Overview

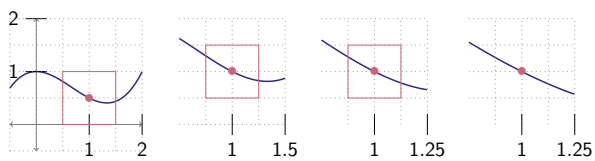
We already saw an algebraic way of thinking about a derivative.

- **Geometric**: zooming into a line
- **Analytic**: continuity and rational functions
- **Computational:** approximations with computers

Notes

Zooming In

For a smooth function, if we zoom in at a point, we see a line:



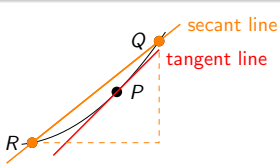
In this example, the slope of our zoomed-in line looks to be about:

$$\frac{\Delta y}{\Delta x} \approx -\frac{1}{2}$$

Notes

Recall

The **secant line** to the curve $y = f(x)$ through points R and Q is a line that passes through R and Q.
We call the slope of the secant line the **average rate of change of** $f(x)$ **from** R **to** Q**.**



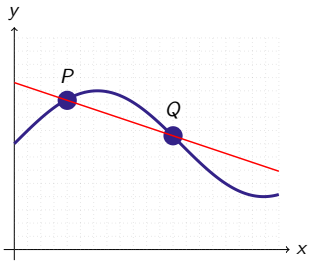
Definition

The straight line that we see when we zoom into the graph of a smooth function at some point P is called the **tangent line** at P.

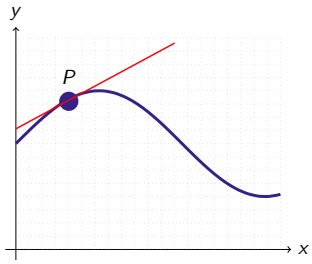
The slope of the tangent line is the **instantaneous rate of change (derivative)**
of $f(x)$ at P.

Notes

On the graph below, draw the secant line to the curve through points P and Q.



On the graph below, draw the tangent line to the curve at point P.

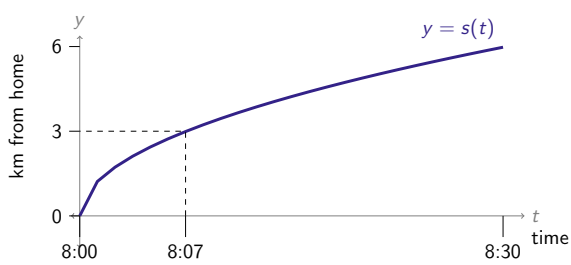


Notes

Derivatives of Familiar Functions

1. The derivative of the function $f(x) = Ax$ is:
   (a) 0
   (b) 1
   (c) A
   (d) x
   (e) Ax
2. The derivative of the function $f(x) = A$ is:
   (a) 0
   (b) 1
   (c) A
   (d) x
   (e) Ax
3. The derivative of the function $f(x) = A + x$ is:
   (a) 0
   (b) 1
   (c) A
   (d) x
   (e) Ax

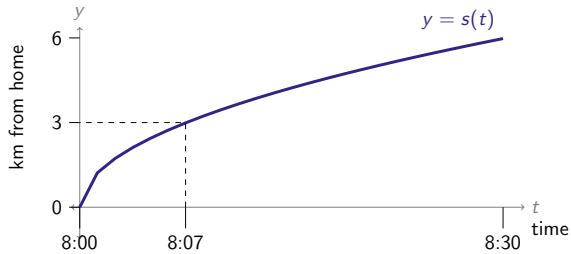
Notes



It takes me half an hour to bike 6 km. So, 12 kph represents the:

A. secant line to $y = s(t)$ from $t = 8:00$ to $t = 8:30$
B. slope of the secant line to $y = s(t)$ from $t = 8:00$ to $t = 8:30$
C. tangent line to $y = s(t)$ at $t = 8:30t = 8:25$
D. slope of the tangent line to $y = s(t)$ at $t = 8:30$

Notes



When I pedal up Crescent Rd at 8:25, the speedometer on my bike tells me I'm going 5kph. This corresponds to the:

A. secant line to $y = s(t)$ from $t = 8:00$ to $t = 8:25$
B. slope of the secant line to $y = s(t)$ from $t = 8:00$ to $t = 8:25$
C. tangent line to $y = s(t)$ at $t = 8:25$
D. slope of the tangent line to $y = s(t)$ at$t = 8:25$

Notes

Zooming in on functions that aren't smooth

For a function with a cusp or a discontinuity, even though we zoom in very closely, we don't see simply a single straight line.

