Overview

- Proof of power rule: Appendix E.1
- Use power rule for derivatives and antiderivatives
Overview

- Proof of power rule: Appendix E.1
- Use power rule for derivatives and antiderivatives
- Derivative as a *linear operation*
Overview

- Proof of power rule: Appendix E.1
- Use power rule for derivatives and antiderivatives
- Derivative as a *linear operation*
- Describe the antiderivative, and explain how it is not unique.
Overview

- Proof of power rule: Appendix E.1
- Use power rule for derivatives and antiderivatives
- Derivative as a *linear operation*
- Describe the antiderivative, and explain how it is not unique.
- Product and quotient rules
- Chain rule
Power Rule

\[
\frac{d}{dx}\{x^n\} = nx^{n-1}
\]
Power Rule

\[
\frac{d}{dx} \left\{ x^n \right\} = nx^{n-1}
\]

- \( \frac{d}{dx} \left\{ x^7 \right\} = 7x^6 \)
Chapter 4: Diff rules, simple antiderivatives, applications

4.1 Rules of differentiation

Power Rule

\[ \frac{d}{dx}\{x^n\} = nx^{n-1} \]

\[ \frac{d}{dx}\{x^7\} = 7x^6 \]

\[ \frac{d}{dx}\{x^{32}\} = \]
Power Rule

\[ \frac{d}{dx}\{x^n\} = nx^{n-1} \]

- \[ \frac{d}{dx}\{x^7\} = 7x^6 \]
- \[ \frac{d}{dx}\{x^{32}\} = 32x^{31} \]
**Power Rule**

\[
\frac{d}{dx} \{x^n\} = nx^{n-1}
\]

- \(\frac{d}{dx} \{x^7\} = 7x^6\)
- \(\frac{d}{dx} \{x^{32}\} = 32x^{31}\)
- \(\frac{d}{dx} \left\{ \frac{1}{x^2} \right\} = -\frac{2}{x^3}\)
- \(\frac{d}{dx} \left\{ \sqrt[5]{x^2} \right\} = \frac{2}{5x^{3/5}}\)
Chapter 4: Diff rules, simple antiderivatives, applications

4.1 Rules of differentiation

**Power Rule**

\[
\frac{d}{dx}\{x^n\} = nx^{n-1}
\]

- \[
\frac{d}{dx}\{x^7\} = 7x^6
\]
- \[
\frac{d}{dx}\{x^{32}\} = 32x^{31}
\]
- \[
\frac{d}{dx}\left\{\frac{1}{x^2}\right\} = \frac{d}{dx}\{x^{-2}\} =
\]
- \[
\frac{d}{dx}\left\{\sqrt[5]{x^2}\right\} =
\]
Chapter 4: Diff rules, simple antiderivatives, applications

4.1 Rules of differentiation

Power Rule

\[ \frac{d}{dx}\{x^n\} = nx^{n-1} \]

- \[ \frac{d}{dx}\{x^7\} = 7x^6 \]
- \[ \frac{d}{dx}\{x^{32}\} = 32x^{31} \]
- \[ \frac{d}{dx}\left\{ \frac{1}{x^2} \right\} = \frac{d}{dx}\{x^{-2}\} = -2x^{-3} \]
- \[ \frac{d}{dx}\left\{ \sqrt[5]{x^2} \right\} = \frac{d}{dx}\{x^{2/5}\} \]
Chapter 4: Diff rules, simple antiderivatives, applications

4.1 Rules of differentiation

**Power Rule**

\[
\frac{d}{dx}\{x^n\} = nx^{n-1}
\]

- \[\frac{d}{dx}\{x^7\} = 7x^6\]
- \[\frac{d}{dx}\{x^{32}\} = 32x^{31}\]
- \[\frac{d}{dx}\left\{\frac{1}{x^2}\right\} = \frac{d}{dx}\{x^{-2}\} = -2x^{-3} = \frac{-2}{x^3}\]
- \[\frac{d}{dx}\left\{\sqrt[5]{x^2}\right\} = \frac{d}{dx}\{x^{2/5}\}\]
Power Rule

\[
\frac{d}{dx}\{x^n\} = nx^{n-1}
\]

- \(\frac{d}{dx}\{x^7\} = 7x^6\)
- \(\frac{d}{dx}\{x^{32}\} = 32x^{31}\)
- \(\frac{d}{dx}\left\{\frac{1}{x^2}\right\} = \frac{d}{dx}\{x^{-2}\} = -2x^{-3} = \frac{-2}{x^3}\)
- \(\frac{d}{dx}\left\{\sqrt[5]{x^2}\right\} = \frac{d}{dx}\{x^{2/5}\} = \frac{2}{5}x^{-3/5} = \frac{2}{5\sqrt[5]{x^3}}\)
Tangent Line

\[ \frac{d}{dx}\{x^n\} = nx^{n-1} \]

Example 1: Write an equation of the tangent line described.

- Tangent line to the function \( y = x^2 \) at the point \((-5, 25)\).
Tangent Line

\[
\frac{d}{dx} \{ x^n \} = nx^{n-1}
\]

Example 1: Write an equation of the tangent line described.

- Tangent line to the function \( y = x^2 \) at the point \((-5, 25)\).

Point-Slope Form

The line with slope \( m \), passing through the point \((x_1, y_1)\), has equation

\[(y - y_1) = m(x - x_1)\]
Chapter 4: Diff rules, simple antiderivatives, applications  
4.1 Rules of differentiation

**Tangent Line**

\[
\frac{d}{dx}\{x^n\} = nx^{n-1}
\]

**Example 1:** Write an equation of the tangent line described.

- Tangent line to the function \( y = x^2 \) at the point \((-5, 25)\).
  \[ y - 25 = -10(x + 5) \]

**Point-Slope Form**

The line with slope \( m \), passing through the point \((x_1, y_1)\), has equation

\[
(y - y_1) = m(x - x_1)
\]
Chapter 4: Diff rules, simple antiderivatives, applications

4.1 Rules of differentiation

**Tangent Line**

\[
\frac{d}{dx}\{x^n\} = nx^{n-1}
\]

**Example 1:** Write an equation of the tangent line described.

- Tangent line to the function \( y = x^2 \) at the point \((-5, 25)\).
  \[
y - 25 = -10(x + 5)
\]
- Tangent line to the function \( y = \frac{1}{x} \) at the point \((3, \frac{1}{3})\).
- Tangent line to the function \( y = \sqrt{x} \) at the point \( x = 4 \).

**Point-Slope Form**

The line with slope \( m \), passing through the point \((x_1, y_1)\), has equation

\[
(y - y_1) = m(x - x_1)
\]
Chapter 4: Diff rules, simple antiderivatives, applications

4.1 Rules of differentiation

**Tangent Line**

\[
\frac{d}{dx}\{x^n\} = nx^{n-1}
\]

**Example 1:** Write an equation of the tangent line described.

- Tangent line to the function \( y = x^2 \) at the point \((-5, 25)\).
  \[
y - 25 = -10(x + 5)
\]
- Tangent line to the function \( y = \frac{1}{x} \) at the point \((3, \frac{1}{3})\).
  \[
y - \frac{1}{3} = -\frac{1}{9}(x - 3)
\]
- Tangent line to the function \( y = \sqrt{x} \) at the point \(x = 4\).
  \[
y - 2 = \frac{1}{4}(x - 4)
\]

**Point-Slope Form**

The line with slope \( m \), passing through the point \((x_1, y_1)\), has equation

\[
(y - y_1) = m(x - x_1)
\]
Chapter 4: Diff rules, simple antiderivatives, applications

4.1 Rules of differentiation

Tangent Line

\[
\frac{d}{dx}\{x^n\} = nx^{n-1}
\]

**Example 1:** Write an equation of the tangent line described.

- **Tangent line to the function** \( y = x^2 \) **at the point** \((-5, 25)\).
  \[
y - 25 = -10(x + 5)
\]
- **Tangent line to the function** \( y = \frac{1}{x} \) **at the point** \((3, \frac{1}{3})\).
  \[
y - \frac{1}{3} = -\frac{1}{9}(x - 3)
\]
- **Tangent line to the function** \( y = \sqrt{x} \) **at the point** \(x = 4\).
  \[
y - 2 = \frac{1}{4}(x - 4)
\]

Point-Slope Form

The line with slope \( m \), passing through the point \((x_1, y_1)\), has equation

\[
(y - y_1) = m(x - x_1)
\]
Tangent line to the function \( y = x^2 \) at the point \((-5, 25)\):
\[ y - 25 = -10(x + 5) \]

Tangent line to the function \( y = \frac{1}{x} \) at the point \((3, \frac{1}{3})\):
\[ y - \frac{1}{3} = -\frac{1}{9}(x - 3) \]

Tangent line to the function \( y = \sqrt{x} \) at the point \( x = 4 \):
\[ y - 2 = \frac{1}{4}(x - 4) \]
Solution 1: Write an equation of the tangent line described.

