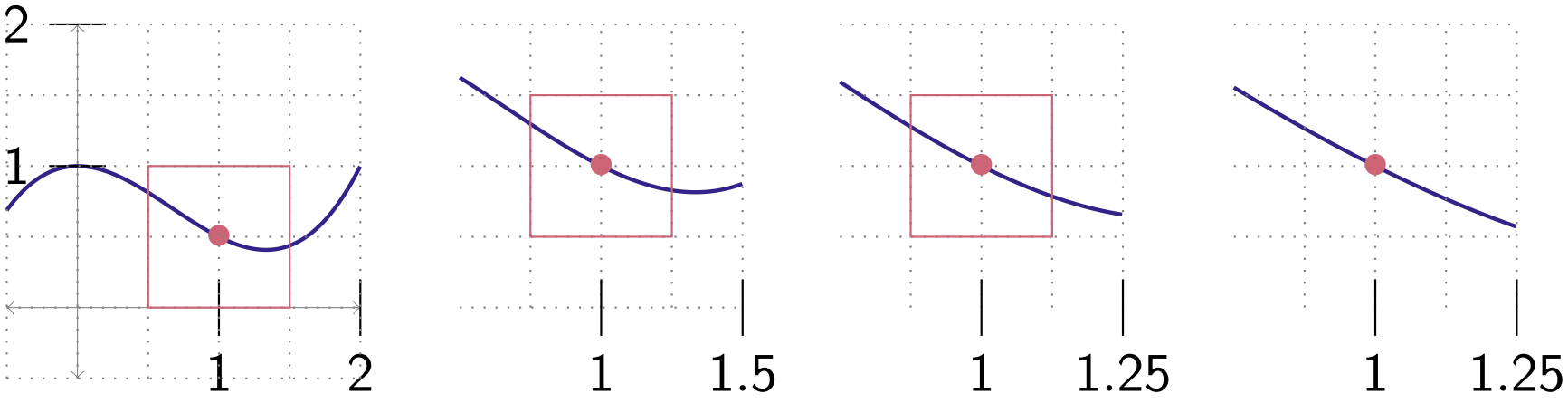
Overview

We already saw an algebraic way of thinking about a derivative.

- **Geometric**: zooming into a line
- **Analytic**: continuity and rational functions
- **Computational:** approximations with computers

Zooming In

For a smooth function, if we zoom in at a point, we see a line:

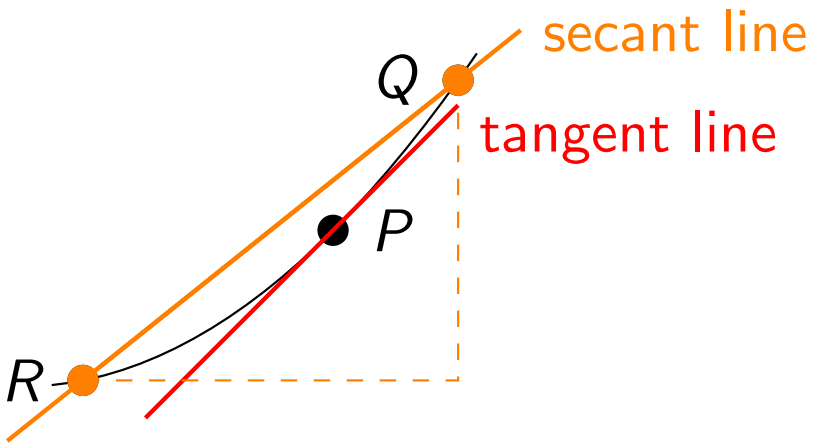


In this example, the slope of our zoomed-in line looks to be about:

$$\frac{\Delta y}{\Delta x} \approx -\frac{1}{2}$$

Recall

The **secant line** to the curve $y = f(x)$ through points R and Q is a line that passes through R and Q.
We call the slope of the secant line the **average rate of change of** $f(x)$ **from** R **to** Q**.**

