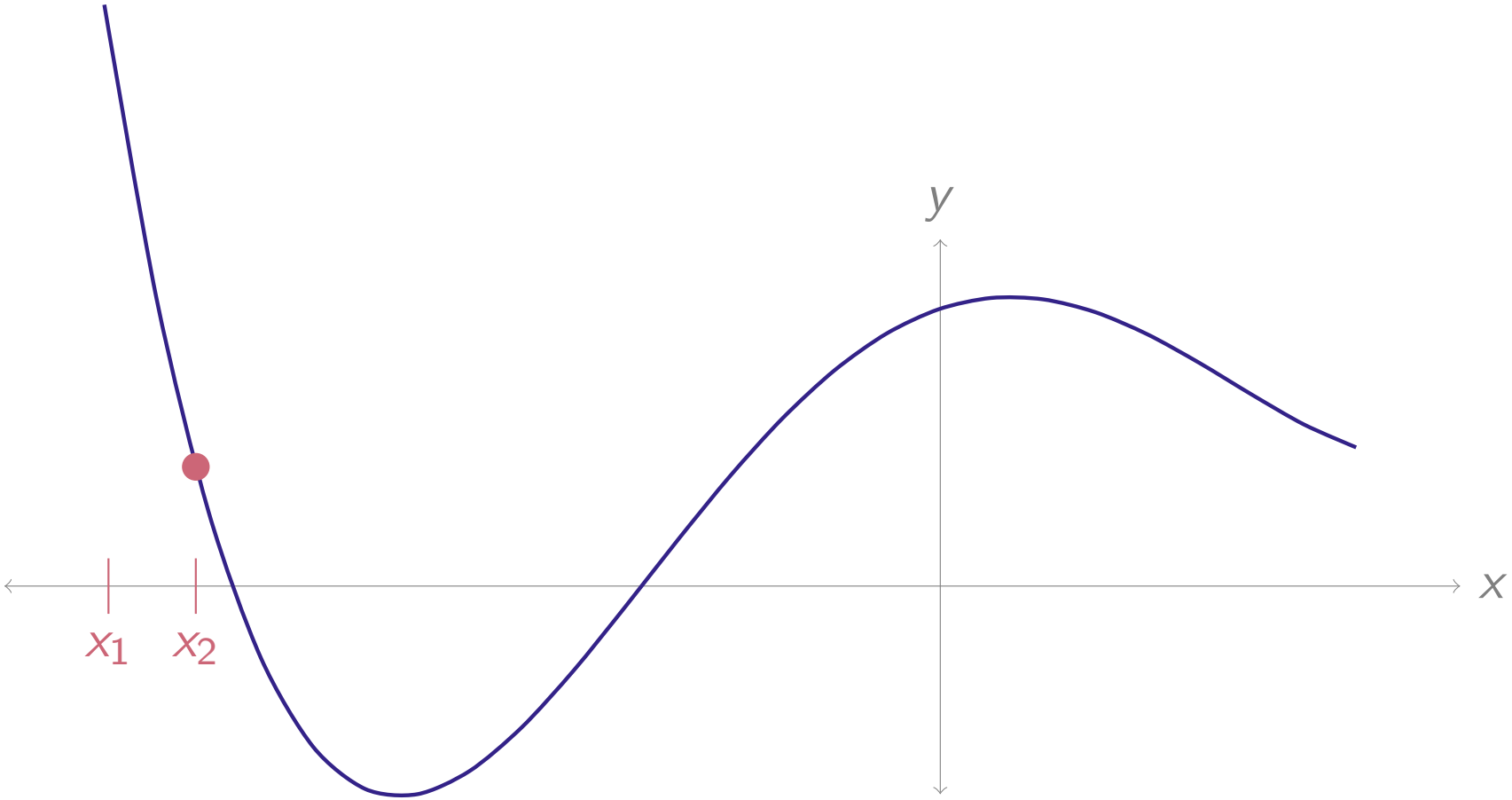

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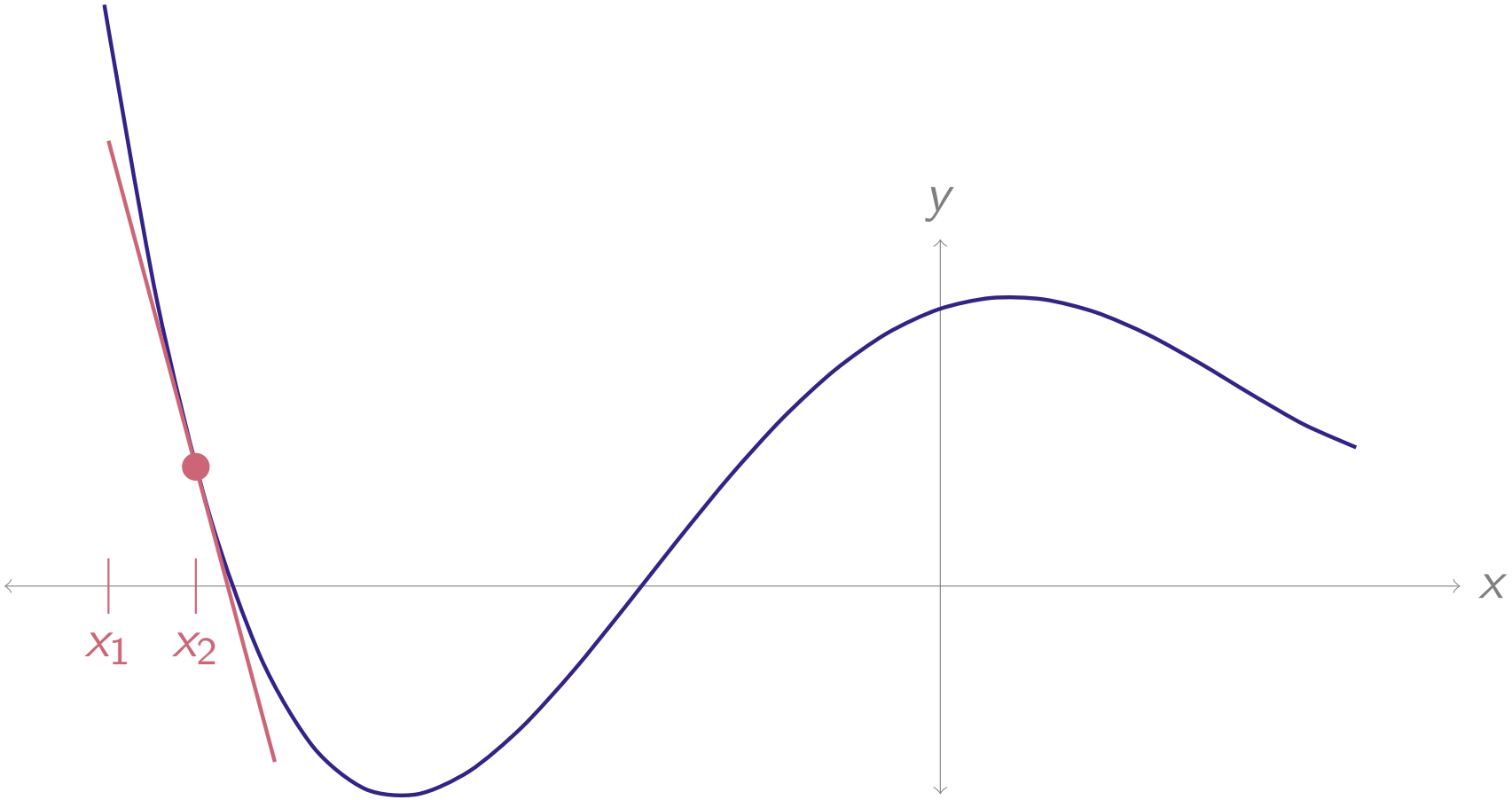

