Overview

- Proof of power rule: Appendix E.1
- Use power rule for derivatives and antiderivatives
- Derivative as a *linear operation*
- Describe the antiderivative, and explain how it is not unique.
- Product and quotient rules
- Chain rule

Power Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\{x^n\} = nx^{n-1}$$

- $\frac{\mathrm{d}}{\mathrm{d}x}\{x^7\} = 7x^6$
- $\frac{\mathrm{d}}{\mathrm{d}x}\{x^{32}\} = 32x^{31}$
- $\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{x^2}\right\} =$
- $\frac{\mathrm{d}}{\mathrm{d}x}\left\{\sqrt[5]{x^2}\right\} =$

Tangent Line

$$\frac{\mathrm{d}}{\mathrm{d}x}\{x^n\} = nx^{n-1}$$

Example 1: Write an equation of the tangent line described.

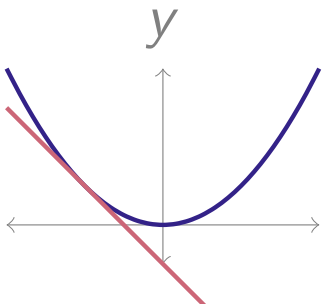
- Tangent line to the function $y = x^2$ at the point $(-5, 25)$.
- Tangent line to the function $y = \frac{1}{x}$ at the point $(3, \frac{1}{3})$.
- Tangent line to the function $y = \sqrt{x}$ at the point $x = 4$.

Point-Slope Form

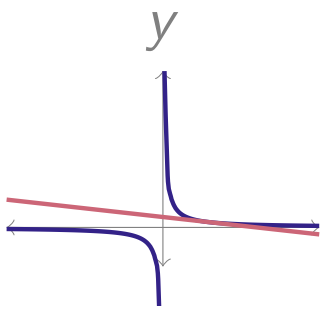
The line with slope m, passing through the point (x_1, y_1), has equation

$$(y - y_1) = m(x - x_1)$$

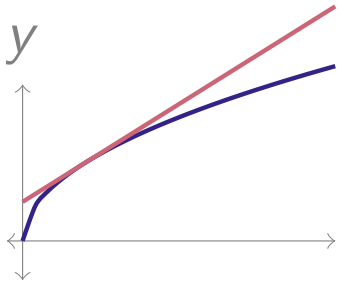
Tangent line to the function $y = x^2$ at the point $(-5, 25)$:
$y - 25 = -10(x + 5)$



Tangent line to the function $y = \frac{1}{x}$ at the point $(3, \frac{1}{3})$:
$y - \frac{1}{3} = -\frac{1}{9}(x - 3)$



Tangent line to the function $y = \sqrt{x}$ at the point $x = 4$:
$y - 2 = \frac{1}{4}(x - 4)$



The derivative is a linear operator

The derivative satisfies several convenient properties, among them:

1. a constant multiplying a function can be brought outside the derivative; and
2. the derivative of a sum of two functions is the same as the sum of the derivatives

$$\frac{\mathrm{d}}{\mathrm{d}x}\{x^2\} = 2x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\{\sqrt{x}\} = \frac{1}{2\sqrt{x}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\{9x^2\} =$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{\sqrt{x}}{3}\right\} =$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{5x^{1/5}\right\} =$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{3}{\sqrt{2x}}\right\} =$$

The derivative is a linear operator

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1. a constant multiplying a function can be brought outside the derivative; and
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$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{x^3 - x^2\right\} = 3x^2 - 2x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{5x^{-2} + \sqrt{3x}\right\} =$$

Higher Order Derivatives

Example 2: Evaluate $\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{\mathrm{d}}{\mathrm{d}x}\{x^5 - 2x^2 + 3\}\right\}$

$$\frac{\mathrm{d}}{\mathrm{d}x}\{x^5 - 2x^2 + 3\} =$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{\mathrm{d}}{\mathrm{d}x}\{x^5 - 2x^2 + 3\}\right\} =$$

Definition

The derivative of a derivative is called the **second derivative**, written

$$f''(x) \qquad \text{or} \qquad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$$

Similarly, the derivative of a second derivative is a third derivative, etc.

Typical Example: Acceleration

- Velocity: rate of change of position
- Acceleration: rate of change of velocity.

Example 3: The position of an object at time t is given by $s(t) = t(5 - t)$

1. Sketch the graph giving the position of the object.
2. What is the velocity of the object when $t = 1$?
3. What is the acceleration of the object when $t = 1$?

Concept Check

True or False: If $f'(1) = 18$, then $f''(1) = 0$, since the derivative of the constant 18 is 0.

Which of the following is always true of a QUADRATIC polynomial $f(x)$?

A. $f(0) = 0$
B. $f'(0) = 0$
C. $f''(0) = 0$
D. $f'''(0) = 0$
E. $f^{(4)}(0) = 0$

Which of the following is always true of a CUBIC polynomial $f(x)$?