- **Tangent line to the function** $y = x^2$ at the point $(-5, 25)$.
  Using the power rule, we find the derivative of this function is $2x$, so when $x = -5$, its derivative is $2(-5) = -10$. Now we can describe the tangent line in point-slope form with $m = -10$, $x_1 = -5$, and $y_1 = 25$:
  \[ y - 25 = -10(x + 5) \]

- **Tangent line to the function** $y = \frac{1}{x}$ at the point $(3, \frac{1}{3})$.
  Note $y = x^{-1}$. Using the power rule, we find the derivative of this function is $-x^{-2}$, so when $x = 3$, its derivative is $-(3^{-2}) = -\frac{1}{9}$. Now we can describe the tangent line in point-slope form with $m = -\frac{1}{9}$, $x_1 = 3$, and $y_1 = \frac{1}{3}$:
  \[ y - \frac{1}{3} = -\frac{1}{9}(x - 3) \]

- **Tangent line to the function** $y = \sqrt{x}$ at the point $x = 4$.
  Note $y = x^{1/2}$, and if $x = 4$, then $y = 2$. Using the power rule, we find the derivative of this function is $\frac{1}{2}x^{-1/2}$, so when $x = 4$, its derivative is $\frac{1}{2} \cdot \frac{1}{4^{1/2}} = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. Now we can describe the tangent line in point-slope form with $m = \frac{1}{4}$, $x_1 = 4$, and $y_1 = 2$:
  \[ y - 2 = \frac{1}{4}(x - 4) \]
The derivative is a linear operator

The derivative satisfies several convenient properties, among them:

1. a constant multiplying a function can be brought outside the derivative; and
2. the derivative of a sum of two functions is the same as the sum of the derivatives
The derivative is a linear operator

The derivative satisfies several convenient properties, among them:

1. A constant multiplying a function can be brought outside the derivative; and
2. The derivative of a sum of two functions is the same as the sum of the derivatives

\[
\frac{d}{dx} \{ x^2 \} = 2x
\]
The derivative is a linear operator

The derivative satisfies several convenient properties, among them:

1. a constant multiplying a function can be brought outside the derivative; and
2. the derivative of a sum of two functions is the same as the sum of the derivatives

\[
\frac{d}{dx}\{x^2\} = 2x \quad \text{and} \quad \frac{d}{dx}\{9x^2\} = 18x
\]

\[
\frac{d}{dx}\{\sqrt{x}\} = \frac{1}{2\sqrt{x}} \quad \text{and} \quad \frac{d}{dx}\{\sqrt{x^3}\} = \frac{3}{2}\sqrt{x}
\]

\[
\frac{d}{dx}\{\frac{5x}{\sqrt[5]{x}}\} = 5(\frac{1}{5}x^{-4/5} - \frac{4}{5}x^{-4/5}) = x - \frac{4}{5}
\]

\[
\frac{d}{dx}\{3\sqrt{2x}\} = 3\sqrt{2} \cdot \frac{1}{2\sqrt{x}} = \frac{3\sqrt{2}}{2\sqrt{x}}
\]
The derivative is a linear operator

The derivative satisfies several convenient properties, among them:

1. a constant multiplying a function can be brought outside the derivative; and
2. the derivative of a sum of two functions is the same as the sum of the derivatives

\[
\frac{d}{dx}\{x^2\} = 2x \quad \quad \frac{d}{dx}\{9x^2\} = 9(2x) = 18x
\]
The derivative is a linear operator

The derivative satisfies several convenient properties, among them:

1. a constant multiplying a function can be brought outside the derivative; and

2. the derivative of a sum of two functions is the same as the sum of the derivatives

\[
\frac{d}{dx}\{x^2\} = 2x \\
\frac{d}{dx}\{9x^2\} = 9(2x) = 18x \\
\frac{d}{dx}\{\sqrt{x}\} = \frac{1}{2\sqrt{x}}
\]
The derivative is a linear operator

The derivative satisfies several convenient properties, among them:

1. a constant multiplying a function can be brought outside the derivative; and
2. the derivative of a sum of two functions is the same as the sum of the derivatives

\[
\frac{d}{dx} \{x^2\} = 2x \\
\frac{d}{dx} \{\sqrt{x}\} = \frac{1}{2\sqrt{x}} \\
\frac{d}{dx} \{9x^2\} = 9(2x) = 18x \\
\frac{d}{dx} \left\{ \frac{\sqrt{x}}{3} \right\} =
\]
The derivative is a linear operator

The derivative satisfies several convenient properties, among them:

1. a constant multiplying a function can be brought outside the derivative; and
2. the derivative of a sum of two functions is the same as the sum of the derivatives

\[
\frac{d}{dx} \{x^2\} = 2x \\
\frac{d}{dx} \{9x^2\} = 9(2x) = 18x \\
\frac{d}{dx} \{\sqrt{x}\} = \frac{1}{2\sqrt{x}} \\
\frac{d}{dx} \{\frac{\sqrt{x}}{3}\} = \frac{1}{3} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{6\sqrt{x}}
\]
The derivative is a linear operator

The derivative satisfies several convenient properties, among them:

1. A constant multiplying a function can be brought outside the derivative; and
2. The derivative of a sum of two functions is the same as the sum of the derivatives

\[
\frac{d}{dx} \{x^2\} = 2x
\]

\[
\frac{d}{dx} \{9x^2\} = 9(2x) = 18x
\]

\[
\frac{d}{dx} \{\sqrt{x}\} = \frac{1}{2\sqrt{x}}
\]

\[
\frac{d}{dx} \left\{\sqrt{\frac{x}{3}}\right\} = \frac{1}{3} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{6\sqrt{x}}
\]

\[
\frac{d}{dx} \left\{5x^{1/5}\right\} =
\]

\[
\frac{d}{dx} \left\{\frac{3}{\sqrt{2x}}\right\} =
\]
The derivative is a linear operator

The derivative satisfies several convenient properties, among them:

1. a constant multiplying a function can be brought outside the derivative; and
2. the derivative of a sum of two functions is the same as the sum of the derivatives

\[
\frac{d}{dx} \{x^2\} = 2x \\
\frac{d}{dx} \{9x^2\} = 9(2x) = 18x \\
\frac{d}{dx} \{\sqrt{x}\} = \frac{1}{2\sqrt{x}} \\
\frac{d}{dx} \left\{ \frac{\sqrt{x}}{3} \right\} = \frac{1}{3} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{6\sqrt{x}} \\
\frac{d}{dx} \left\{ 5x^{1/5} \right\} = 5 \left( \frac{1}{5}x^{-4/5} \right) = x^{-4/5} \\
\frac{d}{dx} \left\{ \frac{3}{\sqrt{2x}} \right\} =
\]
The derivative is a linear operator

The derivative satisfies several convenient properties, among them:

1. a constant multiplying a function can be brought outside the derivative; and

2. the derivative of a sum of two functions is the same as the sum of the derivatives

\[
\frac{d}{dx} \{x^2\} = 2x \\
\frac{d}{dx} \{9x^2\} = 9(2x) = 18x \\
\frac{d}{dx} \{\sqrt{x}\} = \frac{1}{2\sqrt{x}} \\
\frac{d}{dx} \left\{ \frac{\sqrt{x}}{3} \right\} = \frac{1}{3} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{6\sqrt{x}} \\
\frac{d}{dx} \left\{5x^{1/5}\right\} = 5 \left(\frac{1}{5}x^{-4/5}\right) = x^{-4/5} \\
\frac{d}{dx} \left\{ \frac{3}{\sqrt{2x}} \right\} = \frac{d}{dx} \left\{ \frac{3}{\sqrt{2}}x^{-1/2} \right\} = \frac{3}{\sqrt{2}} \cdot \left(-\frac{1}{2}\right)x^{-3/2} = -\frac{3}{2\sqrt{2}x^{3}}
\]
The derivative is a linear operator

The derivative satisfies several convenient properties, among them:

1. a constant multiplying a function can be brought outside the derivative; and
2. the derivative of a sum of two functions is the same as the sum of the derivatives
The derivative is a linear operator

The derivative satisfies several convenient properties, among them:

1. a constant multiplying a function can be brought outside the derivative; and

2. the derivative of a sum of two functions is the same as the sum of the derivatives

\[
\frac{d}{dx} \{x^3 - x^2\} =
\]
The derivative is a linear operator

The derivative satisfies several convenient properties, among them:

1. a constant multiplying a function can be brought outside the derivative; and

2. the derivative of a sum of two functions is the same as the sum of the derivatives

\[
\frac{d}{dx} \{x^3 - x^2\} = 3x^2 - 2x
\]
The derivative is a linear operator

The derivative satisfies several convenient properties, among them:

1. a constant multiplying a function can be brought outside the derivative; and

2. the derivative of a sum of two functions is the same as the sum of the derivatives

\[
\frac{d}{dx} \left\{ x^3 - x^2 \right\} = 3x^2 - 2x
\]

\[
\frac{d}{dx} \left\{ 5x^{-2} + \sqrt{3x} \right\} = \]