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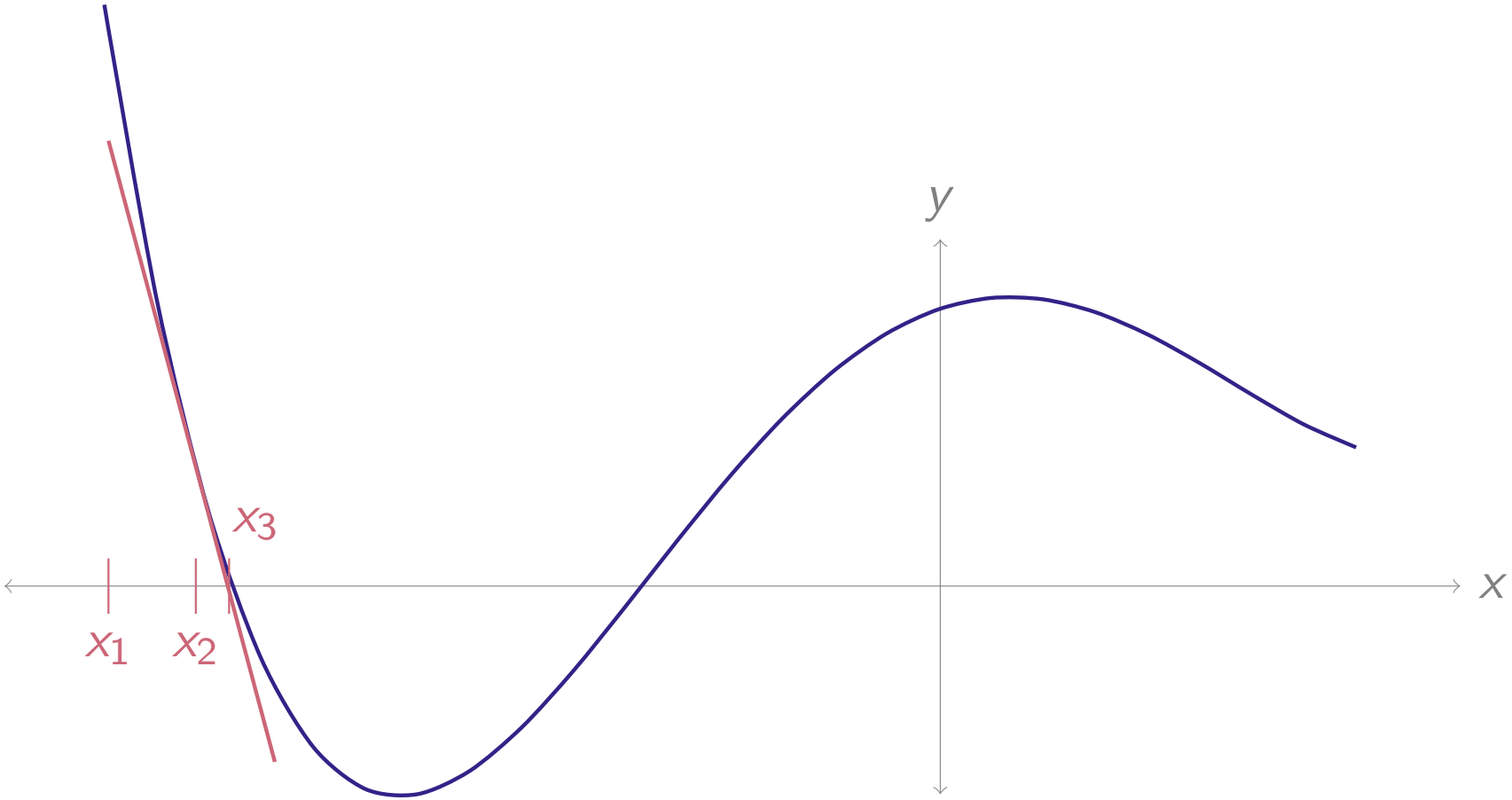


Newton's Method: Finding Roots

Newton's Method

Given an approximation x_k for the root of the equation $f(x) = 0$, we can improve the accuracy of that approximation with another iteration using

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

Example 7:

Use three iterations of Newton's Method to approximate a root of

$$f(x) = x^3 - 2x + 1.$$

with $x_0 = 0$.

Newton's Method: Finding Roots

Solution 7:

Use three iterations of Newton's Method to approximate a root of $f(x) = x^3 - 2x + 1$ with $x_0 = 0$.

$$f'(x) = 3x^2 - 2$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{f(0)}{f'(0)} = 0 - \frac{1}{-2} = \frac{1}{2}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{1}{2} - \frac{f(1/2)}{f'(1/2)} = \frac{1}{2} - \frac{1/8}{-5/4} = \frac{3}{5} = 0.6$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \frac{3}{5} - \frac{f(3/5)}{f'(3/5)} = \frac{3}{5} - \frac{2/125}{-23/25} = \frac{71}{115} \approx 0.617$$

$f(x) = (x-1)(x^2 + x - 1)$; one root: $x = \frac{-1+\sqrt{5}}{2} \approx 0.618$

Newton's Method: Finding Roots

Newton's Method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

Example 8:

Use Newton's method to approximate a root of

$$f(x) = x^5 + x - 1.$$

Newton's Method: Finding Roots

Solution 8:

Use Newton's method to approximate a root of

$$f(x) = x^5 + x - 1.$$

Choosing x_0, and when to terminate, is up to you. We'll choose $x_0 = 0$, but we didn't have to.

$$f'(x) = 5x^4 + 1$$

$$x_1 = 0 - \frac{f(0)}{f'(0)} = 0 - \frac{-1}{1} = 1$$

$$x_2 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1}{6} = \frac{5}{6} = 0.8\overline{33}$$

$$x_3 = \frac{5}{6} - \frac{f(5/6)}{f'(5/6)} = \frac{5}{6} - \frac{(5/6)^5 + (5/6) - 1}{5(5/6)^4 + 1} \approx 0.764$$

Spreadsheet link

Newton's Method: Finding Roots

Example 9:

(a) Use Newton's Method to approximate $\sqrt{50}$.

(b) Use a linear approximation to approximate $\sqrt{50}$.

Newton's Method: Finding Roots

Example 9:

(a) Use Newton's Method to approximate $\sqrt{50}$.

(b) Use a linear approximation to approximate $\sqrt{50}$.

(a) Let $f(x) = x^2 - 50$, so $f'(x) = 2x$, and let $x_0 = 7$.

$$x_1 = 7 - \frac{-1}{14} = 7 + \frac{1}{14} = \frac{99}{14}$$

$$x_2 = \frac{99}{14} - \frac{(99/14)^2 - 50}{99/7} = \frac{19601}{2772}$$

Remark: $\left(7 + \frac{1}{14}\right)^2 \approx 50.005$

Newton's Method: Finding Roots

Example 9:

(a) Use Newton's Method to approximate $\sqrt{50}$.

(b) Use a linear approximation to approximate $\sqrt{50}$.

(b)

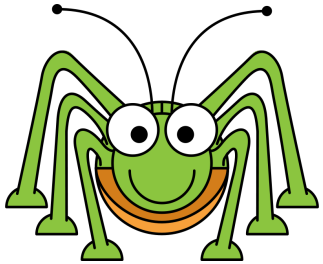
Let $g(x) = \sqrt{x}$, so $g'(x) = \frac{1}{2\sqrt{x}}$, and use the point $(49, 7)$ as an "anchor":

$$g(x) \approx g'(49)(x - 49) + g(49) = \frac{1}{14}(x - 49) + 7$$

$$g(50) \approx \frac{1}{14}(50 - 49) + 7 = 7 + \frac{1}{14}$$

Remark: $\left(7 + \frac{1}{14}\right)^2 \approx 50.005$ $\qquad$ $\left(\frac{19601}{2772}\right)^2 \approx 50.0000001$

Ladybugs and Aphids



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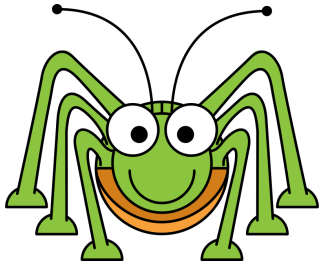

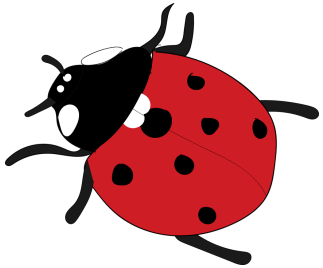

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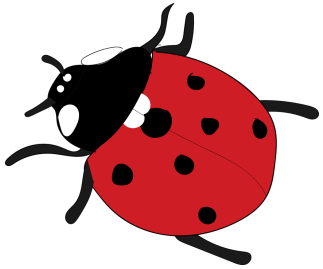

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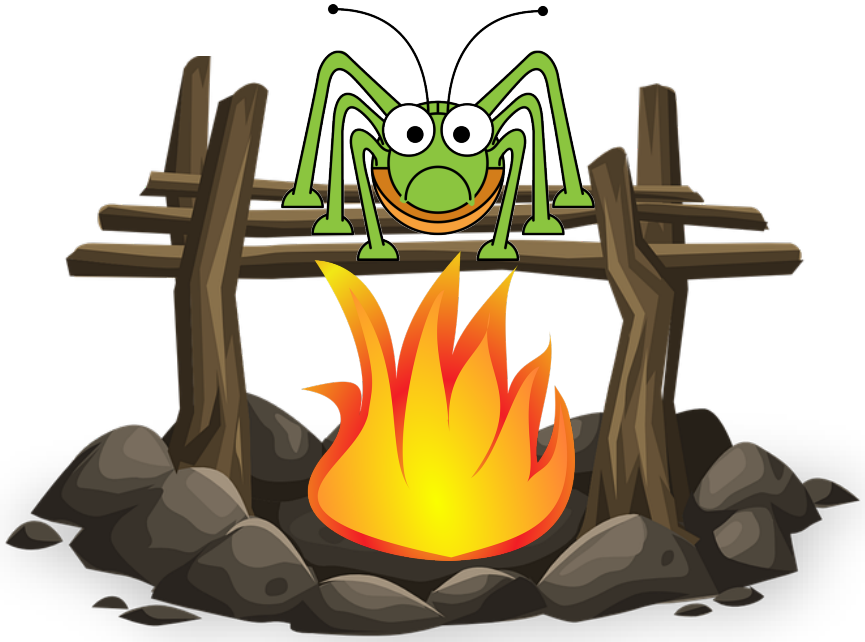