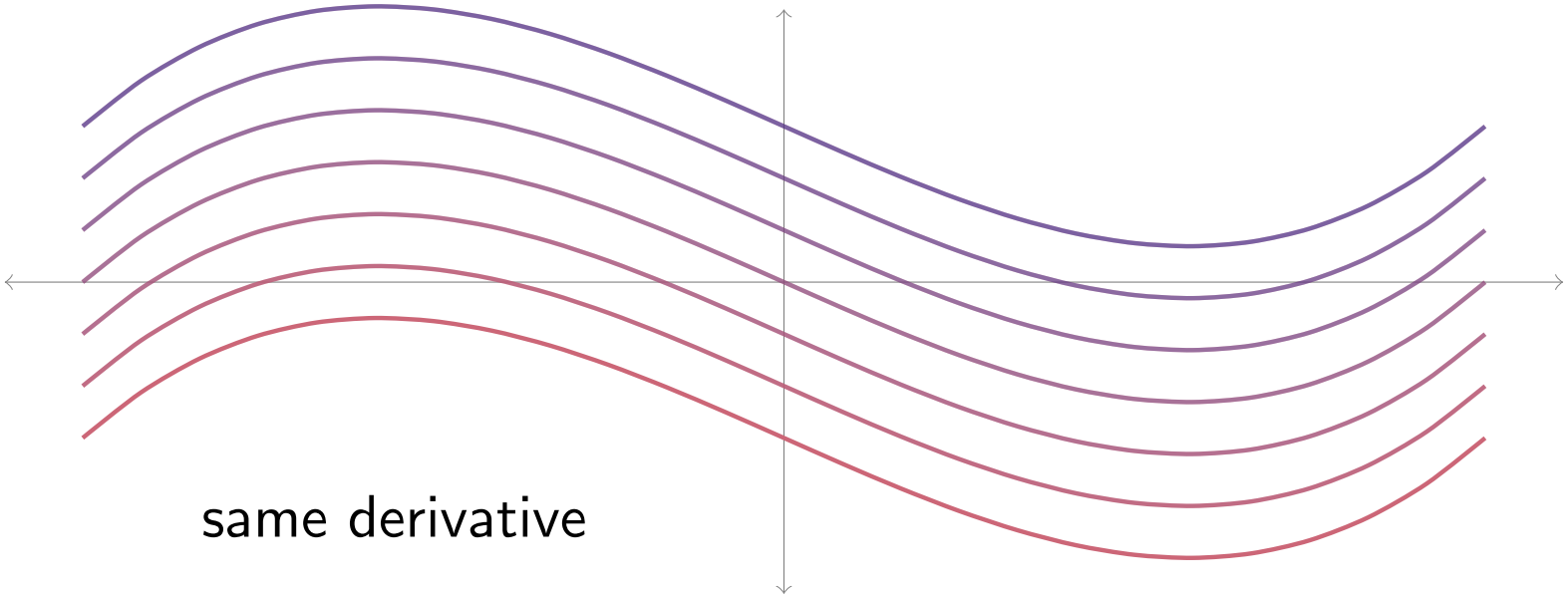
A. $f(0) = 0$
B. $f'(0) = 0$
C. $f''(0) = 0$
D. $f'''(0) = 0$
E. $f^{(4)}(0) = 0$

Antiderivatives of Power Functions and Polynomials

Q: Suppose $f'(x) = 2x$. What was $f(x)$?
A: $f(x) = x^2 + C$, where C is some constant.

Definition

An **antiderivative** of a function $f(x)$ is a function whose derivative is $f(x)$.
We can only determine antiderivatives up to some additive constant.



Antiderivatives of Power Functions and Polynomials

Q: Suppose $f'(x) = x$. What was $f(x)$?

A: $f(x) = \frac{1}{2}x^2 + C$, where C is some constant.

The antiderivative of x is $\frac{1}{2}x^2 + C$.

Q: Suppose $g'(x) = 7x^4$. What was $g(x)$?

Q: Suppose $h'(x) = x^n$, $n \neq -1$. What was $h(x)$?

Antiderivatives of Power Functions and Polynomials

The dorsal fin of a breaching orca accelerates at -10 m/sec^2.

1. Is the velocity increasing or decreasing?
2. Give a general equation for the position of the orca's dorsal fin relative to time.
3. Challenge: If the fin was in the air for 2 seconds, how high did it rise above the water?

Product and Quotient Rules

- True or false:

$$\frac{\mathrm{d}}{\mathrm{d}x}\{x^2 + x^3\} = \frac{\mathrm{d}}{\mathrm{d}x}\{x^2\} + \frac{\mathrm{d}}{\mathrm{d}x}\{x^3\} = 2x + 3x^2$$

- True or false:

$$\frac{\mathrm{d}}{\mathrm{d}x}\{7x^3\} = \frac{\mathrm{d}}{\mathrm{d}x}\{7\} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\{x^3\} = 0 \cdot 3x^2 = 0$$

- True or false:

$$\frac{\mathrm{d}}{\mathrm{d}x}\{x^2\} = \frac{\mathrm{d}}{\mathrm{d}x}\{x \cdot x\} = x \cdot \frac{\mathrm{d}}{\mathrm{d}x}\{x\} = x \cdot 1 = x$$

Where the Product Rule Comes From

You won't be assessed on this–it's only for background knowledge

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big\{f(x)g(x)\Big\} = \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} \quad \text{(def of deriv)}$$

fancy algebra

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h)\overbrace{-f(x+h)g(x)+f(x+h)g(x)}^{0}-f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\left[f(\underbrace{x+h}_{x})\underbrace{\frac{g(x+h)-g(x)}{h}}_{g'(x)} + g(x)\underbrace{\frac{f(x+h)-f(x)}{h}}_{f'(x)}\right]$$

$$= \boxed{f(x)g'(x)+g(x)f'(x)}$$

Product and Quotient Rules

$$f(x) = (x^2 + 5)(x^6 - x^2 + 3x)$$

To differentiate, we could expand, or we could use the product rule.

$$f'(x) = (x^2 + 5)(6x^5 - 2x + 3) + (x^6 - x^2 + 3x)(2x)$$

Where the Quotient Rule Comes From

You won't be assessed on this–it's only for background knowledge

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{f(x)}{g(x)}\right\} = \lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} \quad \text{(def of deriv)}$$

very fancy algebra indeed

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{g(x+h)g(x)}\right]$$

$$= \lim_{h\to 0}\left[\frac{g(x)\left(\frac{f(x+h)-f(x)}{h}\right) - f(x)\left(\frac{g(x+h)-g(x)}{h}\right)}{g(x+h)g(x)}\right]$$

$$= \boxed{\frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}}$$

Product and Quotient Rules

Theorem

If $f(x)$ and $g(x)$ are differentiable, and $g(x) \neq 0$, then:

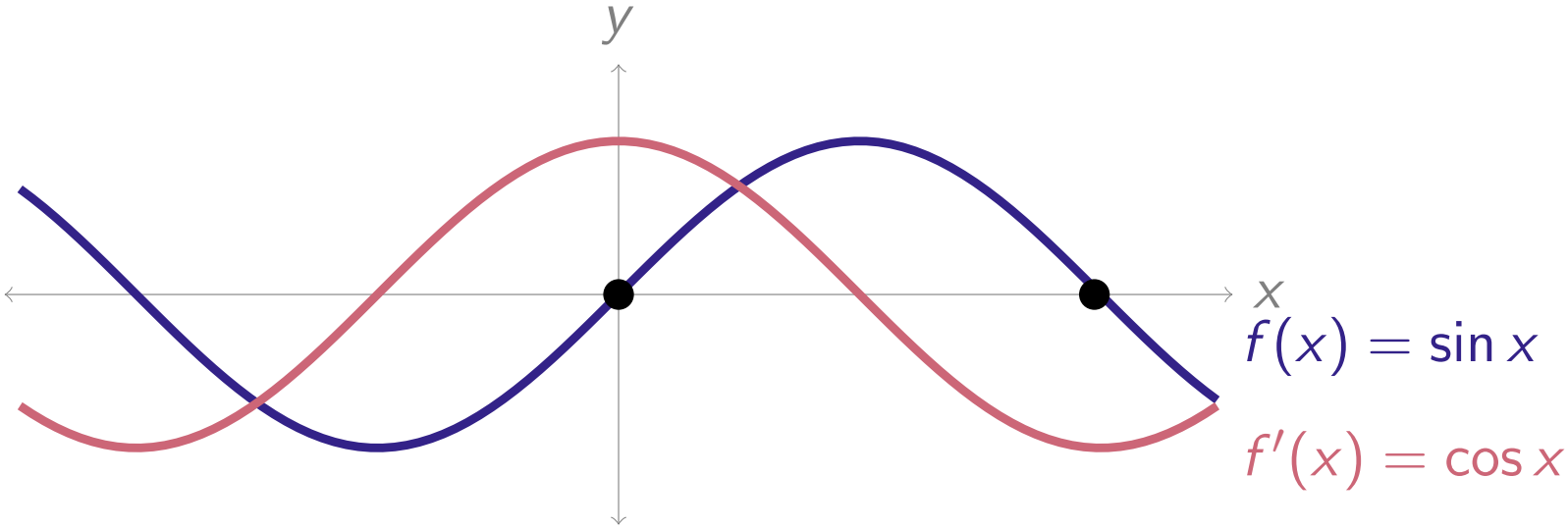
- $\frac{\mathrm{d}}{\mathrm{d}x}\{f(x)g(x)\} = f(x)g'(x) + g(x)f'(x)$ product rule
- $\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$ quotient rule

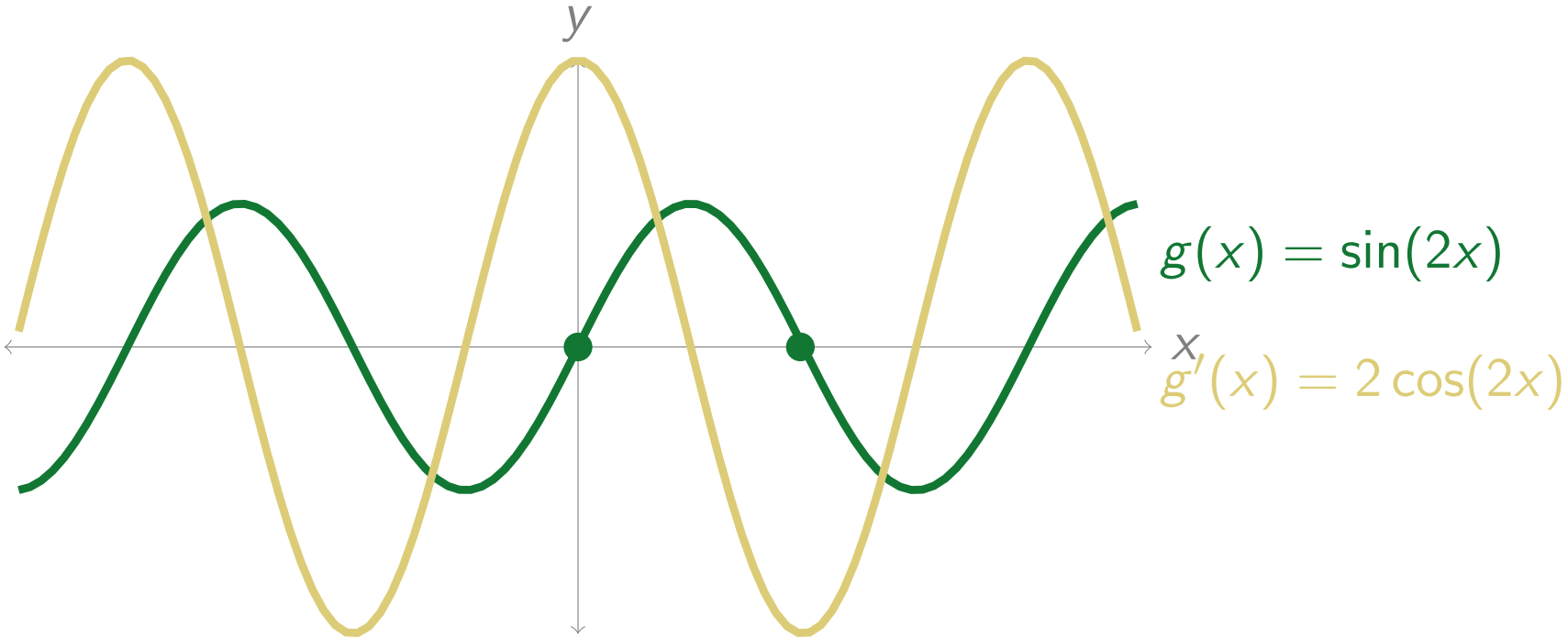
  Mnemonic: *low d-high minus high d-low over low-low*