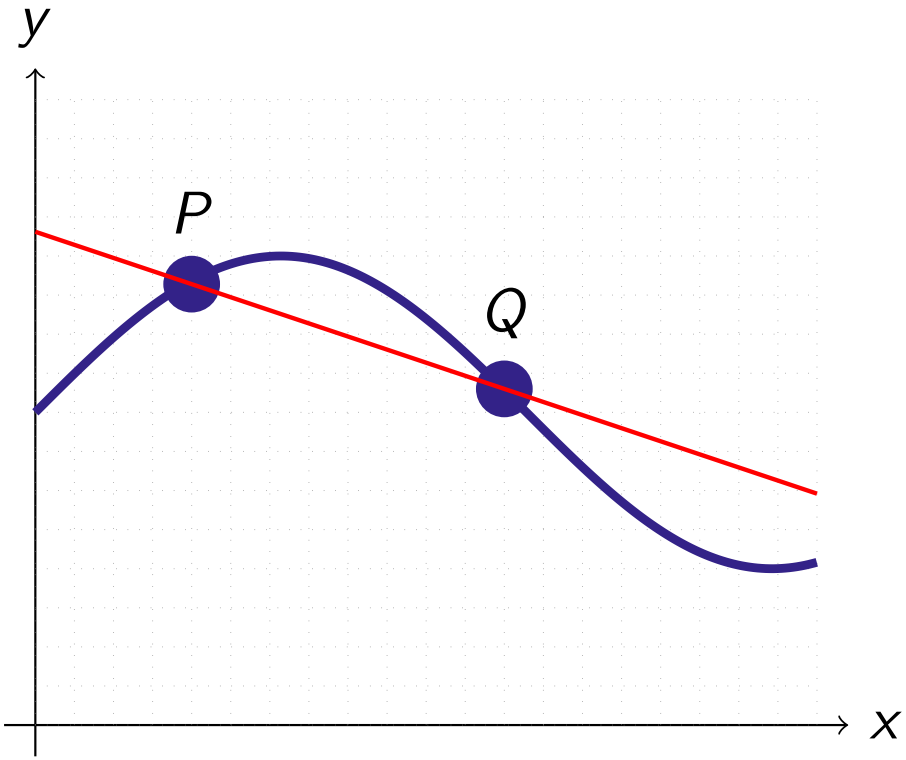


Definition

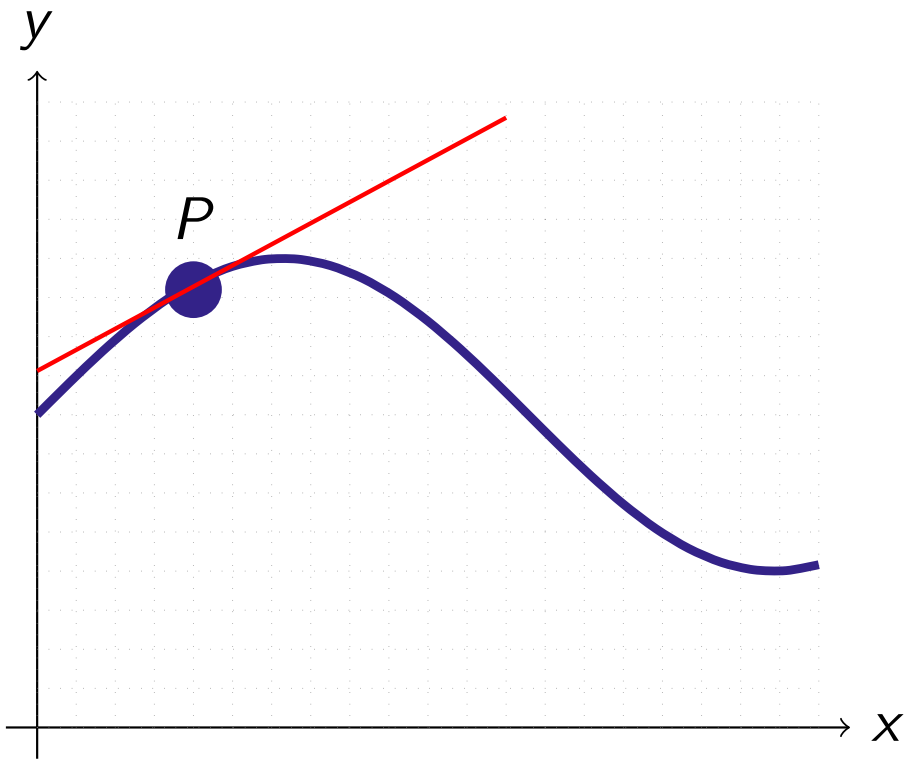
The straight line that we see when we zoom into the graph of a smooth function at some point P is called the **tangent line** at P.