The derivative is a linear operator

The derivative satisfies several convenient properties, among them:

1. A constant multiplying a function can be brought outside the derivative; and

2. The derivative of a sum of two functions is the same as the sum of the derivatives.

\[
\frac{d}{dx}\left\{x^3 - x^2\right\} = 3x^2 - 2x \\
\frac{d}{dx}\left\{5x^{-2} + \sqrt{3x}\right\} = 5\frac{d}{dx}\left\{x^{-2}\right\} + \sqrt{3}\frac{d}{dx}\left\{\sqrt{x}\right\} \\
= 10x^{-3} + \frac{\sqrt{3}}{2\sqrt{x}}
\]
Higher Order Derivatives

Example 2: Evaluate \( \frac{d}{dx} \left( \frac{d}{dx} \{ x^5 - 2x^2 + 3 \} \right) \)
Higher Order Derivatives

Example 2: Evaluate \( \frac{d}{dx} \left\{ \frac{d}{dx} \{x^5 - 2x^2 + 3\} \right\} \)
Higher Order Derivatives

Example 2: Evaluate \( \frac{d}{dx} \left( \frac{d}{dx} \{x^5 - 2x^2 + 3\} \right) \)

\[
\frac{d}{dx} \{x^5 - 2x^2 + 3\} = 5x^4 - 4x
\]
Higher Order Derivatives

Example 2: Evaluate \( \frac{d}{dx} \left\{ \frac{d}{dx} \{x^5 - 2x^2 + 3\} \right\} \)

\[
\frac{d}{dx} \{x^5 - 2x^2 + 3\} = 5x^4 - 4x
\]

\[
\frac{d}{dx} \left\{ \frac{d}{dx} \{x^5 - 2x^2 + 3\} \right\} =
\]

Definition

The derivative of a derivative is called the second derivative, written \( f''(x) \), or \( \frac{d^2y}{dx^2} \). Similarly, the derivative of a second derivative is a third derivative, etc.
Higher Order Derivatives

Example 2: Evaluate \( \frac{d}{dx} \left( \frac{d}{dx} \{x^5 - 2x^2 + 3\} \right) \)

\[
\frac{d}{dx} \{x^5 - 2x^2 + 3\} = 5x^4 - 4x
\]

\[
\frac{d}{dx} \left( \frac{d}{dx} \{x^5 - 2x^2 + 3\} \right) = \frac{d}{dx} \{5x^4 - 4x\}
\]
Example 2: Evaluate $\frac{d}{dx} \left( \frac{d}{dx} \{x^5 - 2x^2 + 3\} \right)$

\[
\frac{d}{dx} \{x^5 - 2x^2 + 3\} = 5x^4 - 4x
\]

\[
\frac{d}{dx} \left( \frac{d}{dx} \{x^5 - 2x^2 + 3\} \right) = \frac{d}{dx} \{5x^4 - 4x\} = 20x^3 - 4
\]
Higher Order Derivatives

Example 2: Evaluate \( \frac{d}{dx} \left\{ \frac{d}{dx} \{x^5 - 2x^2 + 3\} \right\} \)

\[
\frac{d}{dx} \{x^5 - 2x^2 + 3\} = 5x^4 - 4x
\]

\[
\frac{d}{dx} \left\{ \frac{d}{dx} \{x^5 - 2x^2 + 3\} \right\} = \frac{d}{dx} \{5x^4 - 4x\}
\]

\[
= 20x^3 - 4
\]

Definition

The derivative of a derivative is called the **second derivative**, written \( f''(x) \) or \( \frac{d^2 y}{dx^2} \).
Higher Order Derivatives

Example 2: Evaluate \( \frac{d}{dx} \left\{ \frac{d}{dx} \{x^5 - 2x^2 + 3\} \right\} \)

\[
\frac{d}{dx} \left\{ \frac{d}{dx} \{x^5 - 2x^2 + 3\} \right\} = \frac{d}{dx} \{5x^4 - 4x\} = 20x^3 - 4
\]

Definition

The derivative of a derivative is called the **second derivative**, written

\[ f''(x) \quad \text{or} \quad \frac{d^2 y}{dx^2} \]

Similarly, the derivative of a second derivative is a third derivative, etc.
Typical Example: Acceleration

- Velocity: rate of change of position

Example 3: The position of an object at time \( t \) is given by \( s(t) = t(5 - t) \).

1. Sketch the graph giving the position of the object.
2. What is the velocity of the object when \( t = 1 \)?
3. What is the acceleration of the object when \( t = 1 \)?
Typical Example: Acceleration

- Velocity: rate of change of position
- Acceleration: rate of change of velocity.
Typical Example: Acceleration

- Velocity: rate of change of position
- Acceleration: rate of change of velocity.

Example 3: The position of an object at time $t$ is given by $s(t) = t(5 - t)$

1. Sketch the graph giving the position of the object.
2. What is the velocity of the object when $t = 1$?
3. What is the acceleration of the object when $t = 1$?
Typical Example: Acceleration

- Velocity: rate of change of position
- Acceleration: rate of change of velocity.

Example 3: The position of an object at time $t$ is given by $s(t) = t(5 - t)$

1. Sketch the graph giving the position of the object.
2. What is the velocity of the object when $t = 1$?
3. What is the acceleration of the object when $t = 1$?
Typical Example: Acceleration

- Velocity: rate of change of position
- Acceleration: rate of change of velocity.

Example 3: The position of an object at time $t$ is given by $s(t) = t(5 - t)$

1. Sketch the graph giving the position of the object.
2. What is the velocity of the object when $t = 1$?
3. What is the acceleration of the object when $t = 1$?

\[
\begin{align*}
  s(t) &= t(5 - t) = 5t - t^2 \\
  s'(t) &= 5 - 2t \\
  s'(1) &= 5 - 2(1) = 3 = \text{vel}
\end{align*}
\]
Typical Example: Acceleration

- Velocity: rate of change of position
- Acceleration: rate of change of velocity.

Example 3: The position of an object at time $t$ is given by $s(t) = t(5 - t)$

1. Sketch the graph giving the position of the object.
2. What is the velocity of the object when $t = 1$?
3. What is the acceleration of the object when $t = 1$?

\[
\begin{align*}
  s(t) &= t(5 - t) = 5t - t^2 \\
  s'(t) &= 5 - 2t \\
  s'(1) &= 5 - 2(1) = 3 = \text{vel} \\
  s''(t) &= -2 \\
  s''(1) &= -2 = \text{acc}
\end{align*}
\]
Concept Check

True or False: If \( f'(1) = 18 \), then \( f''(1) = 0 \), since the derivative of the constant 18 is 0.

Which of the following is always true of a QUADRATIC polynomial \( f(x) \)?

A. \( f(0) = 0 \)
B. \( f'(0) = 0 \)
C. \( f''(0) = 0 \)
D. \( f'''(0) = 0 \)
E. \( f^{(4)}(0) = 0 \)

Which of the following is always true of a CUBIC polynomial \( f(x) \)?

A. \( f(0) = 0 \)
B. \( f'(0) = 0 \)
C. \( f''(0) = 0 \)
D. \( f'''(0) = 0 \)
E. \( f^{(4)}(0) = 0 \)
Concept Check

**True or False:** If $f'(1) = 18$, then $f''(1) = 0$, since the derivative of the constant 18 is 0.
False: for example, $f(x) = 9x^2$.

Which of the following is always true of a QUADRATIC polynomial $f(x)$?

- **A.** $f(0) = 0$
- **B.** $f'(0) = 0$
- **C.** $f''(0) = 0$
- **D.** $f'''(0) = 0$
- **E.** $f^{(4)}(0) = 0$

Which of the following is always true of a CUBIC polynomial $f(x)$?

- **A.** $f(0) = 0$
- **B.** $f'(0) = 0$
- **C.** $f''(0) = 0$
- **D.** $f'''(0) = 0$
- **E.** $f^{(4)}(0) = 0$
Concept Check

**True or False:** If \( f'(1) = 18 \), then \( f''(1) = 0 \), since the derivative of the constant 18 is 0.
False: for example, \( f(x) = 9x^2 \).

Which of the following is always true of a QUADRATIC polynomial \( f(x) \)?

- **A.** \( f(0) = 0 \)
- **B.** \( f'(0) = 0 \)
- **C.** \( f''(0) = 0 \)
- **D.** \( f'''(0) = 0 \)
- **E.** \( f^{(4)}(0) = 0 \)

Which of the following is always true of a CUBIC polynomial \( f(x) \)?

- **A.** \( f(0) = 0 \)
- **B.** \( f'(0) = 0 \)
- **C.** \( f''(0) = 0 \)
- **D.** \( f'''(0) = 0 \)
- **E.** \( f^{(4)}(0) = 0 \)
Concept Check

True or False: If \( f'(1) = 18 \), then \( f''(1) = 0 \), since the derivative of the constant 18 is 0.
False: for example, \( f(x) = 9x^2 \).