Find the derivatives of the following functions:

$$f(x) = \frac{x^2}{x+1}$$
$$g(x) = (x^3+1)(x^2+2)$$

Intuition: $\sin x$ versus $\sin(2x)$





Compound Functions

Video: 2:27-3:50

Kelp Population

k	kelp population
u	urchin population
o	otter population
p	public policy

$$k = k(u) = k(u(o)) = k(u(o(p)))$$

This is an example of a compound function.

Should $k'(o)$ be positive or negative?

A. positive B. negative C. I'm not sure

Should $k'(u)$ be positive or negative?

A. positive B. negative C. I'm not sure

Chain Rule

Chain Rule

Suppose f and g are differentiable functions. Then

$$\frac{d}{dx}\{f(g(x))\} = f'(g(x))g'(x) = \frac{df}{dg}\frac{dg}{dx}$$

$$\frac{d}{dx}\left\{f\left(\boxed{g(x)}\right)\right\} = f'\left(\boxed{g(x)}\right)g'(x) = \frac{df}{dg}\frac{dg}{dx}$$

In the case of kelp, $\frac{dk}{do} = \frac{dk}{du}\frac{du}{do}$

$$F(v) = \left(\frac{v}{v^3+1}\right)^6$$

Chain Rule Practice

1. Differentiate $f(x) = (x^2 + x^3)^4$
2. Differentiate $g(x) = \sqrt{\frac{x}{x+1}}$
3. Antidifferentiate $h'(x) = 5(x^3 + x^2)^4(3x^2 + 2x)$

Acceleration

Recall

The derivative of velocity with respect to time is acceleration.

An object falling has a constant (negative) acceleration A due to gravity.

$$s''(t) = A$$

What is the equation of its position?

When $t = 0$, what it the object's position? Its velocity?

Acceleration

Position

If an object accelerates at a constant rate a, then its position is given by

$$s(t) = \frac{1}{2}at^2 + v_0 t + s_0$$

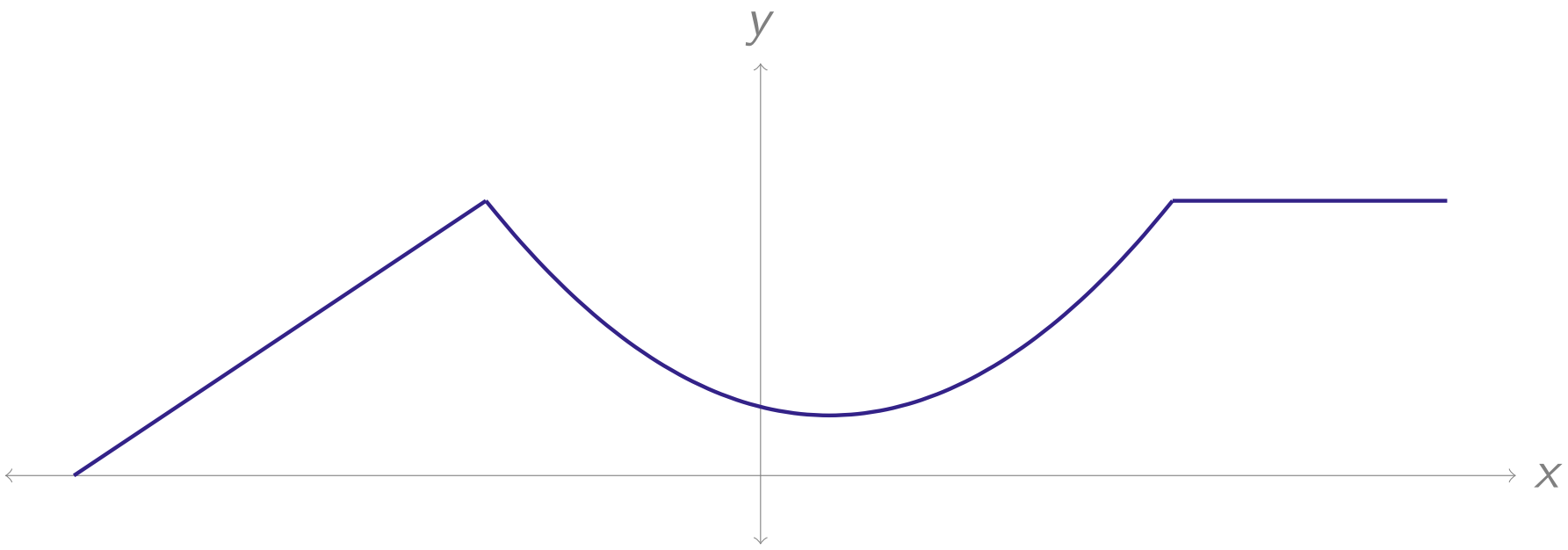
where v_0 is its velocity at time $t = 0$ and s_0 is its position at time $t = 0$.