The slope of the tangent line is the **instantaneous rate of change (derivative)**

On the graph below, draw the secant line to the curve through points P and Q.

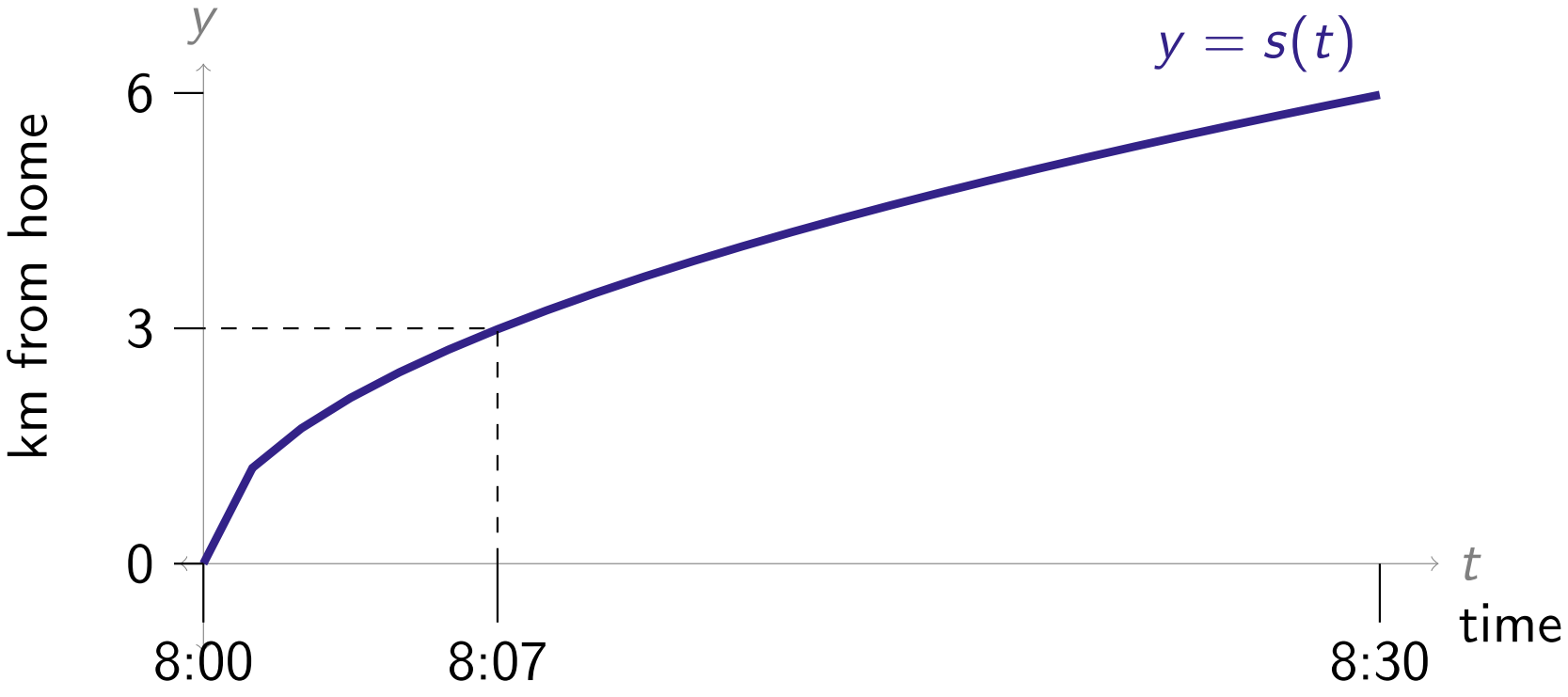


On the graph below, draw the tangent line to the curve at point P.