Which of the following is always true of a QUADRATIC polynomial \( f(x) \)?

- **A.** \( f(0) = 0 \)
- **B.** \( f'(0) = 0 \)
- **C.** \( f'''(0) = 0 \)
- **D.** \( f''(0) = 0 \)
- **E.** \( f''''(0) = 0 \)

Which of the following is always true of a CUBIC polynomial \( f(x) \)?

- **A.** \( f(0) = 0 \)
- **B.** \( f'(0) = 0 \)
- **C.** \( f''(0) = 0 \)
- **D.** \( f'''(0) = 0 \)
- **E.** \( f''''(0) = 0 \)
**Concept Check**

**True or False:** If \( f'(1) = 18 \), then \( f''(1) = 0 \), since the derivative of the constant 18 is 0.
False: for example, \( f(x) = 9x^2 \).

**Which of the following is always true of a QUADRATIC polynomial \( f(x) \)?**

A. \( f(0) = 0 \)  
B. \( f'(0) = 0 \)  
C. \( f''(0) = 0 \)  
D. \( f'''(0) = 0 \)  
E. \( f^{(4)}(0) = 0 \)

**Which of the following is always true of a CUBIC polynomial \( f(x) \)?**

A. \( f(0) = 0 \)  
B. \( f'(0) = 0 \)  
C. \( f''(0) = 0 \)  
D. \( f'''(0) = 0 \)  
E. \( f^{(4)}(0) = 0 \)
Concept Check

**True or False:** If $f'(1) = 18$, then $f''(1) = 0$, since the derivative of the constant 18 is 0.
False: for example, $f(x) = 9x^2$.

Which of the following is always true of a **QUADRATIC** polynomial $f(x)$?

- A. $f(0) = 0$  
- B. $f'(0) = 0$  
- C. $f''(0) = 0$  
- D. $f'''(0) = 0$  
- E. $f^{(4)}(0) = 0$  

Which of the following is always true of a **CUBIC** polynomial $f(x)$?

- A. $f(0) = 0$  
- B. $f'(0) = 0$  
- C. $f''(0) = 0$  
- D. $f'''(0) = 0$  
- E. $f^{(4)}(0) = 0$
Concept Check

**True or False:** If $f'(1) = 18$, then $f''(1) = 0$, since the derivative of the constant 18 is 0.
False: for example, $f(x) = 9x^2$.

Which of the following is always true of a QUADRATIC polynomial $f(x)$?

- **A.** $f(0) = 0$
- **B.** $f'(0) = 0$
- **C.** $f''(0) = 0$
- **D.** $f'''(0) = 0$
- **E.** $f^{(4)}(0) = 0$

Which of the following is always true of a CUBIC polynomial $f(x)$?

- **A.** $f(0) = 0$
- **B.** $f'(0) = 0$
- **C.** $f''(0) = 0$
- **D.** $f'''(0) = 0$
- **E.** $f^{(4)}(0) = 0$
True or False: If $f'(1) = 18$, then $f''(1) = 0$, since the derivative of the constant 18 is 0.
False: for example, $f(x) = 9x^2$.

Which of the following is always true of a QUADRATIC polynomial $f(x)$?

A. $f(0) = 0$  
B. $f'(0) = 0$  
C. $f''(0) = 0$  
D. $f'''(0) = 0$  
E. $f^{(4)}(0) = 0$

f(x) = ax^2 + bx + c

Which of the following is always true of a CUBIC polynomial $f(x)$?

A. $f(0) = 0$  
B. $f'(0) = 0$  
C. $f''(0) = 0$  
D. $f'''(0) = 0$  
E. $f^{(4)}(0) = 0$

f(x) = ax^3 + bx^2 + cx + d
Concept Check

**True or False:** If \( f'(1) = 18 \), then \( f''(1) = 0 \), since the derivative of the constant 18 is 0.
False: for example, \( f(x) = 9x^2 \).

Which of the following is always true of a **QUADRATIC** polynomial \( f(x) \)?

- **A.** \( f(0) = 0 \)
- **B.** \( f'(0) = 0 \)
- **C.** \( f''(0) = 0 \)
- **D.** \( f'''(0) = 0 \)
- **E.** \( f^{(4)}(0) = 0 \)

\( f(x) = ax^2 + bx + c \)

Which of the following is always true of a **CUBIC** polynomial \( f(x) \)?

- **A.** \( f(0) = 0 \)
- **B.** \( f'(0) = 0 \)
- **C.** \( f''(0) = 0 \)
- **D.** \( f'''(0) = 0 \)
- **E.** \( f^{(4)}(0) = 0 \)

\( f(x) = ax^3 + bx^2 + cx + d \)
Concept Check

**True or False:** If \( f'(1) = 18 \), then \( f''(1) = 0 \), since the derivative of the constant 18 is 0.
False: for example, \( f(x) = 9x^2 \).

Which of the following is always true of a QUADRATIC polynomial \( f(x) \)?

<table>
<thead>
<tr>
<th>Option</th>
<th>Condition</th>
<th>Polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>A.</td>
<td>( f(0) = 0 )</td>
<td>( f(x) = ax^2 + bx + c )</td>
</tr>
<tr>
<td>B.</td>
<td>( f'(0) = 0 )</td>
<td>( f'(x) = 2ax + b )</td>
</tr>
<tr>
<td>C.</td>
<td>( f''(0) = 0 )</td>
<td>( f''(x) = 2a )</td>
</tr>
<tr>
<td>D.</td>
<td>( f'''(0) = 0 )</td>
<td>( f'''(x) = 0 ) ✓</td>
</tr>
<tr>
<td>E.</td>
<td>( f^{(4)}(0) = 0 )</td>
<td>( f^{(4)}(x) = 0 ) ✓</td>
</tr>
</tbody>
</table>

Which of the following is always true of a CUBIC polynomial \( f(x) \)?

<table>
<thead>
<tr>
<th>Option</th>
<th>Condition</th>
<th>Polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>A.</td>
<td>( f(0) = 0 )</td>
<td>( f(x) = ax^3 + bx^2 + cx + d )</td>
</tr>
<tr>
<td>B.</td>
<td>( f'(0) = 0 )</td>
<td>( f'(x) = 3ax^2 + 2bx + c )</td>
</tr>
<tr>
<td>C.</td>
<td>( f''(0) = 0 )</td>
<td>( f''(x) = 6ax + 2b )</td>
</tr>
<tr>
<td>D.</td>
<td>( f'''(0) = 0 )</td>
<td>( f'''(x) = 0 )</td>
</tr>
<tr>
<td>E.</td>
<td>( f^{(4)}(0) = 0 )</td>
<td>( f^{(4)}(x) = 0 )</td>
</tr>
</tbody>
</table>
Chapter 4: Diff rules, simple antiderivatives, applications

4.1 Rules of differentiation

Concept Check

True or False: If $f'(1) = 18$, then $f''(1) = 0$, since the derivative of the constant 18 is 0.
False: for example, $f(x) = 9x^2$.

Which of the following is always true of a QUADRATIC polynomial $f(x)$?

A. $f(0) = 0$  
B. $f'(0) = 0$  
C. $f''(0) = 0$  
D. $f'''(0) = 0$  
E. $f^{(4)}(0) = 0$

Which of the following is always true of a CUBIC polynomial $f(x)$?

A. $f(0) = 0$  
B. $f'(0) = 0$  
C. $f''(0) = 0$  
D. $f'''(0) = 0$  
E. $f^{(4)}(0) = 0$
Chapter 4: Diff rules, simple antiderivatives, applications

4.1 Rules of differentiation

Concept Check

True or False: If \( f'(1) = 18 \), then \( f''(1) = 0 \), since the derivative of the constant 18 is 0.
False: for example, \( f(x) = 9x^2 \).

Which of the following is always true of a QUADRATIC polynomial \( f(x) \)?

A. \( f(0) = 0 \)  
B. \( f'(0) = 0 \)  
C. \( f''(0) = 0 \)  
D. \( f'''(0) = 0 \)  
E. \( f^{(4)}(0) = 0 \)

\( f(x) = ax^2 + bx + c \)

Which of the following is always true of a CUBIC polynomial \( f(x) \)?

A. \( f(0) = 0 \)  
B. \( f'(0) = 0 \)  
C. \( f''(0) = 0 \)  
D. \( f'''(0) = 0 \)  
E. \( f^{(4)}(0) = 0 \)

\( f(x) = ax^3 + bx^2 + cx + d \)
Antiderivatives of Power Functions and Polynomials

Q: Suppose $f'(x) = 2x$. What was $f(x)$?
Antiderivatives of Power Functions and Polynomials

Q: Suppose $f'(x) = 2x$. What was $f(x)$?
A: $f(x) = x^2 + C$, where $C$ is some constant.
Q: Suppose $f'(x) = 2x$. What was $f(x)$?
A: $f(x) = x^2 + C$, where $C$ is some constant.