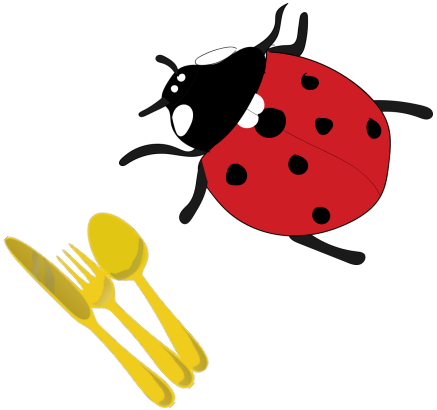

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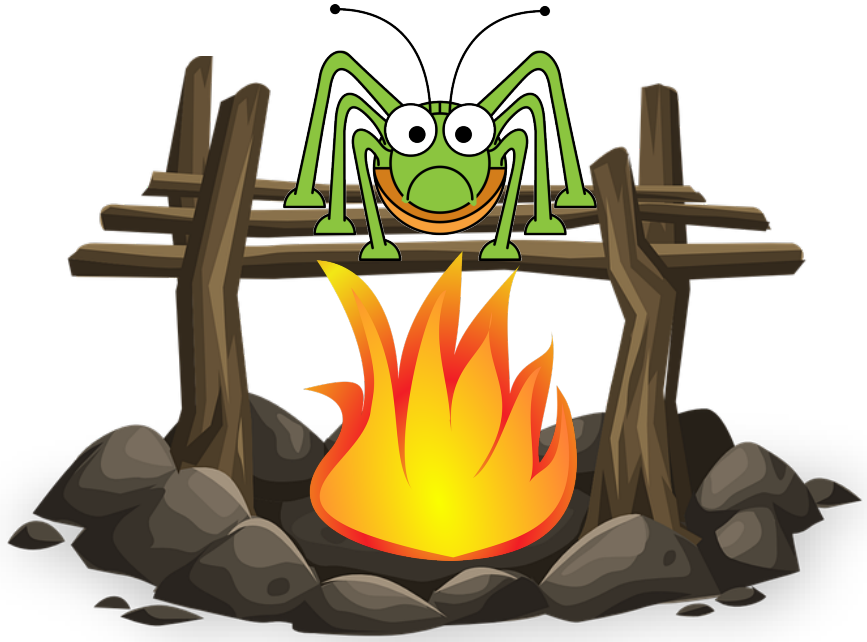
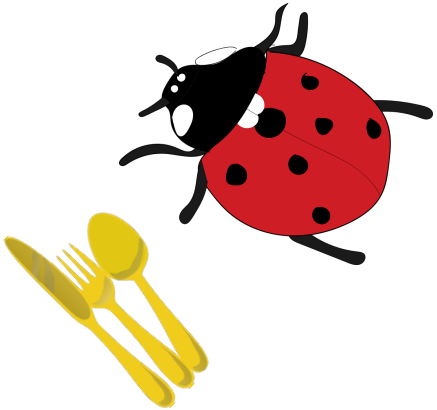


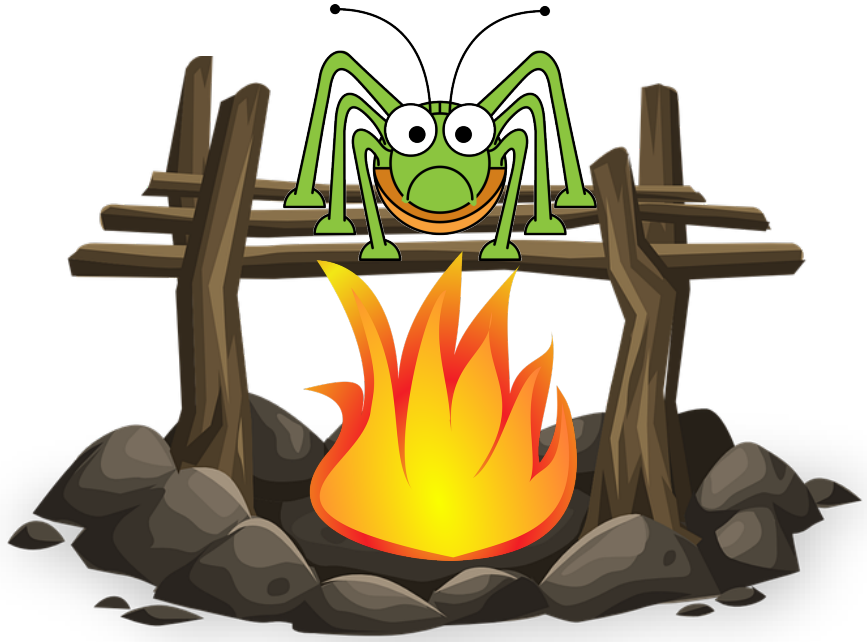


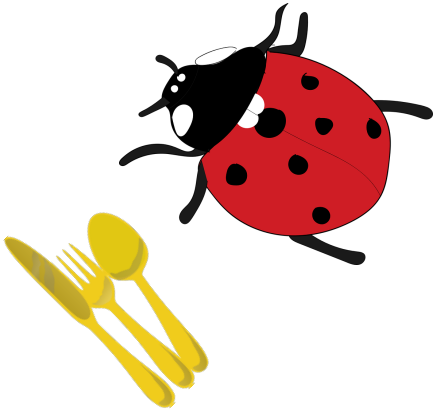
Size of aphid population: x

Ladybug predation rate: $P(x) = \dfrac{30x^3}{20^3 + x^3}$

Aphid population growth rate: $G(x) = 0.5x$

Ladybugs and Aphids





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$$60x^2 = 20^3 + x^3$$

Ladybugs and Aphids

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$$\frac{60x^2}{20^3 + x^3} = 1$$

$$60x^2 = 20^3 + x^3$$

$$x^3 - 60x^2 + 20^3 = 0$$

Ladybugs and Aphids

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Aphid population growth rate: $G(x) = 0.5x$

$P(x) = G(x)$

$$\frac{30x^3}{20^3 + x^3} = 0.5x$$
$$\frac{60x^2}{20^3 + x^3} = 1$$
$$60x^2 = 20^3 + x^3$$
$$x^3 - 60x^2 + 20^3 = 0$$

Hard to solve explicitly– use Newton's Method

Ladybugs and Aphids

$$x^3 - 60x^2 + 20^3 = 0$$

$$f(x) = x^3 - 60x^2 + 20^3$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

Ladybugs and Aphids

$$x^3 - 60x^2 + 20^3 = 0$$

$$f(x) = x^3 - 60x^2 + 20^3$$
$$f'(x) = 3x^2 - 120x$$
$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

Spreadsheet (link)