Example 4:

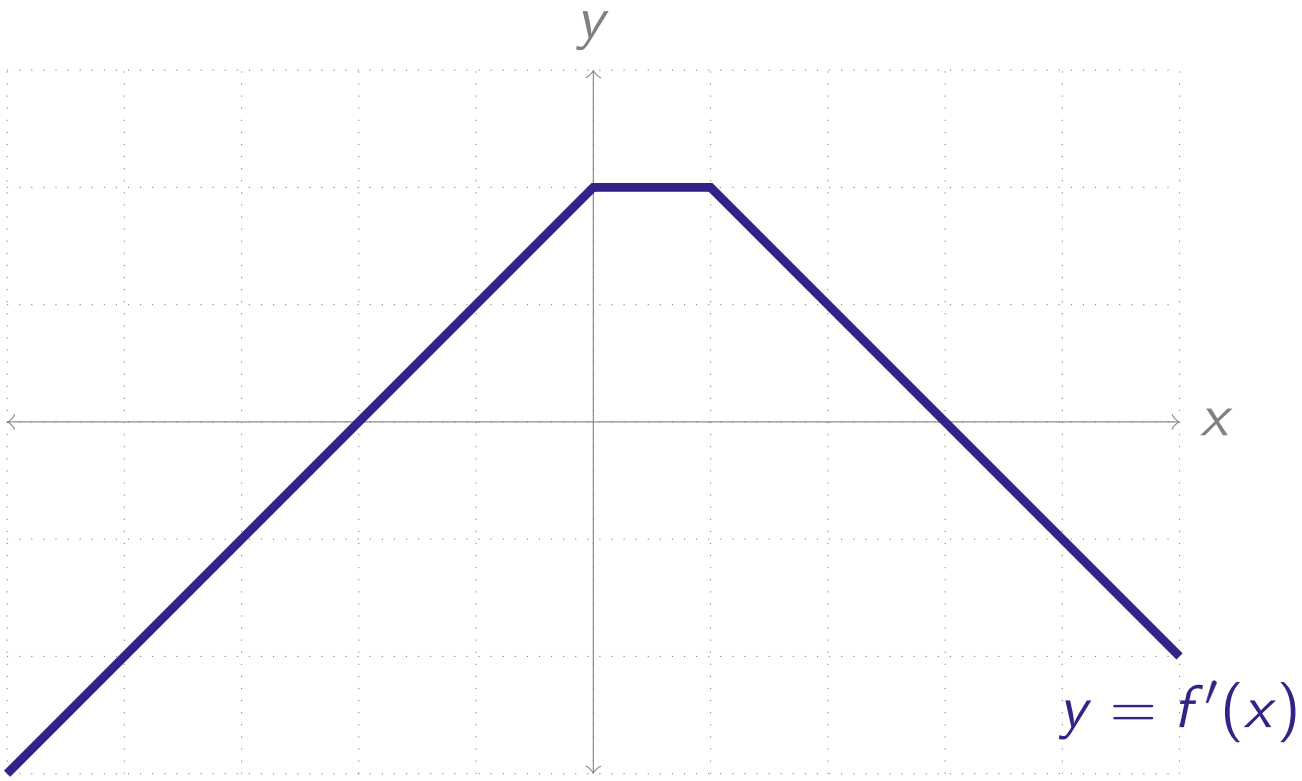
(a) You toss a ball in the air with an initial upwards velocity of 3 metres per second. Assuming the acceleration due to gravity is a constant $a = -9$ m/sec^2, how long is the ball in the air before it falls back in your hand?
(b) You toss a ball in the air and the acceleration due to gravity is constant. If the ball falls back into your hand two seconds later, at what time did the ball reach its highest point?

A frog leaps up from its lily pad, reaches its highest point, then falls back down, landing on a flower that is 50 cm higher than the lily pad. If the frog was in the air for a total of one second, and its vertical acceleration was a constant -10 m/sec^2, what was its velocity upon takeoff? How high did it leap?