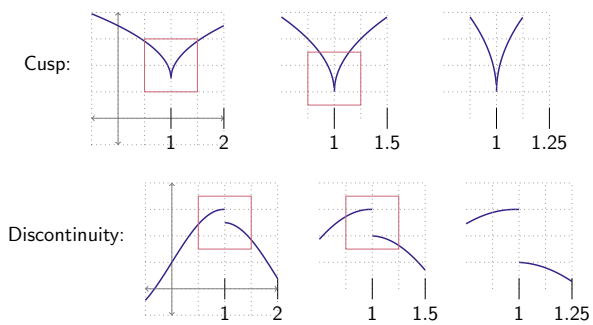
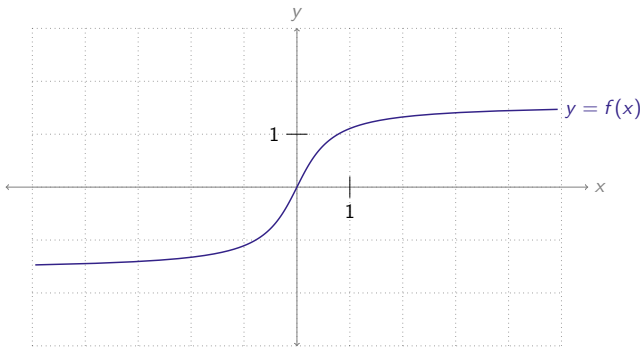


Photo credit: Pixabay

Notes

Sketch a derivative

Sketch the derivative of the function shown.



Notes

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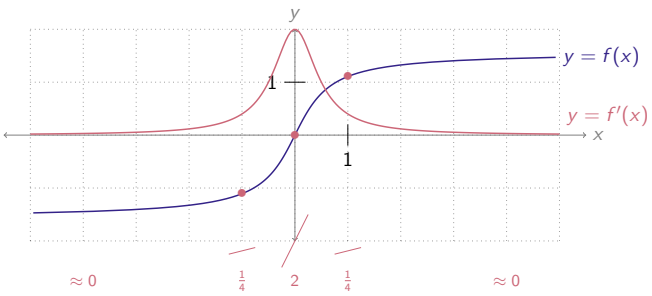
Sketch the derivative of the function shown.

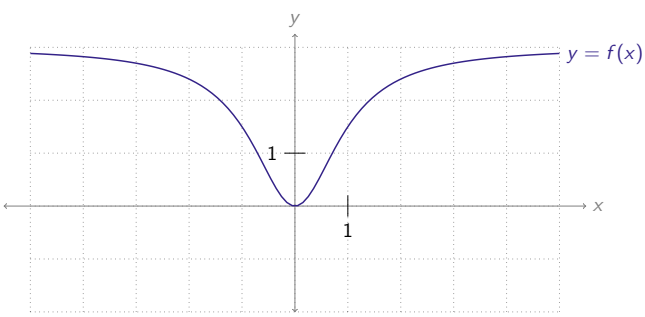


Notes

Chapter 3: Three faces of the derivative 3.1 The geometric view: zooming into the graph of a function

Sketch a derivative

Sketch the derivative of the function shown.

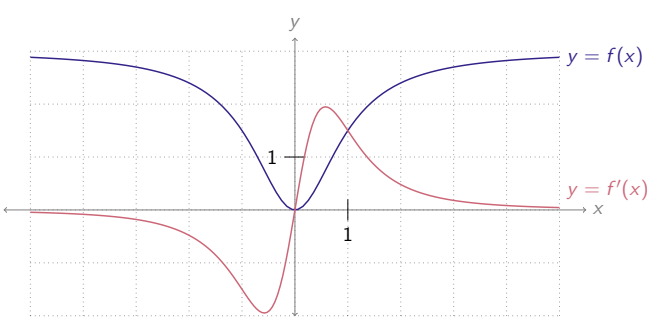


Notes

Chapter 3: Three faces of the derivative 3.1 The geometric view: zooming into the graph of a function

Sketch a derivative

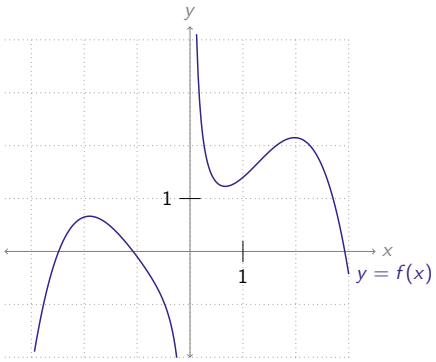
Sketch the derivative of the function shown.