Definition

An **antiderivative** of a function $f(x)$ is a function whose derivative is $f(x)$. We can only determine antiderivatives up to some additive constant.
Q: Suppose $f'(x) = 2x$. What was $f(x)$?
A: $f(x) = x^2 + C$, where $C$ is some constant.

Definition

An \textbf{antiderivative} of a function $f(x)$ is a function whose derivative is $f(x)$. We can only determine antiderivatives up to some additive constant.
Antiderivatives of Power Functions and Polynomials

**Q:** Suppose $f'(x) = 2x$. What was $f(x)$?

**A:** $f(x) = x^2 + C$, where $C$ is some constant.

**Definition**

An **antiderivative** of a function $f(x)$ is a function whose derivative is $f(x)$. We can only determine antiderivatives up to some additive constant.
Antiderivatives of Power Functions and Polynomials

Q: Suppose $f'(x) = x$. What was $f(x)$?
Antiderivatives of Power Functions and Polynomials

Q: Suppose $f'(x) = x$. What was $f(x)$?

A: $f(x) = \frac{1}{2}x^2 + C$, where $C$ is some constant.

_The antiderivative of $x$ is $\frac{1}{2}x^2 + C$. _
Antiderivatives of Power Functions and Polynomials

Q: Suppose $f'(x) = x$. What was $f(x)$?

A: $f(x) = \frac{1}{2}x^2 + C$, where $C$ is some constant.

The antiderivative of $x$ is $\frac{1}{2}x^2 + C$.

Q: Suppose $g'(x) = 7x^4$. What was $g(x)$?

Q: Suppose $h'(x) = x^n$, $n \neq -1$. What was $h(x)$?
Antiderivatives of Power Functions and Polynomials

Q: Suppose \( f'(x) = x \). What was \( f(x) \)?

A: \( f(x) = \frac{1}{2}x^2 + C \), where \( C \) is some constant.

*The antiderivative of \( x \) is \( \frac{1}{2}x^2 + C \).*

Q: Suppose \( g'(x) = 7x^4 \). What was \( g(x) \)?

A: \( g(x) = \frac{7}{5}x^5 + C \), where \( C \) is some constant.

*The antiderivative of \( 7x^4 \) is \( \frac{7}{5}x^5 + C \).*

Q: Suppose \( h'(x) = x^n \), \( n \neq -1 \). What was \( h(x) \)?
Antiderivatives of Power Functions and Polynomials

Q: Suppose \( f'(x) = x \). What was \( f(x) \)?

A: \( f(x) = \frac{1}{2}x^2 + C \), where \( C \) is some constant.

\text{The antiderivative of } x \text{ is } \frac{1}{2}x^2 + C.

Q: Suppose \( g'(x) = 7x^4 \). What was \( g(x) \)?

A: \( g(x) = \frac{7}{5}x^5 + C \), where \( C \) is some constant.

\text{The antiderivative of } 7x^4 \text{ is } \frac{7}{5}x^5 + C.

Q: Suppose \( h'(x) = x^n \), \( n \neq -1 \). What was \( h(x) \)?

A: \( h(x) = \frac{1}{n+1}x^{n+1} + C \), where \( C \) is some constant.

\text{The antiderivative of } x^n \text{ is } \frac{x^{n+1}}{n+1} + C.
Antiderivatives of Power Functions and Polynomials

The dorsal fin of a breaching orca accelerates at $-10 \text{ m/sec}^2$. 

Antiderivatives of Power Functions and Polynomials

The dorsal fin of a breaching orca accelerates at $-10 \text{ m/sec}^2$.

1. Is the velocity increasing or decreasing?
The dorsal fin of a breaching orca accelerates at \(-10 \text{ m/sec}^2\).

1. Is the velocity increasing or decreasing?

2. Give a general equation for the position of the orca’s dorsal fin relative to time.
Antiderivatives of Power Functions and Polynomials

The dorsal fin of a breaching orca accelerates at $-10 \text{ m/sec}^2$.

1. Is the velocity increasing or decreasing?

2. Give a general equation for the position of the orca’s dorsal fin relative to time.

3. Challenge: If the fin was in the air for 2 seconds, how high did it rise above the water?
The dorsal fin of a breaching orca accelerates at $-10 \text{ m/sec}^2$.

1. Is the velocity increasing or decreasing?
   Decreasing: acceleration is rate of change of velocity. Since acceleration is negative, change in velocity is negative.

2. Give a general equation for the position of the orca’s dorsal fin relative to time.

3. Challenge: If the fin was in the air for 2 seconds, how high did it rise above the water?
Antiderivatives of Power Functions and Polynomials

The dorsal fin of a breaching orca accelerates at $-10 \text{ m/sec}^2$.

1. **Is the velocity increasing or decreasing?**
   Decreasing: acceleration is rate of change of velocity. Since acceleration is negative, change in velocity is negative.

2. **Give a general equation for the position of the orca's dorsal fin relative to time.**
   
   $s''(t) = -10$ \hspace{1cm} $s'(t) = -10t + C$ \hspace{1cm} $s(t) = -5t^2 + Ct + D$

3. **Challenge:** If the fin was in the air for 2 seconds, how high did it rise above the water?
The dorsal fin of a breaching orca accelerates at $-10 \text{ m/sec}^2$.

1. **Is the velocity increasing or decreasing?**
   Decreasing: acceleration is rate of change of velocity. Since acceleration is negative, change in velocity is negative.

2. **Give a general equation for the position of the orca’s dorsal fin relative to time.**
   
   
   
   $s''(t) = -10$  
   $s'(t) = -10t + C$  
   $s(t) = -5t^2 + Ct + D$

3. **Challenge: If the fin was in the air for 2 seconds, how high did it rise above the water?**
   
   
   
   $s \left( \frac{C}{10} \right) - s \left( \frac{C}{10} - 1 \right) = 5 \text{ m}$
Facing a Long Problem

Keep in mind:

- **What (exactly) are you doing?** (Can you describe it precisely?)
- **Why are you doing it?** (How does it fit into the solution?)
- **How does it help you?** (What will you do with the outcome when you obtain it?)

Product and Quotient Rules

- True or false:
  \[ \frac{d}{dx} \{x^2 + x^3\} = \frac{d}{dx} \{x^2\} + \frac{d}{dx} \{x^3\} = 2x + 3x^2 \]

- True or false:
  \[ \frac{d}{dx} \{7x^3\} = \frac{d}{dx} \{7\} \cdot \frac{d}{dx} \{x^3\} = 0 \cdot 3x^2 = 0 \]

- True or false:
  \[ \frac{d}{dx} \{x^2\} = \frac{d}{dx} \{x \cdot x\} = x \cdot \frac{d}{dx} \{x\} = x \cdot 1 = x \]

- True or false:
  \[ \frac{d}{dx} \{x^2\} = \frac{d}{dx} \{x \cdot x\} = \frac{d}{dx} \{x\} \cdot \frac{d}{dx} \{x\} = 1 \cdot 1 = 1 \]
Product and Quotient Rules

- True or false: true
  \[ \frac{d}{dx} \{x^2 + x^3\} = \frac{d}{dx} \{x^2\} + \frac{d}{dx} \{x^3\} = 2x + 3x^2 \]

- True or false: false
  \[ \frac{d}{dx} \{7x^3\} = \frac{d}{dx} \{7\} \cdot \frac{d}{dx} \{x^3\} = 0 \cdot 3x^2 = 0 \]

- True or false: false
  \[ \frac{d}{dx} \{x^2\} = \frac{d}{dx} \{x \cdot x\} = x \cdot \frac{d}{dx} \{x\} = x \cdot 1 = x \]

- True or false: false
  \[ \frac{d}{dx} \{x^2\} = \frac{d}{dx} \{x \cdot x\} = \frac{d}{dx} \{x\} \cdot \frac{d}{dx} \{x\} = 1 \cdot 1 = 1 \]
Where the Product Rule Comes From

You won’t be assessed on this—it’s only for background knowledge

\[
\frac{d}{dx}\{f(x)g(x)\} = \lim_{h \to 0} \frac{f(x + h)g(x + h) - f(x)g(x)}{h}
\]
Where the Product Rule Comes From

You won’t be assessed on this—it’s only for background knowledge

\[
\frac{d}{dx} \{ f(x)g(x) \} = \lim_{h \to 0} \frac{f(x + h)g(x + h) - f(x)g(x)}{h} 
\]