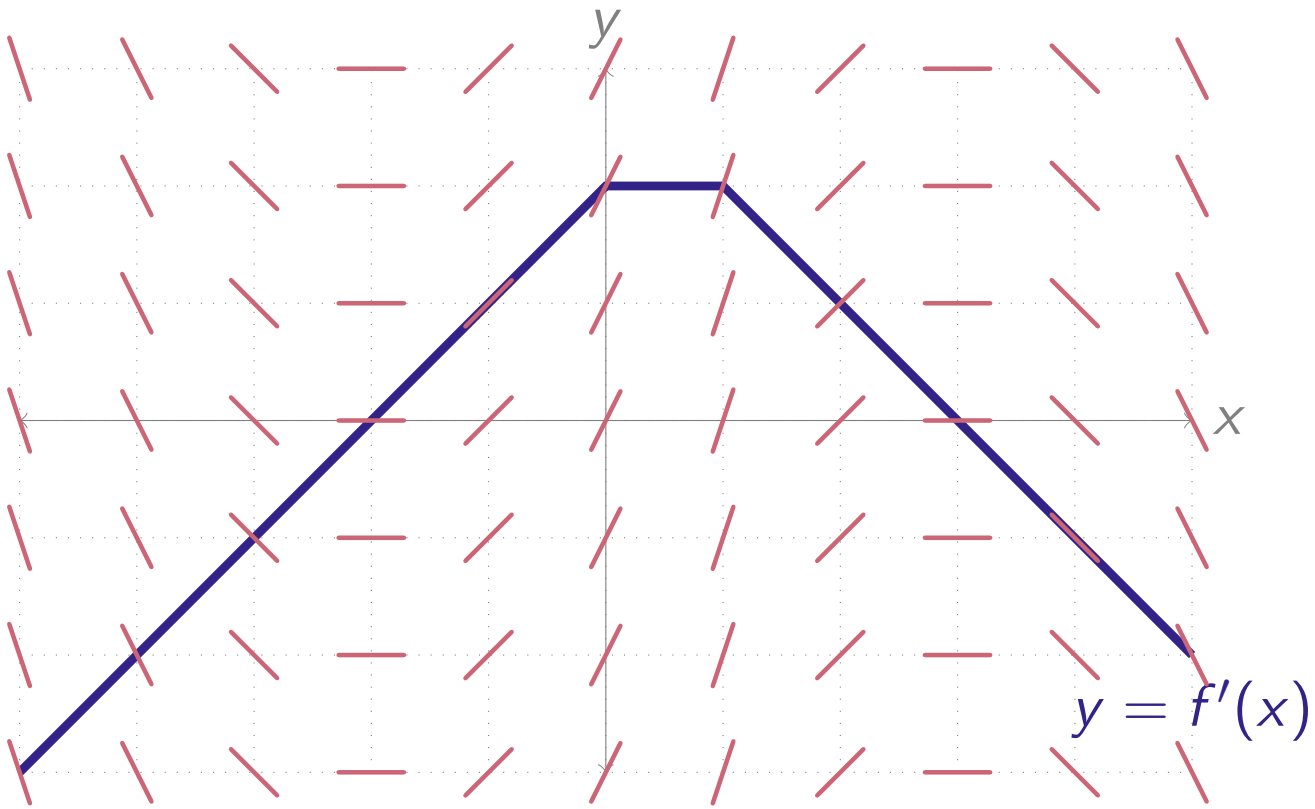
Sketching Higher-Order Derivatives



Sketching Antiderivatives



Sketching Antiderivatives



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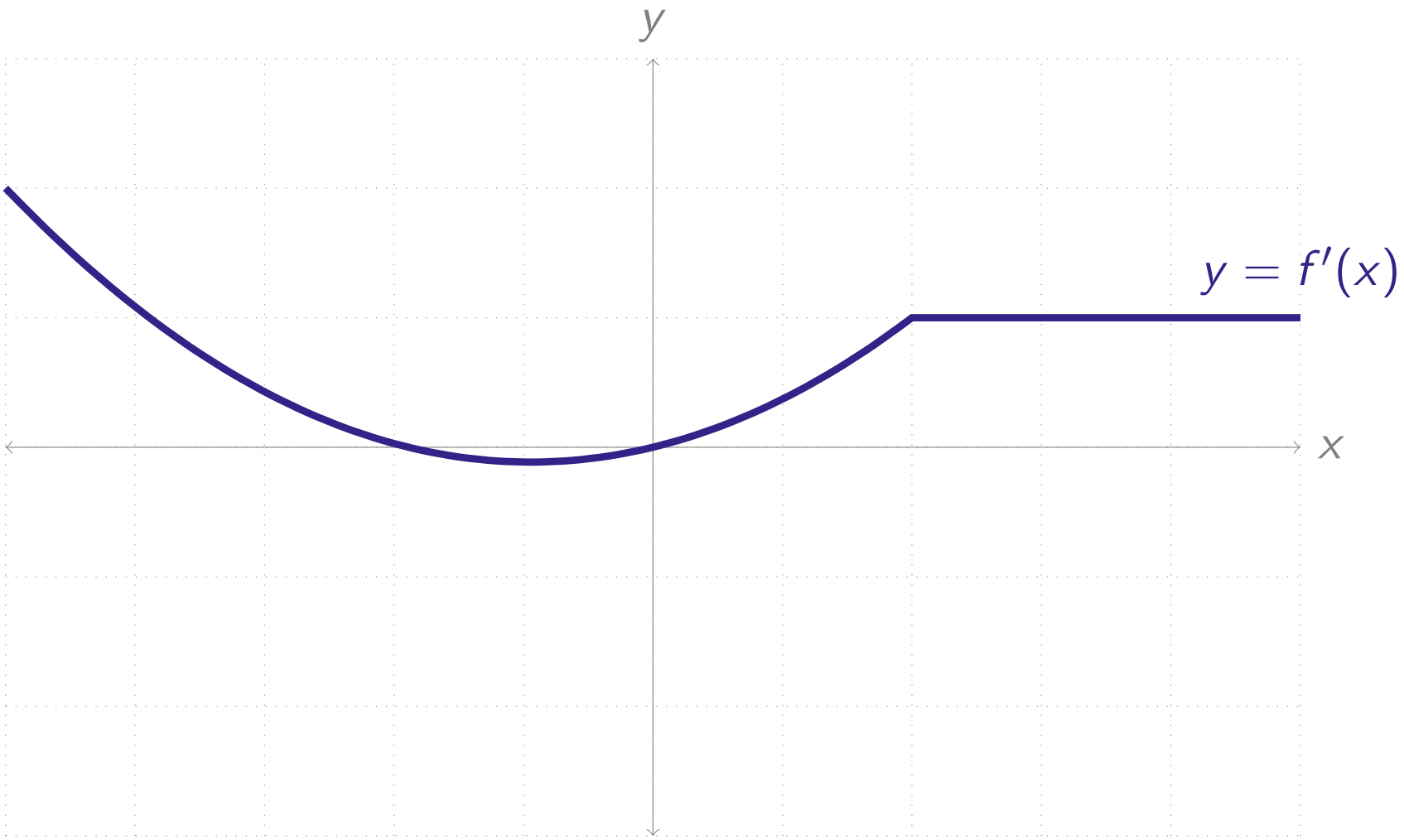