Notes

Sketch a derivative

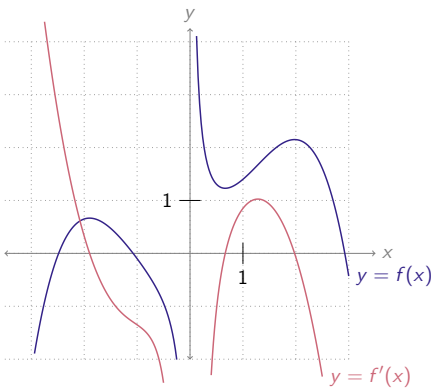
Sketch the derivative of the function shown.



Notes

Sketch a derivative

Sketch the derivative of the function shown.



Notes

Sketch a derivative

Sketch the approximate behaviour of the function, then use your sketch of the function to sketch its derivative.

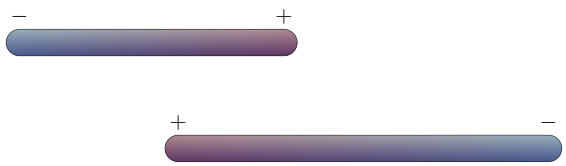
$$f(x) = \frac{-5x^2}{10 + x^2}$$

Notes

Kinesin

Kinesin: link

- Kinesin transports vesicles along *microtubules*, which have a plus and minus end
- Kinesin only travels towards the plus end
- Kinesin can hop on and off microtubules

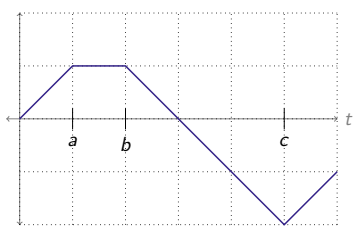


More explanation of Kinesin's walking: link

Notes

Kinesin

Below is a sketch of the position of a kinesin molecule over time, as it travels around a network of microtubules of fixed position.

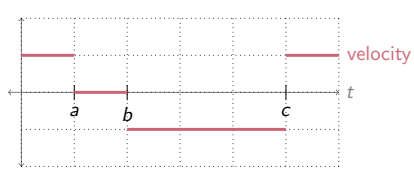


1. Sketch the velocity of the kinesin molecule over the same time period.
2. Describe what happened to the kinesin molecule.

Notes

Kinesin

Below is a sketch of the position of a kinesin molecule over time, as it travels around a network of microtubules of fixed position.



$(0, a)$ The molecule travels along a microtubule at a velocity of 1 in the positive direction.

(a, b) Then, it falls off or gets stuck.

(b, c) Then it travels on along a mictorubule with the *opposite orientation*, so it moves in the negative direction.

$t > c$ Finally, it hops onto another microtubule (possibly the same as the first) and travels in the positive direction.

Notes

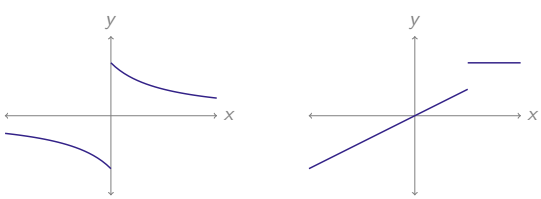
Overview

- Explain the definition of a continuous function.
- Identify functions with various types of discontinuities.
- Evaluate simple limits of rational functions.
- Calculate the derivative of a simple function using the definition of the derivative

Notes

Intuitive Continuity

Intuitively, we think of a point on a function as **continuous** if it's "connected" to the points around it.

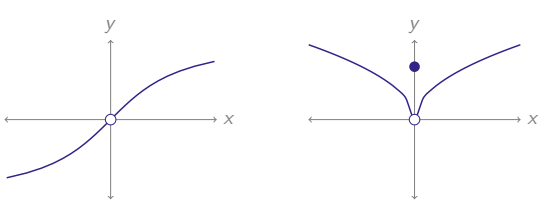


Jump Discontinuity

Notes

Intuitive Continuity

Intuitively, we think of a point on a function as **continuous** if it's "connected" to the points around it.