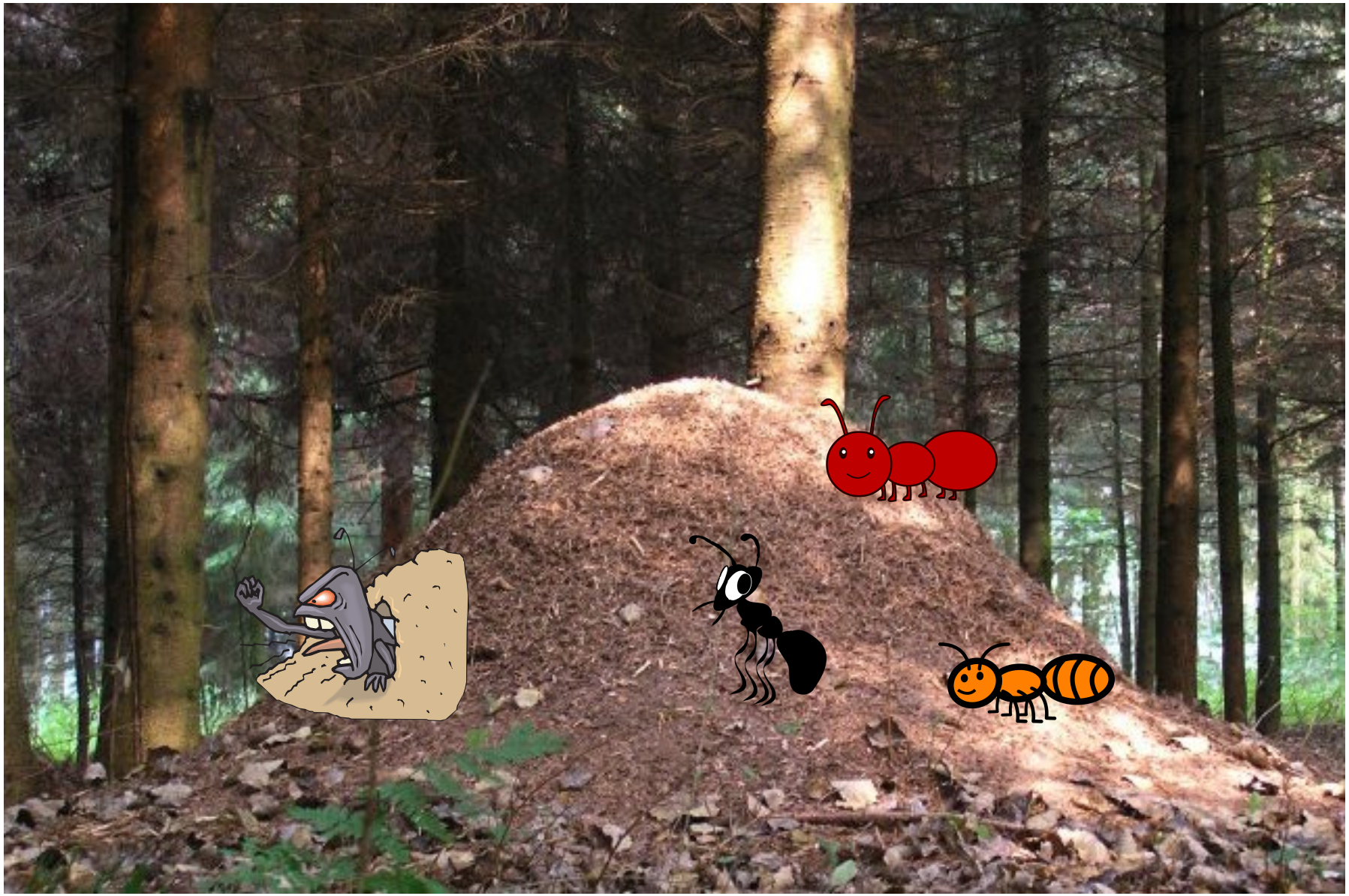
Anthill problem

Anthill problem

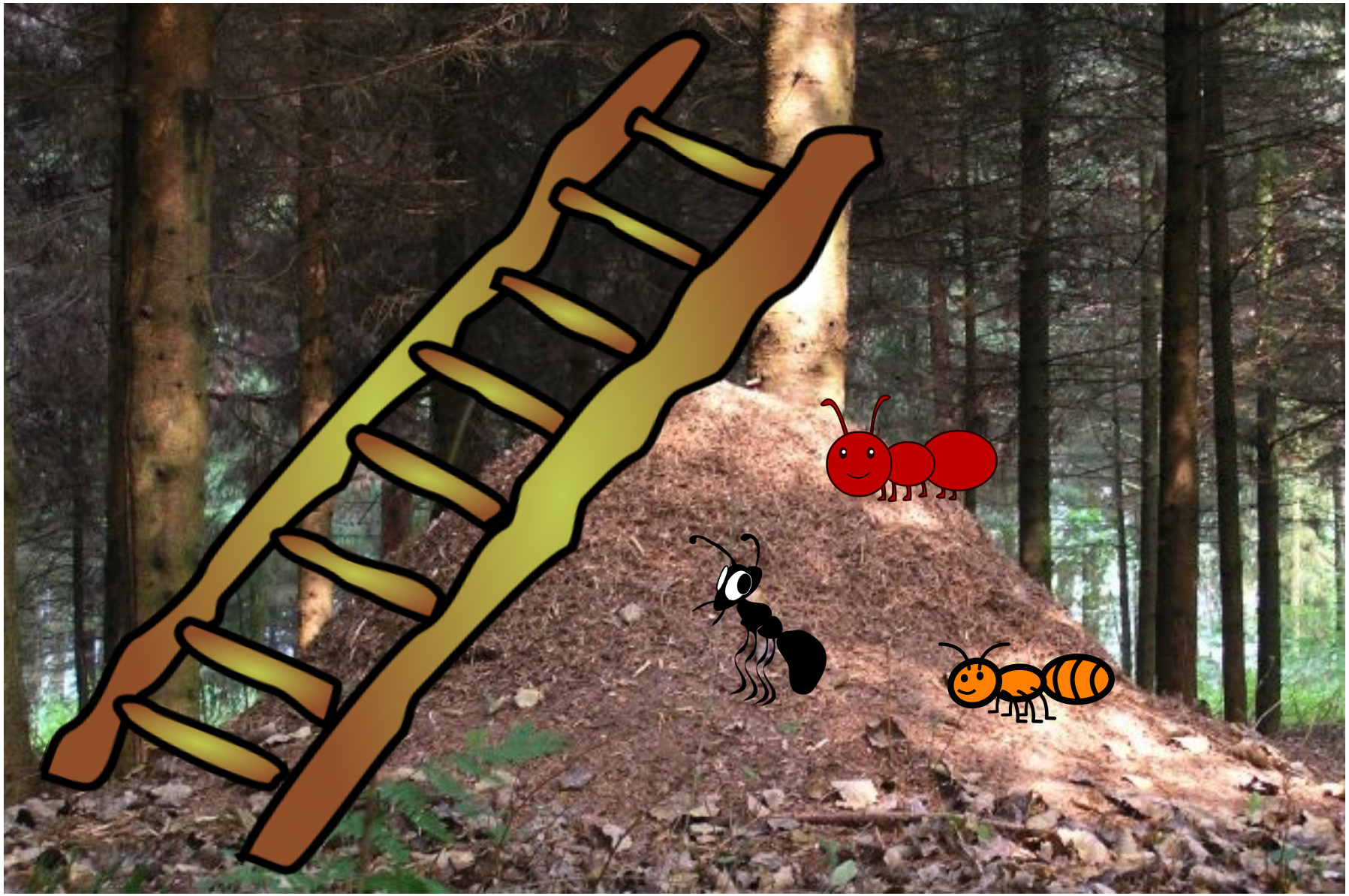
Anthill problem

Anthill problem

Anthill problem



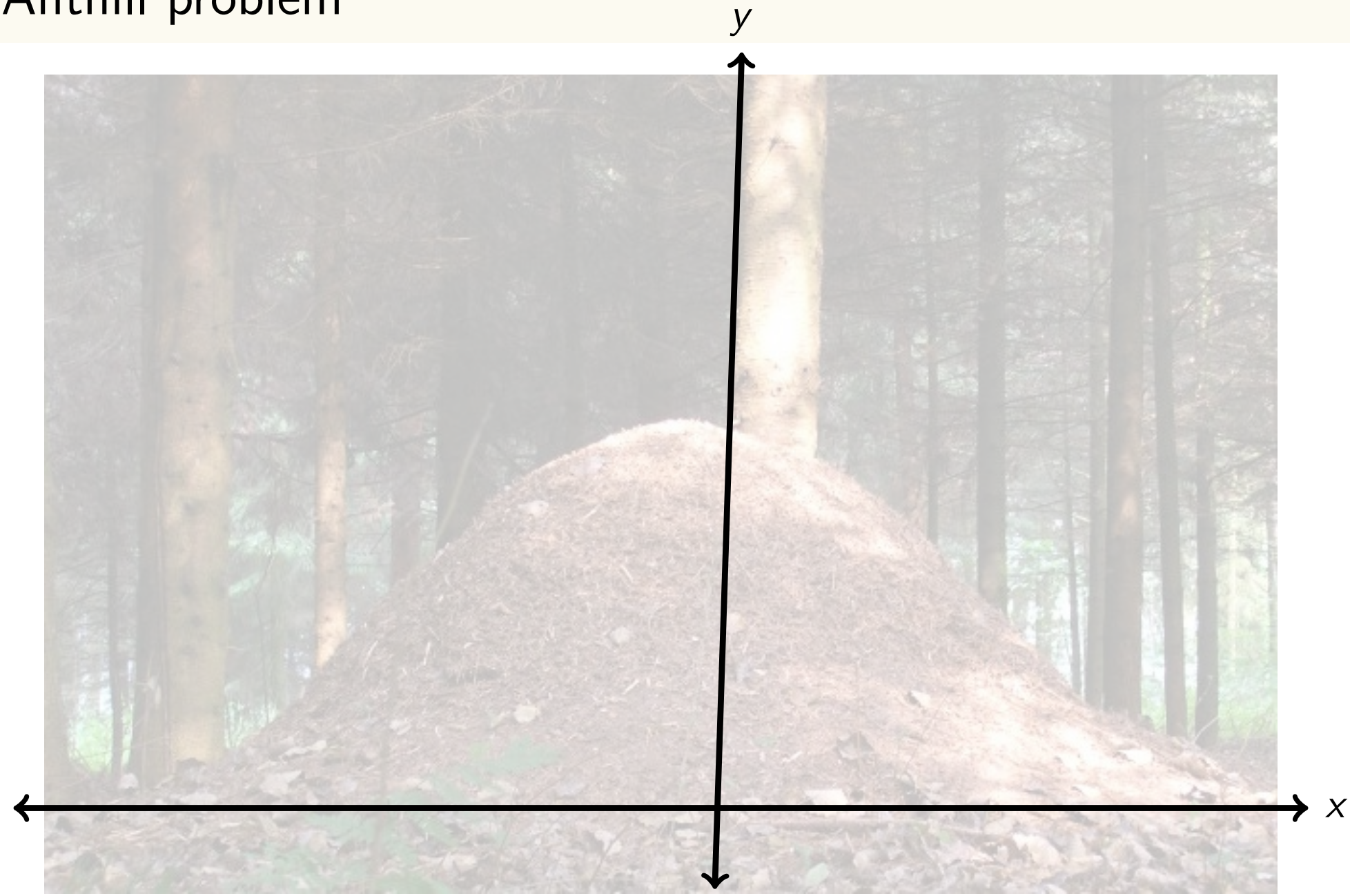
Anthill problem