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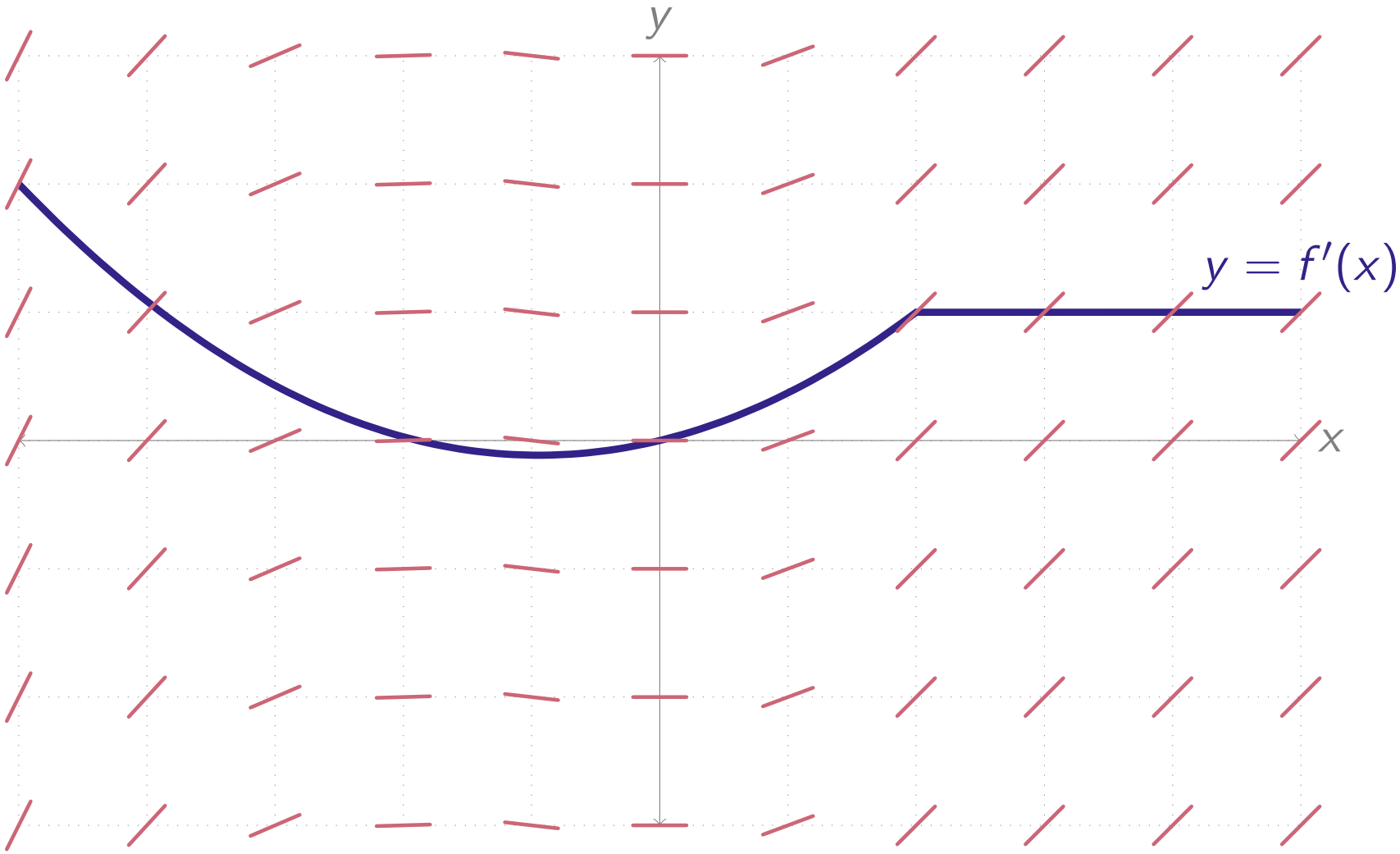

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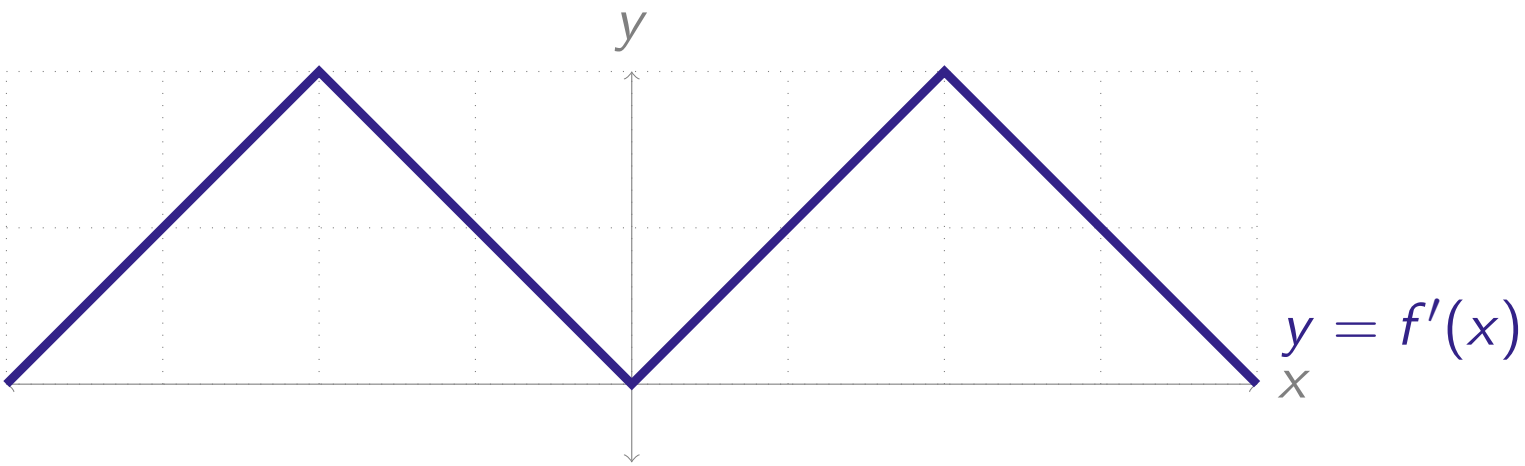