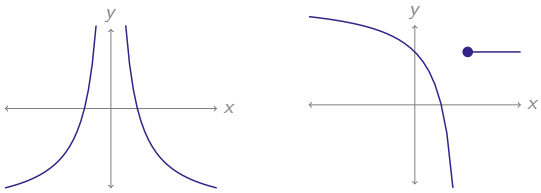


Removable Discontinuity

Notes

Intuitive Continuity

Intuitively, we think of a point on a function as **continuous** if it's "connected" to the points around it.

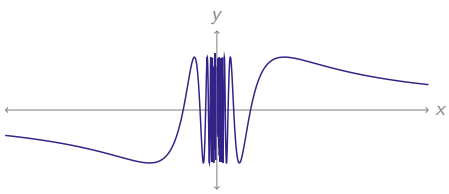


Blow-Up (Infinite) Discontinuity

If a graph has a discontinuity at a point, it has no tangent line at that point (and so no derivative at that point).

Notes

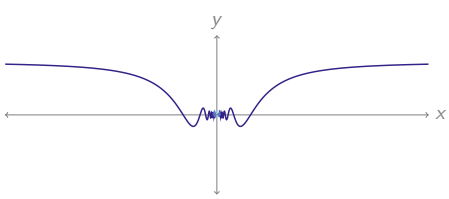
Intuition Failing



$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 01 & \text{if } x = 0 \end{cases}$$

Notes

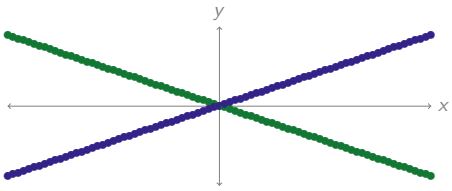
Intuition Failing



$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Notes

Intuition Failing



$$f(x) = \begin{cases} \frac{x}{3} & \text{if } x \text{ is rational} \\ -\frac{x}{3} & \text{if } x \text{ is irrational} \end{cases}$$

Notes

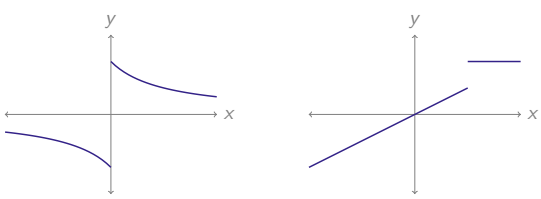
A More Rigorous Definition

Definition

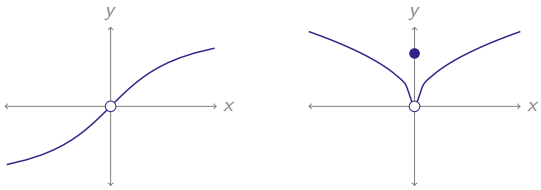
A function $f(x)$ is **continuous** at a point a in its domain if

$$\lim_{x \to a} f(x) = f(a)$$

- $f(a)$ must exist
- As x approaches a, there are no "surprises."



Jump: Limit doesn't exist (left and right don't match)



Notes

Continuity Proofs

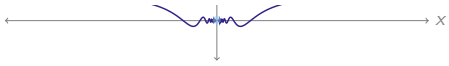
Decide whether $f(x) = \dfrac{x^2 - 4}{x - 2}$ is continuous or discontinuous at $x = 2$. Justify your result.

It is discontinuous, because 2 is not in the domain of $f(x)$. That is, $f(2)$ does not exist.

Example 1:

For which value(s) of a is $f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{if } x \neq 2 \\ a & \text{if } x = 2 \end{cases}$ continuous at $x = 2$?

Prove your result.



$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Notes

Rational Functions with Holes

Example 2: Let $f(x) = \dfrac{(x+1)x(x-1)(x-2)}{(x-1)(x-2)}$

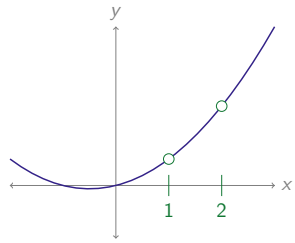
Note the domain of $f(x)$ does not include $x = 1$ or $x = 2$. Evaluate the following:

(a) $\lim_{x \to 0} f(x)$

(b) $\lim_{x \to 1} f(x)$

(c) $\lim_{x \to 2} f(x)$

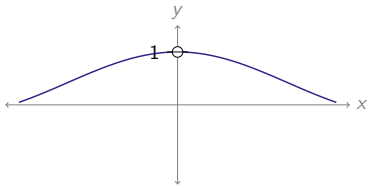
(d) $\lim_{x \to 3} f(x)$



Notes

Using Graphs

Example 3: Approximate $\lim_{x \to 0} \dfrac{\sin x}{x}$ using the graph below.