Anthill problem

Anthill problem

Anthill problem



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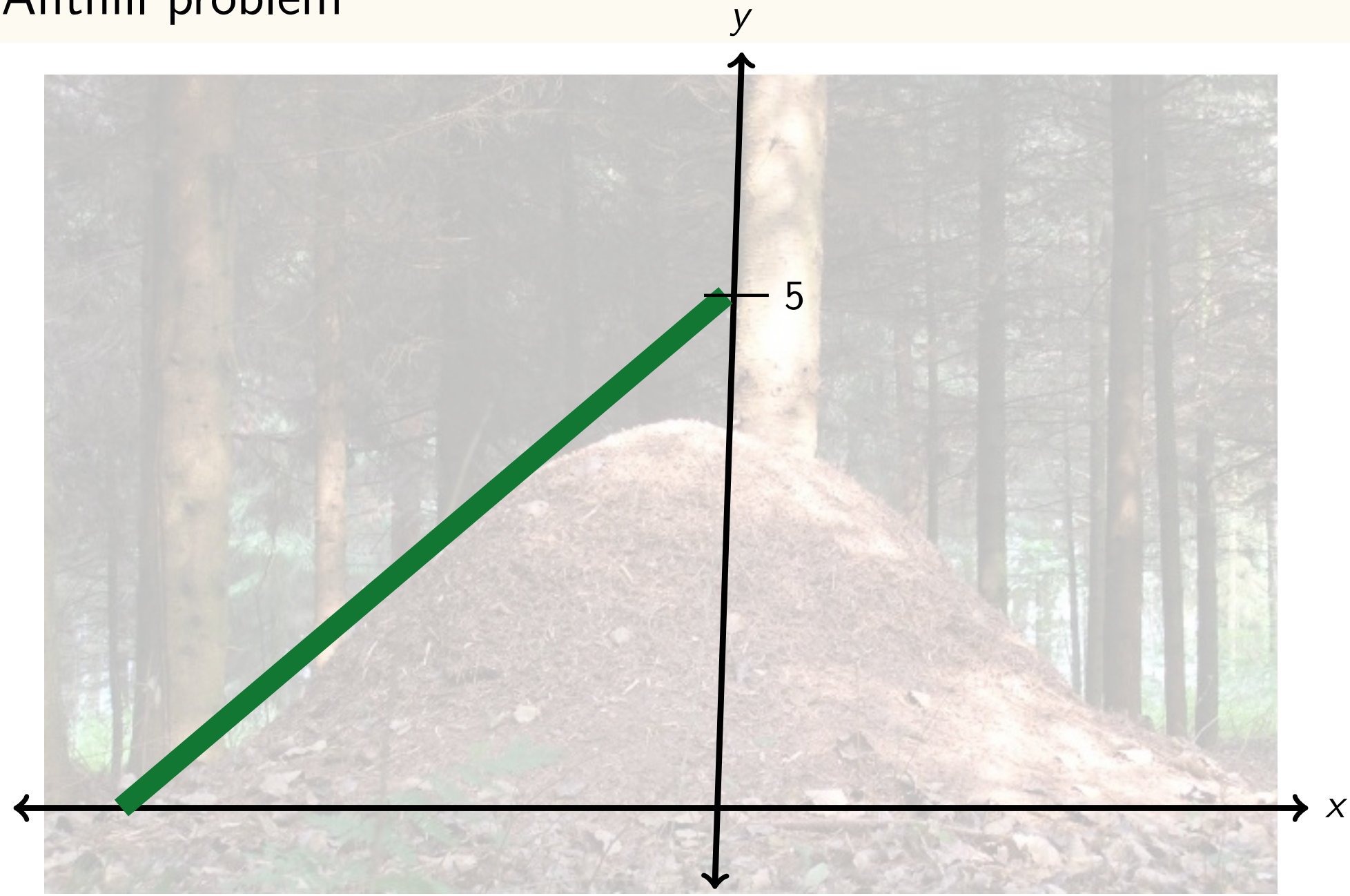
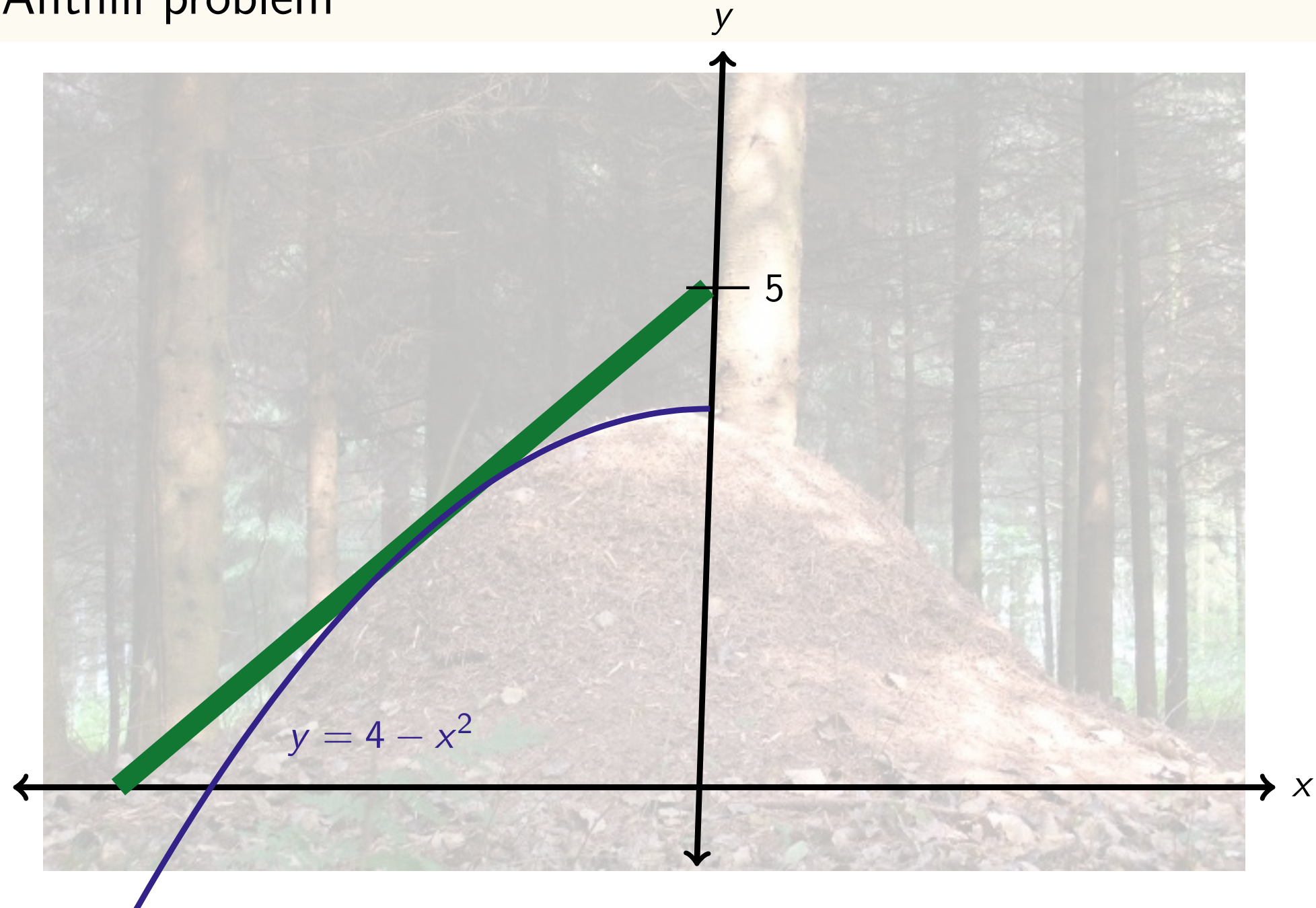
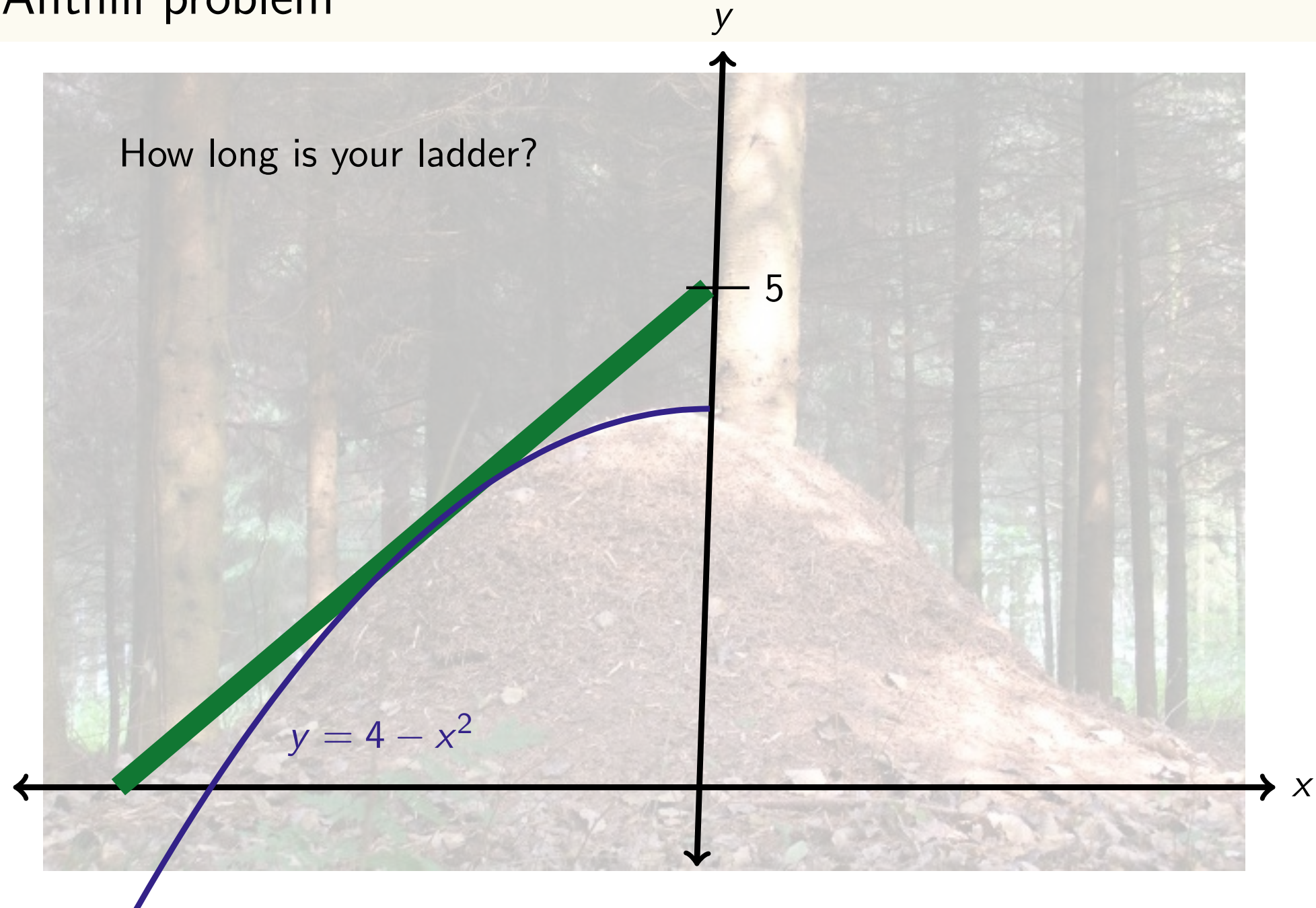