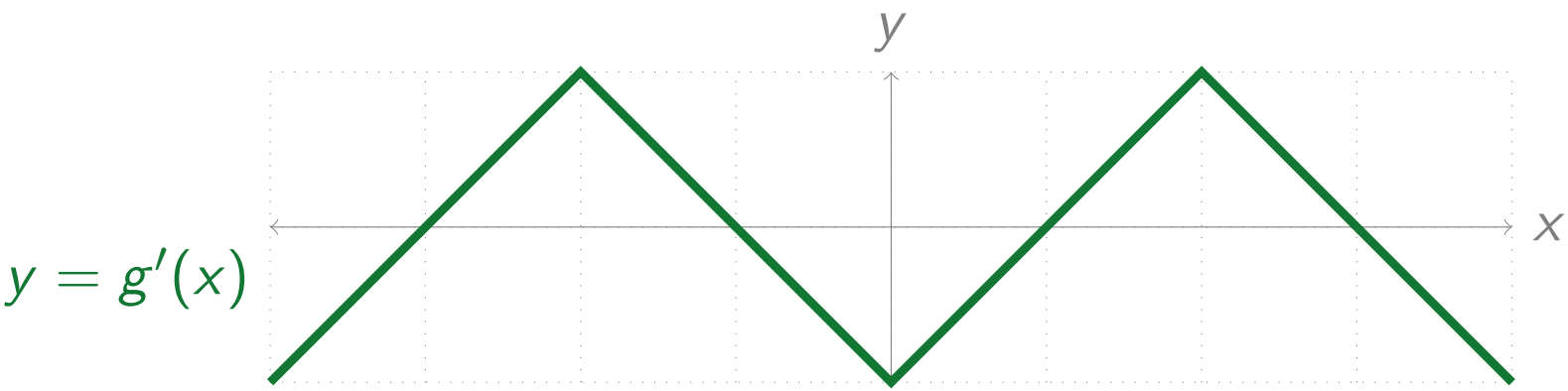

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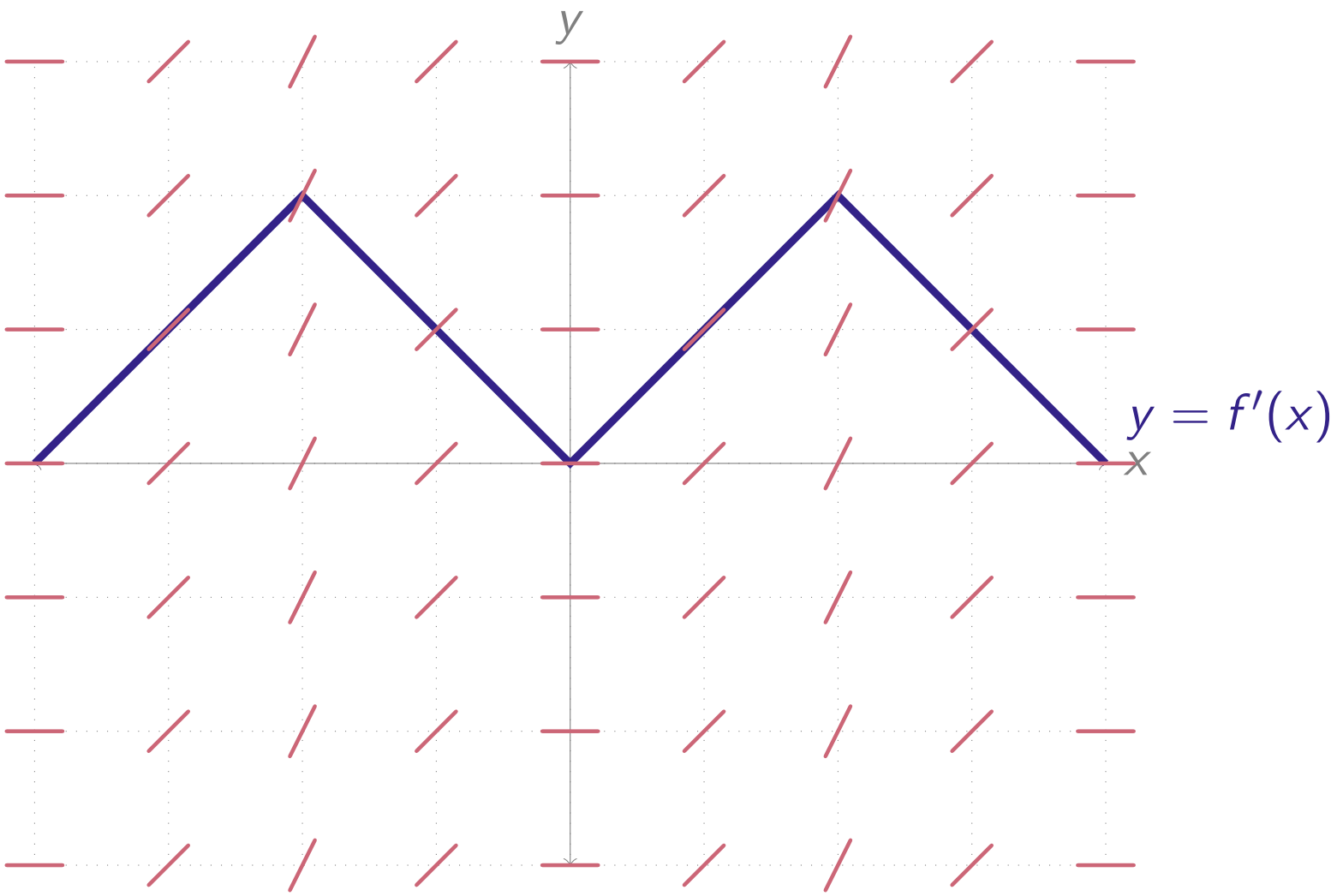

Sketching Antiderivatives