Notes

Using limits to evaluate derivatives

Evaluate the derivative of $f(x) = \sqrt{x}$ using the definition of the derivative,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) \\
&= \lim_{h \to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h \to 0} \frac{1}{(\sqrt{x+h} + \sqrt{x})} \\
&= \frac{1}{\sqrt{x+0} + \sqrt{x}} = \frac{1}{2\sqrt{x}}
\end{aligned}$$

Notes

Using limits to evaluate derivatives

Example 4:

Find the derivatives of the following functions using the definition of a derivative:

(a) $f(x) = \frac{1}{x}$

(b) $g(x) = \frac{1}{\sqrt{x}}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Notes

Concept Check

True or False: If $f(x)$ is not defined at $x = 1$, then $\lim_{x \to 1} f(x)$ does not exist.

True or False: If $\lim_{x \to 1} f(x)$ exists, then it's equal to $f(1)$.

True or False: If $f(x)$ is continuous at $x = 1$, then $f'(1)$ exists.

True or False: If $f'(1)$ exists, then $f(x)$ is continuous at $x = 1$.

Notes

Overview

1. Use software to numerically compute an **approximation** to the derivative.
2. Explain that the approximation replaces a (true) tangent line with an (approximating) secant line.
3. Explain using words how the derivative shape is connected with the shape of the original function.
4. Interpret the differences between two types of biochemical kinetics: Michaelis-Menten and Hill function.

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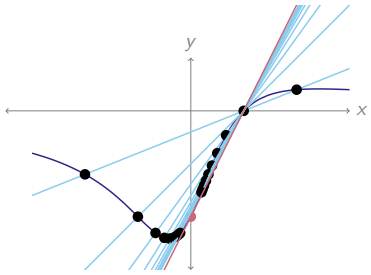
Notes

Idea: Approximate Tangent Line with Secant Line



Derivative	(slope of tangent line)	$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$
Approximate derivative	Slope of secant line	$\frac{f(x+h)-f(x)}{h}$

Notes

Approximate Derivative

$$f(x) = \frac{2x^5}{2^5 + x^5}$$

$$f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \frac{320}{33^2} \approx 0.29385...$$

$\mathbf{h}$	$\frac{\mathbf{f(1+h)-f(1)}}{\mathbf{h}}$
1	0.93939...
0.1	0.35228...
0.01	0.29932...
0.001	0.29439...
0.0001	0.29390...
0.00001	0.29385...

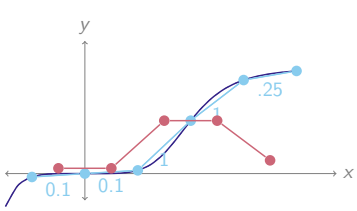
spreadsheet link

Notes

Approximate Derivative

$$f(x) = \frac{2x^5}{2^5 + x^5} \qquad f'(1) \approx \frac{f(1+h0.1)-f(1)}{h0.1}$$

Using very small values of h can get us an approximation of the derivative at a particular point.
We can also use a spreadsheet to guess the derivative at a number of different points, using tiny secant lines.

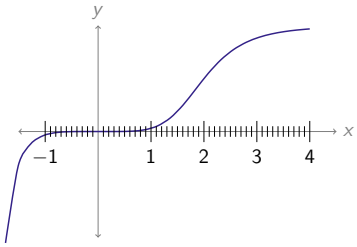


Notes

Approximate derivative at many points

$$f(x) = \frac{2x^5}{2^5 + x^5}$$

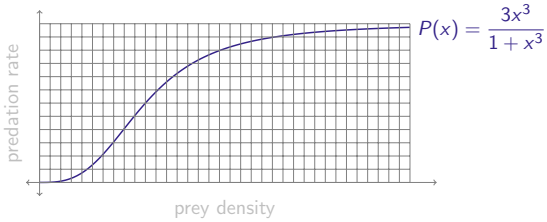
Interval: $[-1, 4]$
Small h: say $h = 0.1$



spreadsheet link

Notes

Predator-Prey Interactions: Holling Predator Response



Type III: I can hardly find the prey when the prey density is low, but I also get satiated at high prey density.

(a) Label the asymptote on the y-axis.
(b) Label the half-max activation level on the x-axis. (c) At what prey density is the predation rate changing the fastest?
That is, at which density is each additional prey animal responsible for the most increase in feeding by the predators?

Make a prediction, then verify with a spreadsheet.

Notes

more coming soon

Notes