Anthill problem



Anthill problem



Anthill Problem

- The ladder is a tangent line to the anthill.
- The ladder intersects the hill at point $(p, 4 - p^2)$ for some constant p.
- Since $\frac{\mathrm{d}}{\mathrm{d}x}\{4 - x^2\} = -2x$, the slope of the tangent line is $-2p$.
- Since the tangent line passes through the point $(0, 5)$, it has equation

$$(y - 5) = -2px$$

- Also, the tangent line passes through the point $(p, 4 - p^2)$. So:

$$\underbrace{(4 - p^2)}_{y} - 5 = -2p\,\underbrace{(p)}_{x}$$

$$p^2 = 1 \qquad p = -1$$

- Then the equation of the tangent line is

$$y - 5 = 2x$$

- The tangent line has x-intercept $x = -\frac{2}{5}$. So, the ladder is the hypotenuse of a triangle with side lengths 5 and $-\frac{2}{5}$. By the Pythagorean Theorem, the ladder has length $\sqrt{5^2 + \left(\frac{2}{5}\right)^2} = \frac{5\sqrt{5}}{2}$.

Linear Approximation

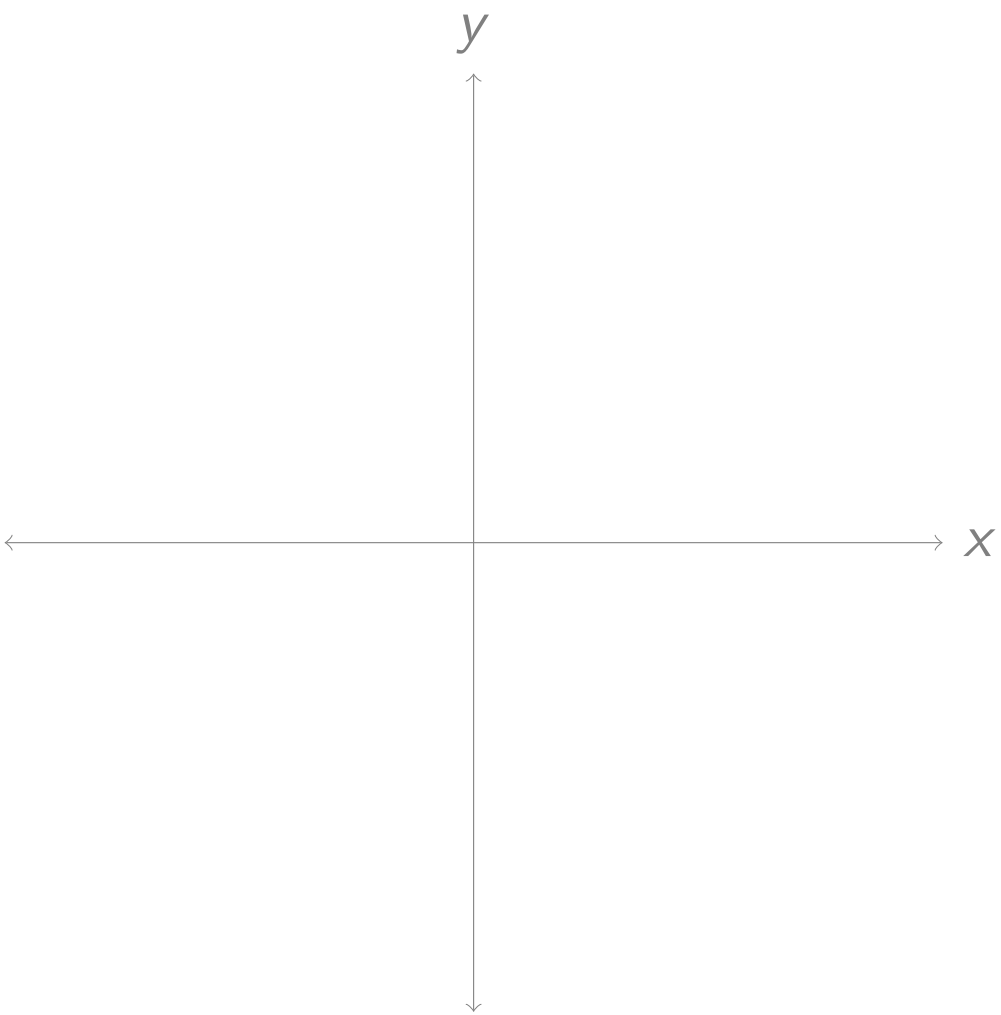
Example 10:

$$f(x) = x^3 - 4x$$

1. Sketch $f(x)$, using the methods we learned earlier.
2. Give an equation of the tangent line to $y = f(x)$ when $x = -1$.
3. Give all points where the tangent line intersects the curve.