(def of deriv)
Where the Product Rule Comes From

You won’t be assessed on this—it’s only for background knowledge

\[
\frac{d}{dx} \{f(x)g(x)\} = \lim_{h \to 0} \frac{f(x + h)g(x + h) - f(x)g(x)}{h} \tag{def of deriv}
\]

\[
= \lim_{h \to 0} \left[ f(x + h) \frac{g(x + h) - g(x)}{h} + g(x) \frac{f(x + h) - f(x)}{h} \right]
\]

fancy algebra
Where the Product Rule Comes From

You won’t be assessed on this—it’s only for background knowledge

\[
\frac{d}{dx} \left\{ f(x)g(x) \right\} = \lim_{h \to 0} \frac{f(x + h)g(x + h) - f(x)g(x)}{h} \quad \text{(def of deriv)}
\]

\[
= \lim_{h \to 0} \left[ f(x + h) \frac{g(x + h) - g(x)}{h} + g(x) \frac{f(x + h) - f(x)}{h} \right]
\]
Where the Product Rule Comes From

You won’t be assessed on this—it’s only for background knowledge

\[
\frac{d}{dx} \{f(x)g(x)\} = \lim_{h \to 0} \frac{f(x + h)g(x + h) - f(x)g(x)}{h}
\]

(def of deriv)
Where the Product Rule Comes From

You won’t be assessed on this—it’s only for background knowledge

\[
\frac{d}{dx} \{ f(x)g(x) \} = \lim_{h \to 0} \frac{f(x + h)g(x + h) - f(x)g(x)}{h}
\]

(def of deriv)
Where the Product Rule Comes From

You won't be assessed on this—it's only for background knowledge

\[
\frac{d}{dx} \{ f(x)g(x) \} = \lim_{h \to 0} \frac{f(x + h)g(x + h) - f(x)g(x)}{h} \quad \text{(def of deriv)}
\]

\[
= \lim_{h \to 0} \left[ f(x + h) \frac{g(x + h) - g(x)}{h} + g(x) \frac{f(x + h) - f(x)}{h} \right]
\]

\[
= f(x)g'(x) + g(x)f'(x)
\]
Product and Quotient Rules

\[ f(x) = (x^2 + 5)(x^6 - x^2 + 3x) \]
Product and Quotient Rules

\[ f(x) = (x^2 + 5)(x^6 - x^2 + 3x) \]

To differentiate, we could expand, or we could use the product rule.
Product and Quotient Rules

\[ f(x) = (x^2 + 5)(x^6 - x^2 + 3x) \]

To differentiate, we could expand, or we could use the product rule.

\[ f'(x) = (x^2 + 5)(6x^5 - 2x + 3) + (x^6 - x^2 + 3x)(2x) \]
Where the Quotient Rule Comes From

You won’t be assessed on this—it’s only for background knowledge

\[
\frac{d}{dx} \left\{ \frac{f(x)}{g(x)} \right\} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{g(x+h) - g(x)} \quad \text{(def of deriv)}
\]
Where the Quotient Rule Comes From

You won’t be assessed on this—it’s only for background knowledge

\[
\frac{d}{dx} \left\{ \frac{f(x)}{g(x)} \right\} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{(def of deriv)}
\]

\[
= \lim_{h \to 0} \frac{1}{h} \left[ f(x + h)g(x) - f(x)g(x + h) \right]
\]

\[
= \lim_{h \to 0} \frac{1}{h} \left[ f(x + h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x + h) \right]
\]

\[
= \lim_{h \to 0} \left[ g(x) \left( \frac{f(x+h)-f(x)}{h} \right) - f(x) \left( \frac{g(x+h)-g(x)}{h} \right) \right]
\]

very fancy algebra indeed
Where the Quotient Rule Comes From

You won’t be assessed on this—it’s only for background knowledge.

\[
\frac{d}{dx} \left\{ \frac{f(x)}{g(x)} \right\} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{g(x+h) - g(x)} \quad \text{(def of deriv)}
\]

very fancy algebra indeed

\[
= \lim_{h \to 0} \frac{1}{h} \left[ \frac{(f(x+h)g(x) - f(x)g(x+h))}{g(x+h)g(x)} \right]
\]

\[
= \lim_{h \to 0} \frac{1}{h} \left[ \frac{(f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h))}{g(x+h)g(x)} \right]
\]

\[
= \lim_{h \to 0} \left[ \frac{g(x) \left( \frac{f(x+h) - f(x)}{h} \right)}{g(x+h)g(x)} - f(x) \left( \frac{g(x+h) - g(x)}{h} \right) \right]
\]

\[
= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}
\]
Product and Quotient Rules

Theorem

If \( f(x) \) and \( g(x) \) are differentiable, and \( g(x) \neq 0 \), then:

\[
\frac{d}{dx} \{ f(x)g(x) \} = f(x)g'(x) + g(x)f'(x)
\]

Product rule

\[
\frac{d}{dx} \left\{ \frac{f(x)}{g(x)} \right\} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}
\]

Quotient rule

Find the derivatives of the following functions:

\[
f(x) = \frac{x^2 + 1}{x}
\]

\[
f'(x) = \frac{(x+1)^2}{x} - \frac{x^2}{(x+1)^2}
\]

\[
g(x) = (x^3 + 1)(x^2 + 2)
\]

\[
g'(x) = (x^3 + 1)(2x) + (x^2 + 2)(3x^2)
\]

\[
g'(x) = 5x^4 + 6x^2 + 2x
\]
Product and Quotient Rules

Theorem
If \( f(x) \) and \( g(x) \) are differentiable, and \( g(x) \neq 0 \), then:

\[
\frac{d}{dx} \{ f(x)g(x) \} = f(x)g'(x) + g(x)f'(x)
\]

\[
\frac{d}{dx} \left\{ \frac{f(x)}{g(x)} \right\} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}
\]

Mnemonic: \textit{low d-high minus high d-low over low-low}
Product and Quotient Rules

Theorem

If $f(x)$ and $g(x)$ are differentiable, and $g(x) \neq 0$, then:

1. \[ \frac{d}{dx}\{f(x)g(x)\} = f(x)g'(x) + g(x)f'(x) \] product rule

2. \[ \frac{d}{dx}\left\{ \frac{f(x)}{g(x)} \right\} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} \] quotient rule

Mnemonic: low d-high minus high d-low over low-low

Find the derivatives of the following functions:

\[ f(x) = \frac{x^2}{x + 1} \]
\[ g(x) = (x^3 + 1)(x^2 + 2) \]


### Product and Quotient Rules

**Theorem**

If \( f(x) \) and \( g(x) \) are differentiable, and \( g(x) \neq 0 \), then:

- \[
  \frac{d}{dx} \{ f(x)g(x) \} = f(x)g'(x) + g(x)f'(x)
\]
  **product rule**

- \[
  \frac{d}{dx} \left\{ \frac{f(x)}{g(x)} \right\} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}
\]
  **quotient rule**

Mnemonic: \textit{low d-high minus high d-low over low-low}

Find the derivatives of the following functions:

\[
\begin{align*}
f(x) &= \frac{x^2}{x + 1} & f'(x) &= \frac{(x + 1)2x - x^2(1)}{(x + 1)^2} = \frac{x^2 + 2x}{(x + 1)^2} \\
g(x) &= (x^3 + 1)(x^2 + 2)
\end{align*}
\]
Chapter 4: Diff rules, simple antiderivatives, applications

4.1 Rules of differentiation

Product and Quotient Rules

**Theorem**

If \( f(x) \) and \( g(x) \) are differentiable, and \( g(x) \neq 0 \), then:

\[
\frac{d}{dx} \{ f(x)g(x) \} = f(x)g'(x) + g(x)f'(x) \quad \text{product rule}
\]

\[
\frac{d}{dx} \left\{ \frac{f(x)}{g(x)} \right\} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} \quad \text{quotient rule}
\]

Mnemonic: *low d-high minus high d-low over low-low*

Find the derivatives of the following functions:

\[
f(x) = \frac{x^2}{x + 1} \quad f'(x) = \frac{(x + 1)2x - x^2(1)}{(x + 1)^2} = \frac{x^2 + 2x}{(x + 1)^2}
\]

\[
g(x) = (x^3 + 1)(x^2 + 2) \quad g'(x) = (x^3 + 1)(2x) + (x^2 + 2)(3x^2) = 5x^4 + 6x^2 + 2x
\]
Intuition: $\sin x$ versus $\sin(2x)$

$f(x) = \sin x$

$g(x) = \sin(2x)$

$g'(x) = 2 \cos(2x)$
Intuition: $\sin x$ versus $\sin(2x)$

$\sin x$ and $\sin(2x)$ graphs are shown. The graph of $f(x) = \sin x$ is a wave that increases and decreases with amplitude 1. The graph of $g(x) = \sin(2x)$ is a wave that increases and decreases more rapidly, with amplitude 1, and is plotted with a green line.