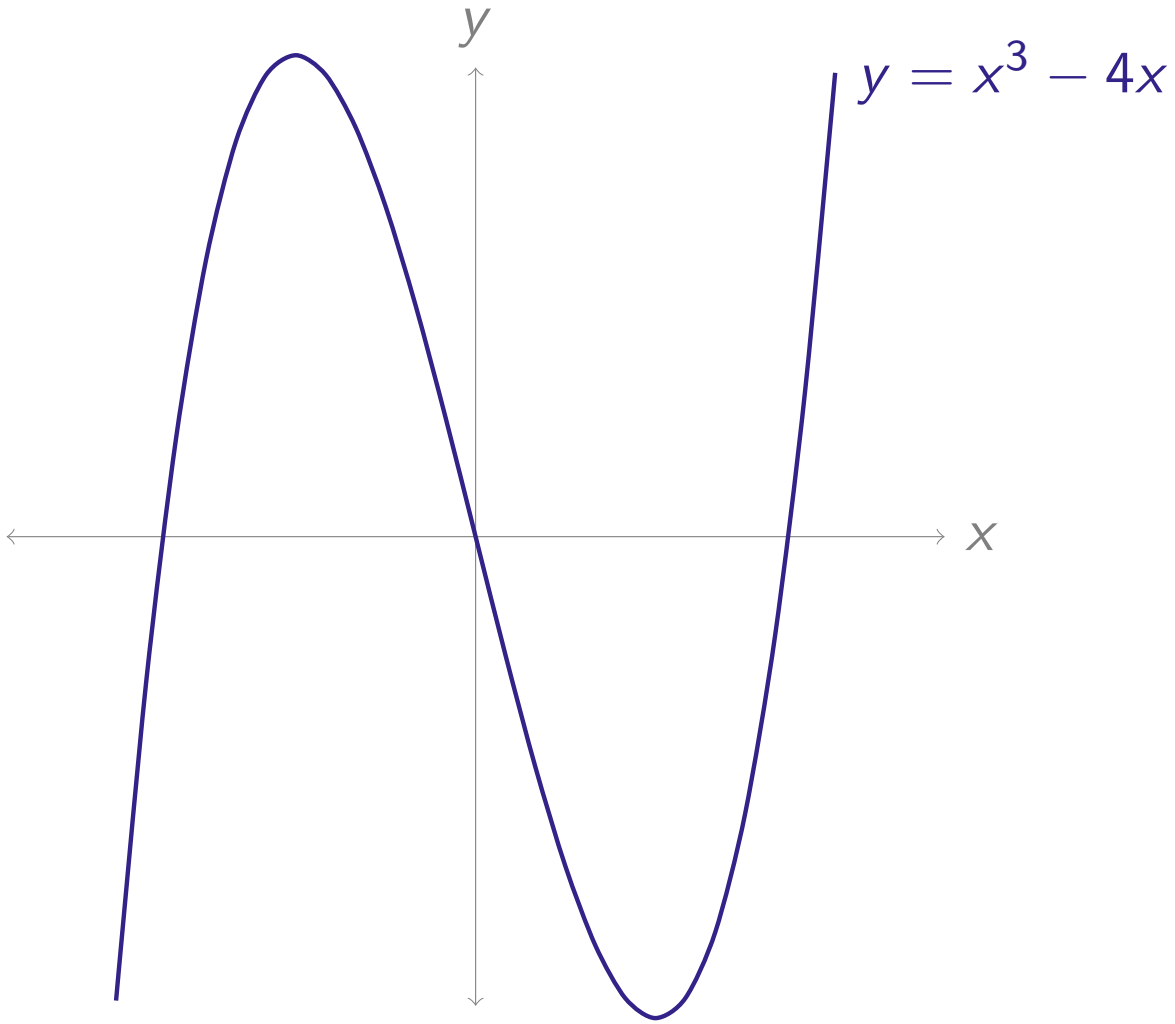
Linear Approximation

Solution 10:



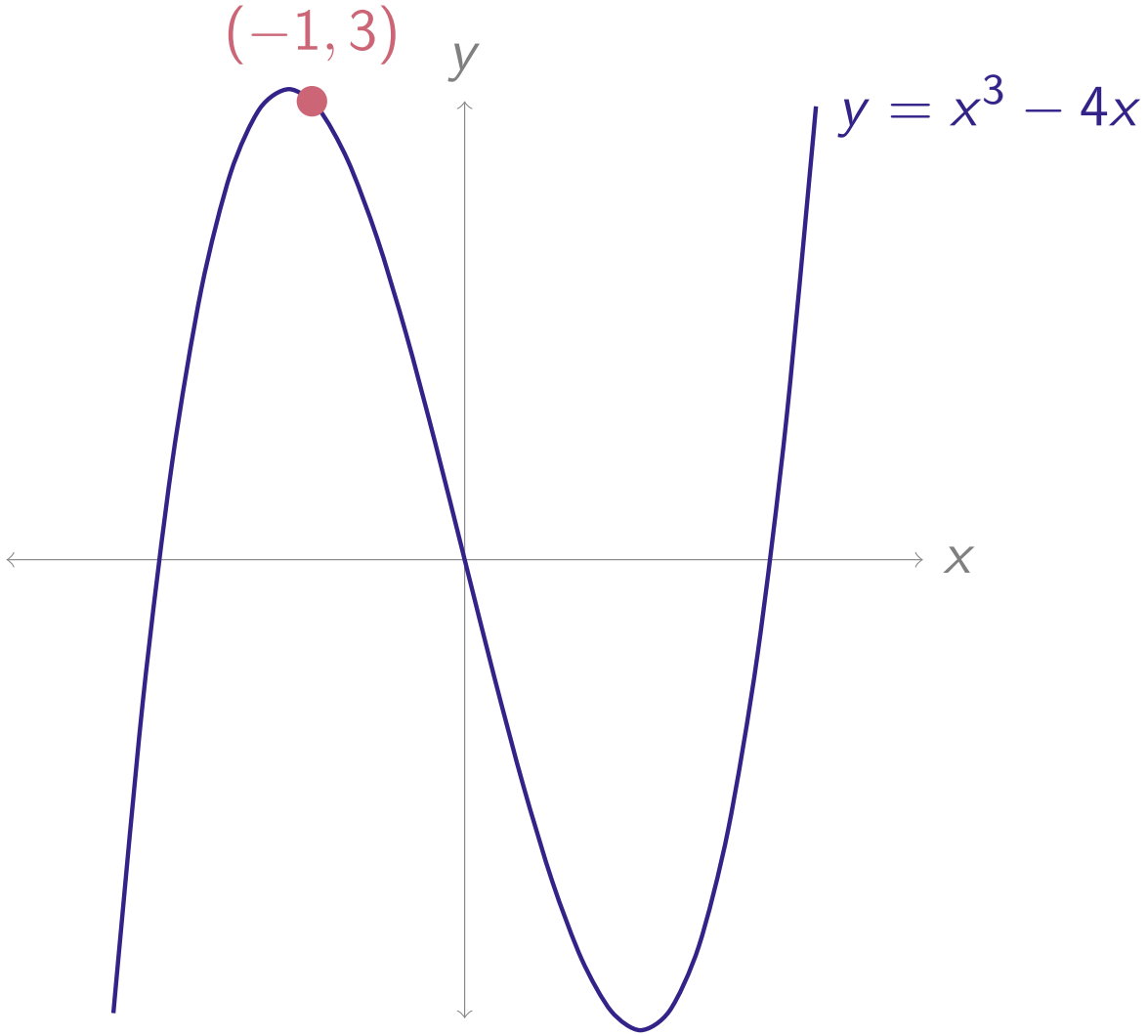
Linear Approximation

Solution 10:



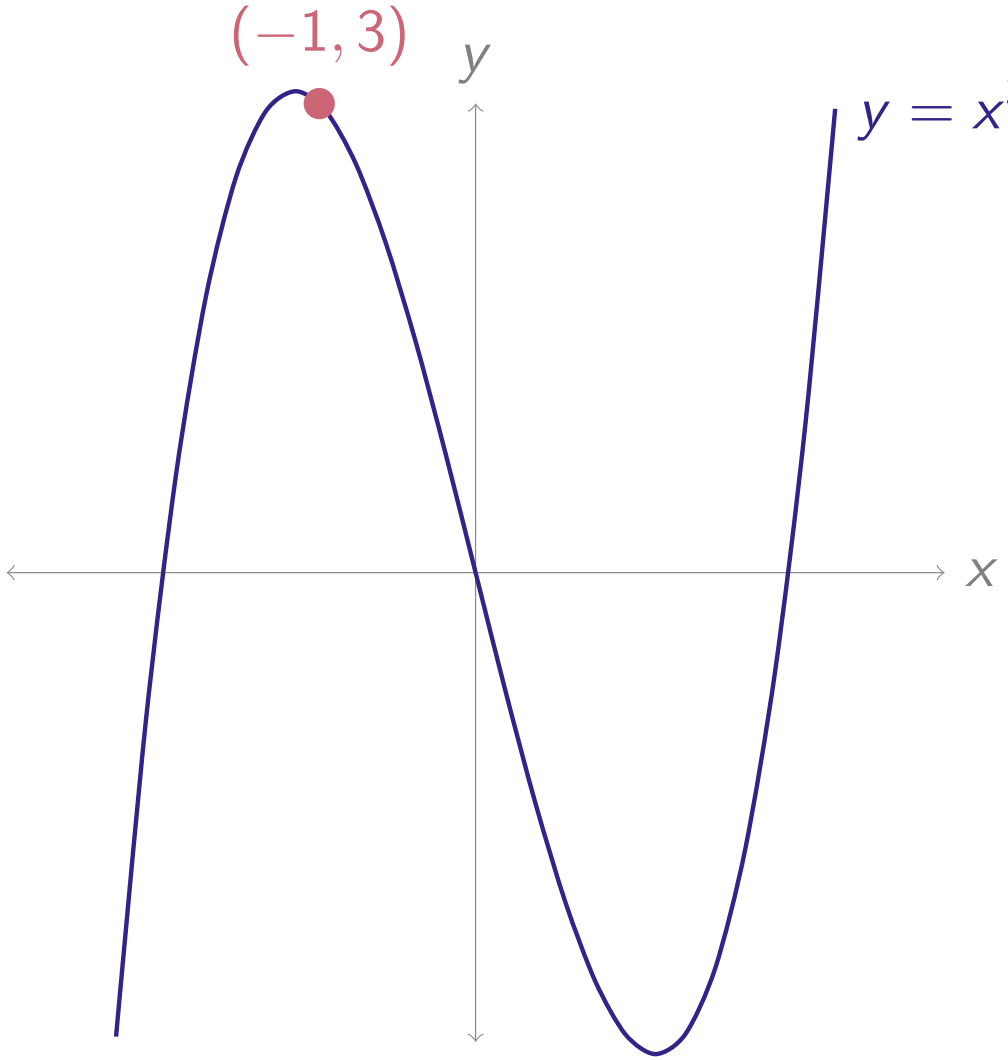
Linear Approximation

Solution 10:



Linear Approximation

Solution 10:

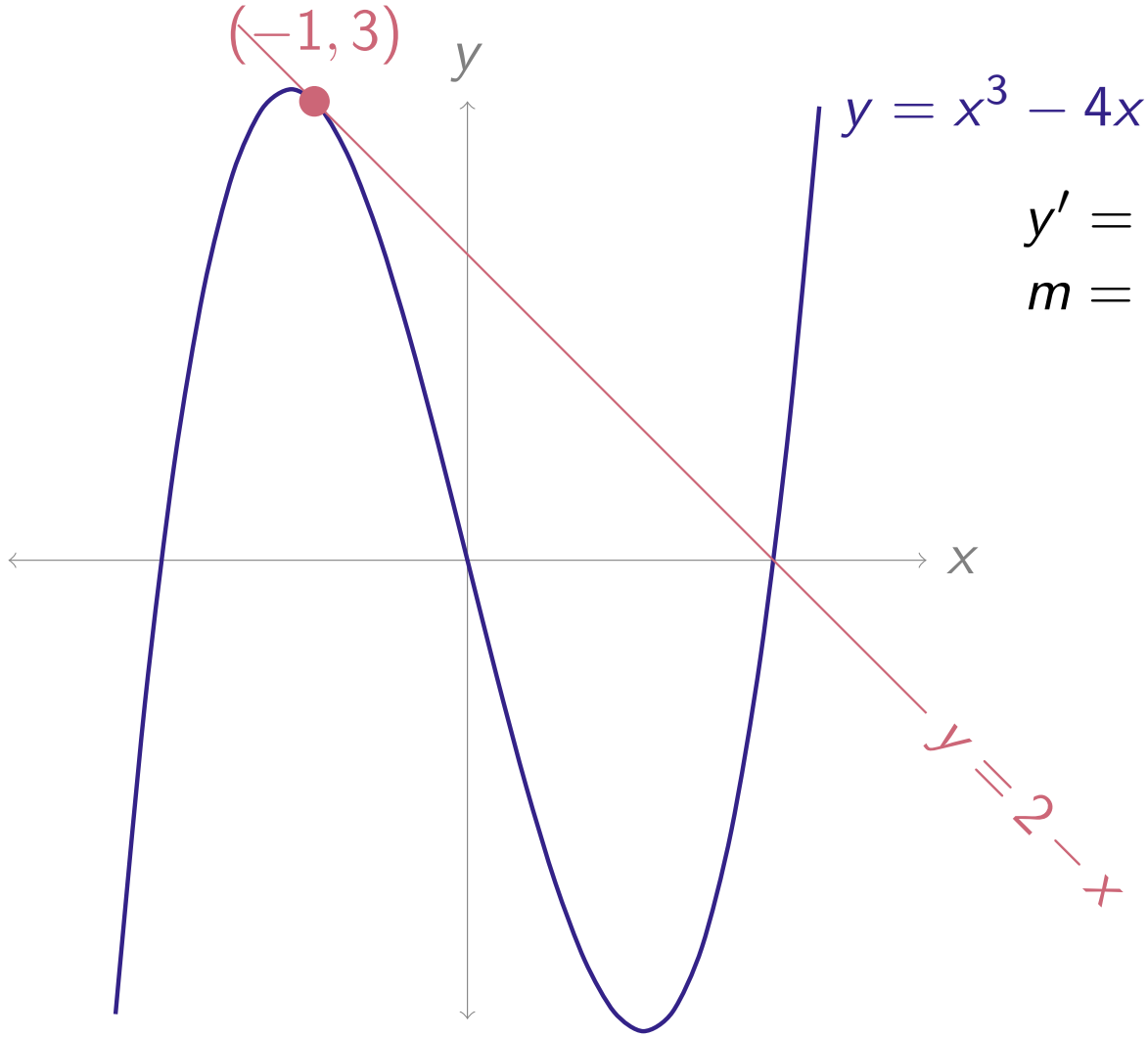


$y = x^3 - 4x$

$y' = 3x^2 - 4$

Linear Approximation

Solution 10:



$$y' = 3x^2 - 4$$
$$m = -1$$

Linear Approximation

To find where the curves $y = x^3 - 4x$ and $y = 2 - x$ meet, we start by setting them equal to each other.

$$x^3 - 4x = 2 - x$$
$$x^3 - 3x - 2 = 0$$

We know that one place these curves meet is $x = -1$. So, $x = -1$ is a solution to the above equation; hence $(x + 1)$ is a factor.

$$\begin{array}{r|rrrr}
 & & x^2 & -x & -2 \\
\hline
x+1 & x^3 & & -3x & -2 \\
 & -x^3 & -x^2 & & \\
\hline
 & & -x^2 & -3x & \\
 & & x^2 & +x & \\
\hline
 & & & -2x & -2 \\
 & & & 2x & +2 \\
\hline
 & & & & 0
\end{array}$$

Then,

$$\begin{aligned} x^3 - 3x - 2 &= (x+1)(x^2 - x - 2) \\ &= (x+1)(x-2)(x+1) \\ &= (x+1)^2(x-2) \end{aligned}$$

The curves meet at $x = -1$ and $x = 2$.

Extra Practice

Example 11:

Use Newton's Method to approximate $\sqrt{10}$.

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Spreadsheet link

Extra Practice

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Use Newton's Method to approximate $\sqrt{10}$.

- Choose $f(x)$ such that $f(\sqrt{10}) = 0$
- Choose x_0 (preferably close to $\sqrt{10}$)

Spreadsheet link

$$f(x) = x^2 - 10 \qquad f'(x) = 2x$$

$$x_0 = 3$$

$$x_1 = 3 - \frac{-1}{6} = \frac{19}{6} = 3.1\overline{66}$$

$$x_2 = \frac{19}{6} - \frac{1/36}{38/6} = \frac{721}{228} \approx 3.16228$$