$f(x) = \sin x$

$g(x) = \sin(2x)$
Intuition: \( \sin x \) versus \( \sin(2x) \)

\[
f(x) = \sin x \quad \Rightarrow \quad f'(x) = \cos x
\]

\[
g(x) = \sin(2x) \quad \Rightarrow \quad g'(x) = 2 \cos(2x)
\]
Intuition: $\sin x$ versus $\sin(2x)$

$f(x) = \sin x$

$g(x) = \sin(2x)$
Intuition: $\sin x$ versus $\sin(2x)$

$f(x) = \sin x$
$f'(x) = \cos x$

$g(x) = \sin(2x)$
Intuition: $\sin x$ versus $\sin(2x)$

- $f(x) = \sin x$
  - $f'(x) = \cos x$

- $g(x) = \sin(2x)$
  - $g'(x) = 2\cos(2x)$
Compound Functions

Video: 2:27-3:50

Photo: Mike Baird
Kelp Population

\[ k = k(u(o(p))) \]

This is an example of a compound function.

Should \(k'(o)\) be positive or negative?
A. positive  
B. negative  
C. I'm not sure

Should \(k'(u)\) be positive or negative?
A. positive  
B. negative  
C. I'm not sure
Kelp Population

\[ k = k(u) \]

\[ k = k(u) \cdot u \cdot o \cdot p \]

This is an example of a compound function.

Should \( k'(o) \) be positive or negative?

A. positive  
B. negative  
C. I'm not sure

Should \( k'(u) \) be positive or negative?

A. positive  
B. negative  
C. I'm not sure
Kelp Population

\[ k \quad \text{kelp population} \]
\[ u \quad \text{urchin population} \]
\[ o \quad \text{otter population} \]

\[ k = k(u) = k(u(o)) \]

This is an example of a compound function. Should \( k'(o) \) be positive or negative?
A. positive  
B. negative  
C. I'm not sure

Should \( k'(u) \) be positive or negative?
A. positive  
B. negative  
C. I'm not sure
Kelp Population

$k$ kelp population
$u$ urchin population
$o$ otter population
$p$ public policy
$c$ carbon

\[ k = k(u) = k(u(o)) = k(u(o(p))) \]
\[ c = c(k(u(o(p)))) \]
Kelp Population

\[ k = k(u) = k(u(o)) = k(u(o(p))) \]
\[ c = c(k(u(o(p)))) \]

This is an example of a compound function.

Should \( k'(o) \) be positive or negative?
A. positive  
B. negative  
C. I'm not sure

Should \( k'(u) \) be positive or negative?
A. positive  
B. negative  
C. I'm not sure
Kelp Population

\[ k = k(u) = k(u(o)) = k(u(o(p))) \]
\[ c = c(k(u(o(p)))) \]

This is an example of a compound function.

Should \( k'(o) \) be positive or negative?
A. positive  B. negative  C. I’m not sure
Kelp Population

\[ k \quad \text{kelp population} \]
\[ u \quad \text{urchin population} \]
\[ o \quad \text{otter population} \]
\[ p \quad \text{public policy} \]
\[ c \quad \text{carbon} \]

\[ k = k(u) = k(u(o)) = k(u(o(p))) \]
\[ c = c(k(u(o(p)))) \]

This is an example of a compound function.

Should \( k'(o) \) be positive or negative?
A. positive \quad B. negative \quad C. I’m not sure
Kelp Population

\[ k = k(u) = k(u(o)) = k(u(o(p))) \]
\[ c = c(k(u(o(p)))) \]

This is an example of a compound function.

Should \( k'(o) \) be positive or negative?
A. positive \hspace{1cm} B. negative \hspace{1cm} C. I’m not sure

Should \( k'(u) \) be positive or negative?
A. positive \hspace{1cm} B. negative \hspace{1cm} C. I’m not sure
Kelp Population

\[ k \quad \text{kelp population} \]
\[ u \quad \text{urchin population} \]
\[ o \quad \text{otter population} \]
\[ p \quad \text{public policy} \]
\[ c \quad \text{carbon} \]

\[ k = k(u) = k(u(o)) = k(u(o(p))) \]
\[ c = c(k(u(o(p)))) \]

This is an example of a compound function.

Should \( k'(o) \) be positive or negative?
A. positive  B. negative  C. I’m not sure

Should \( k'(u) \) be positive or negative?
A. positive  B. negative  C. I’m not sure
Chain Rule

Suppose \( f \) and \( g \) are differentiable functions. Then

\[
\frac{d}{dx} \{ f(g(x)) \} = f'(g(x))g'(x) = \frac{df}{dg} \frac{dg}{dx}
\]
Chain Rule

Suppose $f$ and $g$ are differentiable functions. Then

$$\frac{d}{dx} \left\{ f \left( g(x) \right) \right\} = f' \left( g(x) \right) = \frac{df}{dg} \frac{dg}{dx}$$
Chain Rule

Suppose $f$ and $g$ are differentiable functions. Then

$$
\frac{d}{dx} \left\{ f \left( g(x) \right) \right\} = f' \left( g(x) \right) = \frac{df}{dg} \frac{dg}{dx}
$$
Chain Rule

Suppose $f$ and $g$ are differentiable functions. Then

\[
\frac{d}{dx} \left\{ f \left( g(x) \right) \right\} = f' \left( g(x) \right) g'(x) = \frac{df}{dg} \frac{dg}{dx}
\]
Chain Rule

Suppose $f$ and $g$ are differentiable functions. Then

$$
\frac{d}{dx} \left\{ f \left( g(x) \right) \right\} = f' \left( g(x) \right) g'(x) = \frac{df}{dg} \frac{dg}{dx}
$$

In the case of kelp, \( \frac{dk}{do} = \frac{dk}{du} \frac{du}{do} \)
Chain Rule

Chain Rule

Suppose \( f \) and \( g \) are differentiable functions. Then

\[
\frac{d}{dx} \left\{ f \left( g(x) \right) \right\} = f' \left( g(x) \right) g'(x) = \frac{df}{dg} \frac{dg}{dx}
\]

\[
F(v) = \left( \frac{v}{v^3 + 1} \right)^6
\]
4.1 Rules of differentiation

Chain Rule

Suppose \( f \) and \( g \) are differentiable functions. Then

\[
\frac{d}{dx} \left\{ f \left( g(x) \right) \right\} = f' \left( g(x) \right) \cdot g'(x) = \frac{df}{dg} \cdot \frac{dg}{dx}
\]

\[
F(v) = \left( \frac{v}{v^3 + 1} \right)^6
\]

\[
F'(v) = 6 \left( \frac{v}{v^3 + 1} \right)^5 \cdot \frac{(v^3 + 1)(1) - (v)(3v^2)}{(v^3 + 1)^2}
\]

\[
= 6 \left( \frac{v}{v^3 + 1} \right)^5 \cdot \frac{-2v^3 + 1}{(v^3 + 1)^2}
\]
Chain Rule Practice

1. Differentiate \( f(x) = (x^2 + x^3)^4 \)

2. Differentiate \( g(x) = \sqrt{\frac{x}{x + 1}} \)

3. Antidifferentiate \( h'(x) = 5(x^3 + x^2)^4(3x^2 + 2x) \)
Chapter 4: Diff rules, simple antiderivatives, applications

4.1 Rules of differentiation

Chain Rule Practice

1. Differentiate $f(x) = (x^2 + x^3)^4$

   $f'(x) = 4(x^2 + x^3)^3(2x + 3x^2)$

2. Differentiate $g(x) = \sqrt{\frac{x}{x + 1}}$

3. Antidifferentiate $h'(x) = 5(x^3 + x^2)^4(3x^2 + 2x)$
Chain Rule Practice

1. Differentiate $f(x) = (x^2 + x^3)^4$
   
   $f'(x) = 4(x^2 + x^3)^3(2x + 3x^2)$

2. Differentiate $g(x) = \sqrt{\frac{x}{x+1}}$
   
   $g'(x) = \frac{1}{2} \left( \frac{x}{x+1} \right)^{-1/2} \left( \frac{(x+1)(1)-(x)(1)}{(x+1)^2} \right) = \frac{1}{2} \sqrt{\frac{x+1}{x}} \cdot \frac{1}{(x+1)^2}$

3. Antidifferentiate $h'(x) = 5(x^3 + x^2)^4(3x^2 + 2x)$
Chain Rule Practice

1. Differentiate $f(x) = (x^2 + x^3)^4$

   $f'(x) = 4 (x^2 + x^3)^3 (2x + 3x^2)$

2. Differentiate $g(x) = \sqrt{\frac{x}{x+1}}$

   $g'(x) = \frac{1}{2} \left( \frac{x}{x+1} \right)^{-1/2} \left( \frac{(x+1)(1)-(x)(1)}{(x+1)^2} \right) = \frac{1}{2} \sqrt{\frac{x+1}{x}} \cdot \frac{1}{(x+1)^2}$

3. Antidifferentiate $h'(x) = 5(x^3 + x^2)^4(3x^2 + 2x)$

   By inspection, $h(x) = (x^3 + x^2)^5$
Recall

The derivative of velocity with respect to time is acceleration.
Acceleration

Recall
The derivative of velocity with respect to time is acceleration.

An object falling has a constant (negative) acceleration $A$ due to gravity.

$$s''(t) = A$$

What is the equation of its position?
Recall
The derivative of velocity with respect to time is acceleration.

An object falling has a constant (negative) acceleration $A$ due to gravity.

$$s''(t) = A$$

What is the equation of its position?

$$s'(t) = At + B$$

$$s(t) = \frac{1}{2}At^2 + Bt + C$$

When $t = 0$, what is the object’s position? Its velocity?
## Acceleration

### Recall

The derivative of velocity with respect to time is acceleration.

An object falling has a constant (negative) acceleration $A$ due to gravity.

$$s''(t) = A$$

What is the equation of its position?

$$s'(t) = At + B$$

$$s(t) = \frac{1}{2}At^2 + Bt + C$$

When $t = 0$, what is the object's position? Its velocity?

Velocity: $B$  
Position: $C$
Position

If an object accelerates at a constant rate $a$, then its position is given by

$$s(t) = \frac{1}{2} at^2 + v_0 t + s_0$$

where $v_0$ is its velocity at time $t = 0$ and $s_0$ is its position at time $t = 0$. 

**Example 4:**

(a) You toss a ball in the air with an initial upwards velocity of 3 metres per second. Assuming the acceleration due to gravity is a constant $a = -9 \text{ m/sec}^2$, how long is the ball in the air before it falls back in your hand?

(b) You toss a ball in the air and the acceleration due to gravity is constant. If the ball falls back into your hand two seconds later, at what time did the ball reach its highest point?
Chapter 4: Diff rules, simple antiderivatives, applications

4.2 Application of the second derivative to acceleration

Acceleration

Position

If an object accelerates at a constant rate \( a \), then its position is given by

\[
s(t) = \frac{1}{2} at^2 + v_0 t + s_0
\]

where \( v_0 \) is its velocity at time \( t = 0 \) and \( s_0 \) is its position at time \( t = 0 \).

Example 4:

(a) You toss a ball in the air with an initial upwards velocity of 3 metres per second. Assuming the acceleration due to gravity is a constant \( a = -9 \) m/sec\(^2\), how long is the ball in the air before it falls back in your hand?
(b) You toss a ball in the air and the acceleration due to gravity is constant. If the ball falls back into your hand two seconds later, at what time did the ball reach its highest point?
Acceleration

Solution 4:

(a) You toss a ball in the air with an initial upwards velocity of 3 metres per second. Assuming the acceleration due to gravity is a constant $a = -9 \text{ m/sec}^2$, how long is the ball in the air before it falls back in your hand?

If your hand is at height 0, and you throw the ball at time $t = 0$, then $s(t) = -\frac{9}{2} t^2 + 3t = t \left( 3 - \frac{9}{2} t \right)$, so the ball is back at your hand when $0 = \left( 3 - \frac{9}{2} t \right)$, i.e. at time $t = \frac{2}{3}$. The ball is in the air for two-thirds of a second.

(b) You toss a ball in the air and the acceleration due to gravity is constant. If the ball falls back into your hand two seconds later, at what time did the ball reach its highest point?

If $a$ is constant, then the position function is a parabola, and parabolas are symmetric, so it reached its highest point after 1 second.
Example 5:

A frog leaps up from its lily pad, reaches its highest point, then falls back down, landing on a flower that is 50 cm higher than the lily pad. If the frog was in the air for a total of one second, and its vertical acceleration was a constant $-10 \text{ m/sec}^2$, what was its velocity upon takeoff? How high did it leap?
The position of the frog is

\[ s(t) = \frac{1}{2}(-10)t^2 + v_0 t = -5t^2 + v_0 t, \]

if we take its initial height to be 0.
After one second, its height is

\[ s(1) = -5 + v_0 t = \frac{1}{2} \]

Solving, we see its initial velocity was

\[ v_0 = 5.5 \text{ m/sec} \]

To find its maximum height, we observe that this happens when its velocity is zero. Differentiating,

\[ s'(t) = -10t + v_0 = -10t + 5.5 \]

So, its velocity is zero when \( t = 0.55 \). At this point, its height is

\[ s(0.55) = -5(0.55)^2 + 5.5(0.55) = 1.5125 \text{ m} \]
Sketching Antiderivatives

\[ y = f'(x) \]
Sketching Antiderivatives
Sketching Antiderivatives

\[ y = f'(x) \]
Sketching Antiderivatives
Sketching Antiderivatives

$y = f'(x)$
Sketching Antiderivatives
Sketching Antiderivatives

$y = f'(x)$
Sketching Antiderivatives

\[ y = f'(x) \]
Sketching Antiderivatives

\[ y = f'(x) \]
Sketching Antiderivatives

The graph shows a plot of the function $y = f'(x)$, where $f'$ is the first derivative of the function $f$. The graph illustrates how the antiderivative $f$ can be sketched based on the behavior of $f'$. The slope of $f'$ at any point $x$ indicates the rate of change of $f$ at that point.
Sketching Antiderivatives

\[ y = f'(x) \]
Sketching Antiderivatives

\[ y = f'(x) \]
Sketching Antiderivatives

\[ y = f'(x) \]
Sketching Antiderivatives

$y = f'(x)$
Sketching Antiderivatives

$y = f'(x)$
4.3 Sketching first and second derivatives, antiderivatives

Sketching Antiderivatives

$y = f'(x)$
Sketching Antiderivatives

\[ y = f'(x) \]
Sketching Antiderivatives

\[ y = f'(x) \]
Sketching Antiderivatives

$y = f'(x)$
Sketching Antiderivatives

\[ y = f'(x) \]
Sketching Antiderivatives

The graph shows a function $y = f'(x)$, which represents the first derivative of another function $f(x)$. The antiderivative of $f'(x)$ is the original function $f(x)$. The graph illustrates how the shape of the function changes as we move along the x-axis, with the y-values corresponding to $f'(x)$ at various points. The red dotted lines indicate points of interest on the graph, helping to visualize the behavior of the function and its derivative.
Sketching Antiderivatives

\[ y = f'(x) \]
Sketching Antiderivatives

\[ y = f'(x) \]
Sketching Antiderivatives

\[ y = f'(x) \]
Sketching Antiderivatives

\[ y = f'(x) \]
Sketching Antiderivatives

\[ y = f'(x) \]
Sketching Antiderivatives

\[ y = f'(x) \]
Sketching Antiderivatives

$y = f'(x)$
**Sketching Antiderivatives**

Given a function $y = f'(x)$, the antiderivative $y = f(x)$ can be sketched by considering the behavior of $f'(x)$. If $f'(x)$ is positive, $f(x)$ is increasing; if $f'(x)$ is negative, $f(x)$ is decreasing. The graph of $y = f'(x)$ indicates the slope of $f(x)$ at various points.

The graph shows that as $f'(x)$ decreases from positive to negative, $f(x)$ changes from increasing to decreasing. This change in slope is reflected in the antiderivative graph, illustrating the relationship between derivatives and antiderivatives.
Sketching Antiderivatives

\[ y = f'(x) \]
Sketching Antiderivatives
Sketching Antiderivatives

\[ y = f'(x) \]
Sketching Antiderivatives
Sketching Antiderivatives

\[ y = f'(x) \]
Sketching Antiderivatives

$y = f'(x)$

$y = g'(x)$
Sketching Antiderivatives

\[ y = f'(x) \]
Sketching Antiderivatives

\[ y = f'(x) \]
Sketching Antiderivatives
Sketching Antiderivatives
Sketching Antiderivatives

\[ y = f'(x) \]
Sketching Antiderivatives

\[
y = f'(x)
\]
Sketching Antiderivatives

$y = f'(x)$
Sketching Antiderivatives
Sketching Antiderivatives

\[ y = f'(x) \]
Sketching Antiderivatives

\[ y = f'(x) \]
Sketching Antiderivatives

\[ y = f'(x) \]
Sketching Antiderivatives

\[ y = f'(x) \]
Sketching Antiderivatives

\[ y = g'(x) \]
Sketching Antiderivatives

$y = g'(x)$
Sketching Antiderivatives

\[ y = g'(x) \]
Sketching Antiderivatives

$y = g'(x)$
Sketching Antiderivatives

$y = g'(x)$
Sketching Antiderivatives

$y = g'(x)$
Sketching Antiderivatives

\[ y = g'(x) \]
Sketching Antiderivatives
Sketching Antiderivatives

\[ y = g'(x) \]
Sketching Antiderivatives

\[ y = g'(x) \]
Sketching Antiderivatives

\[ y = g'(x) \]
Sketching Antiderivatives

\[ y = g'(x) \]
Sketching Antiderivatives

\[ y = g'(x) \]
Sketching Antiderivatives

\[ y = g'(x) \]

Andy Warhol, Wallpaper Cow
Sketching Antiderivatives

\[ y = f'(x) \]
\[ y = g'(x) \]
Sketching Antiderivatives

$y = f(x)$

$y = f'(x)$

$y = g'(x)$

$y = g(x)$