Derivatives of Familiar Functions

1. The derivative of the function $f(x) = Ax$ is:
   (a) 0
   (b) 1
   (c) A
   (d) x
   (e) Ax
2. The derivative of the function $f(x) = A$ is:
   (a) 0
   (b) 1
   (c) A
   (d) x
   (e) Ax
3. The derivative of the function $f(x) = A + x$ is:
   (a) 0
   (b) 1
   (c) A
   (d) x
   (e) Ax



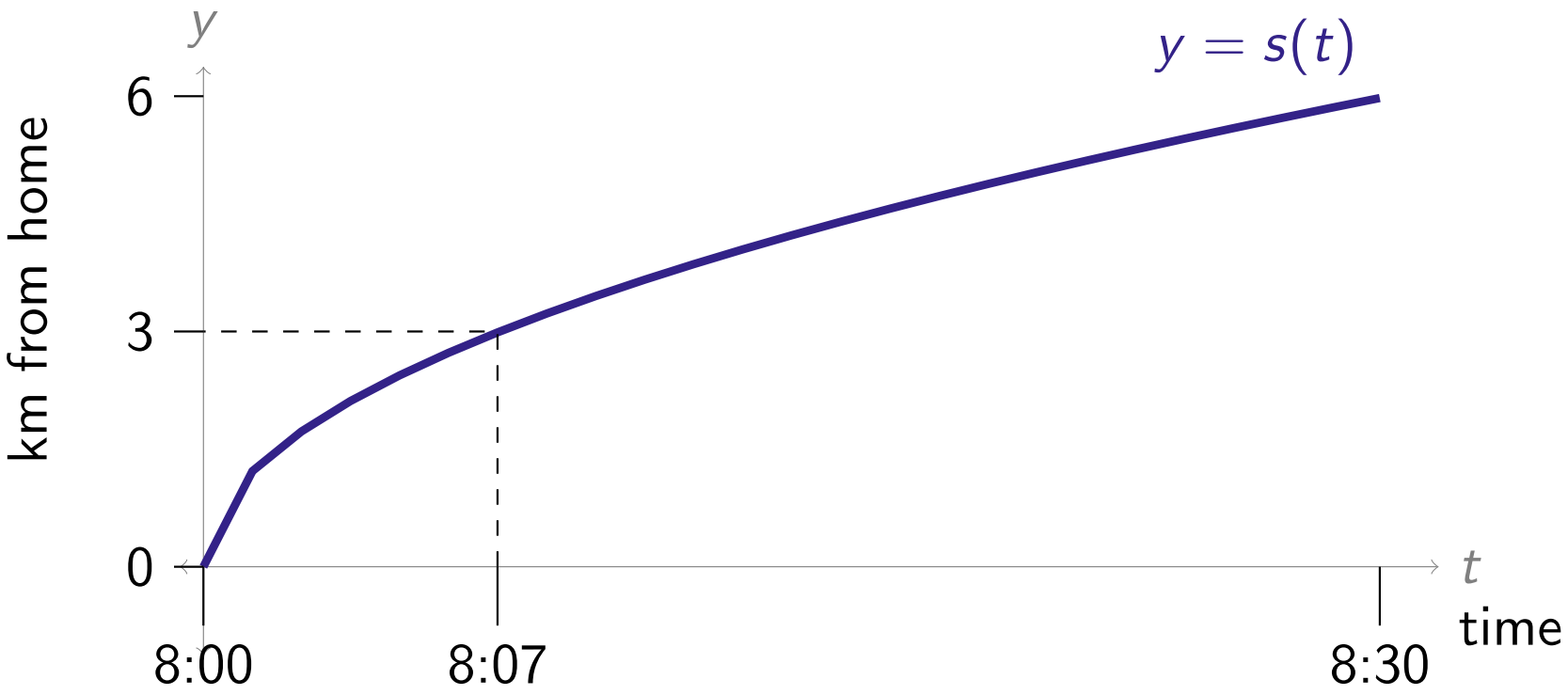
It takes me half an hour to bike 6 km. So, 12 kph represents the:

A. secant line to $y = s(t)$ from $t = 8:00$ to $t = 8:30$

B. slope of the secant line to $y = s(t)$ from $t = 8:00$ to $t = 8:30$

C. tangent line to $y = s(t)$ at $t = 8:30t = 8:25$

D. slope of the tangent line to $y = s(t)$ at $t = 8:30$



When I pedal up Crescent Rd at 8:25, the speedometer on my bike tells me I'm going 5kph. This corresponds to the:

A. secant line to $y = s(t)$ from $t = 8:00$ to $t = 8:25$

B. slope of the secant line to $y = s(t)$ from $t = 8:00$ to $t = 8:25$

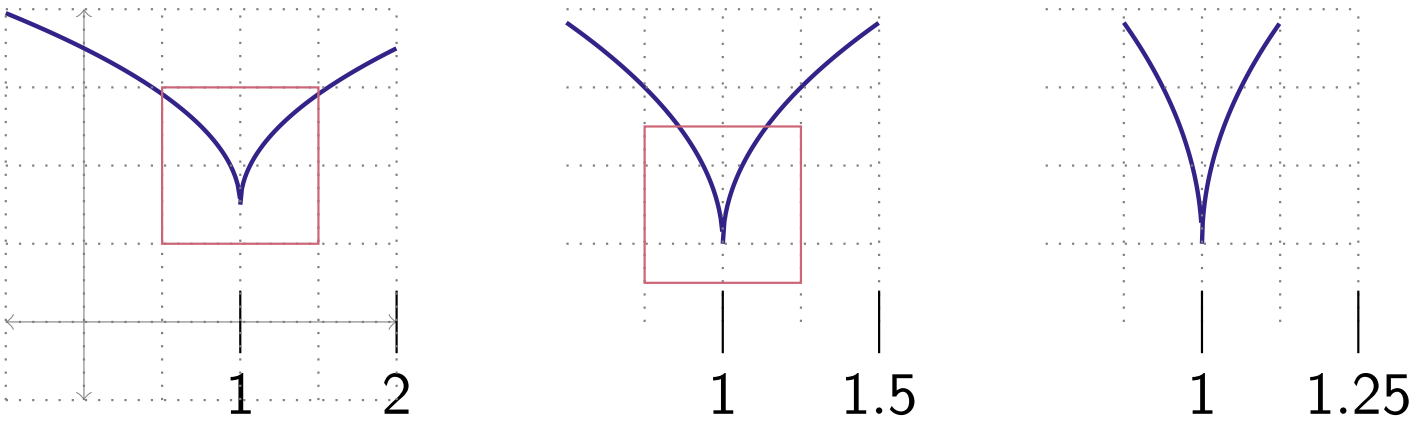
C. tangent line to $y = s(t)$ at $t = 8:25$

D. slope of the tangent line to $y = s(t)$ at$t = 8:25$

Zooming in on functions that aren't smooth

For a function with a cusp or a discontinuity, even though we zoom in very closely, we don't see simply a single straight line.

Cusp:



Discontinuity:

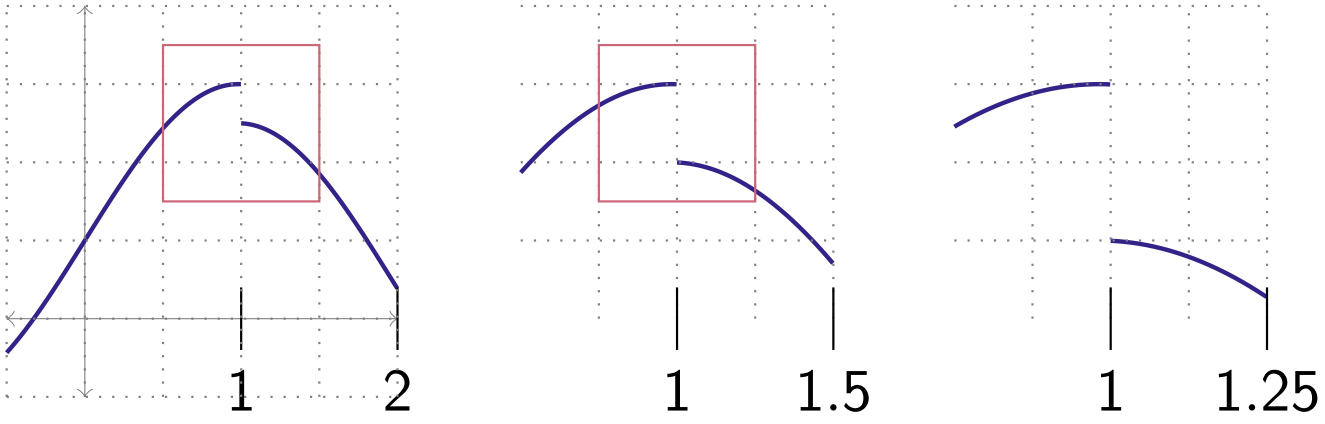
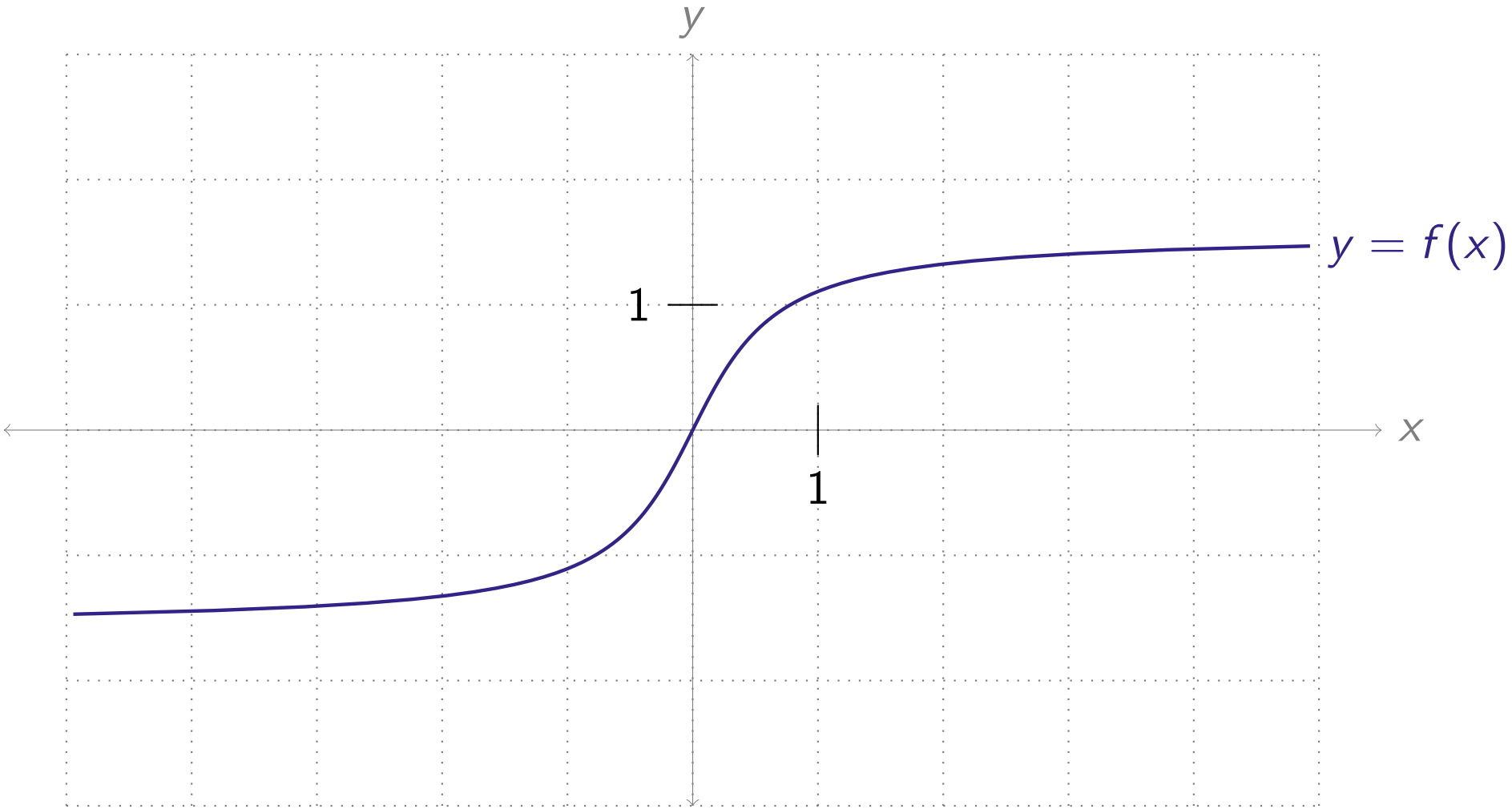


Photo credit: Pixabay

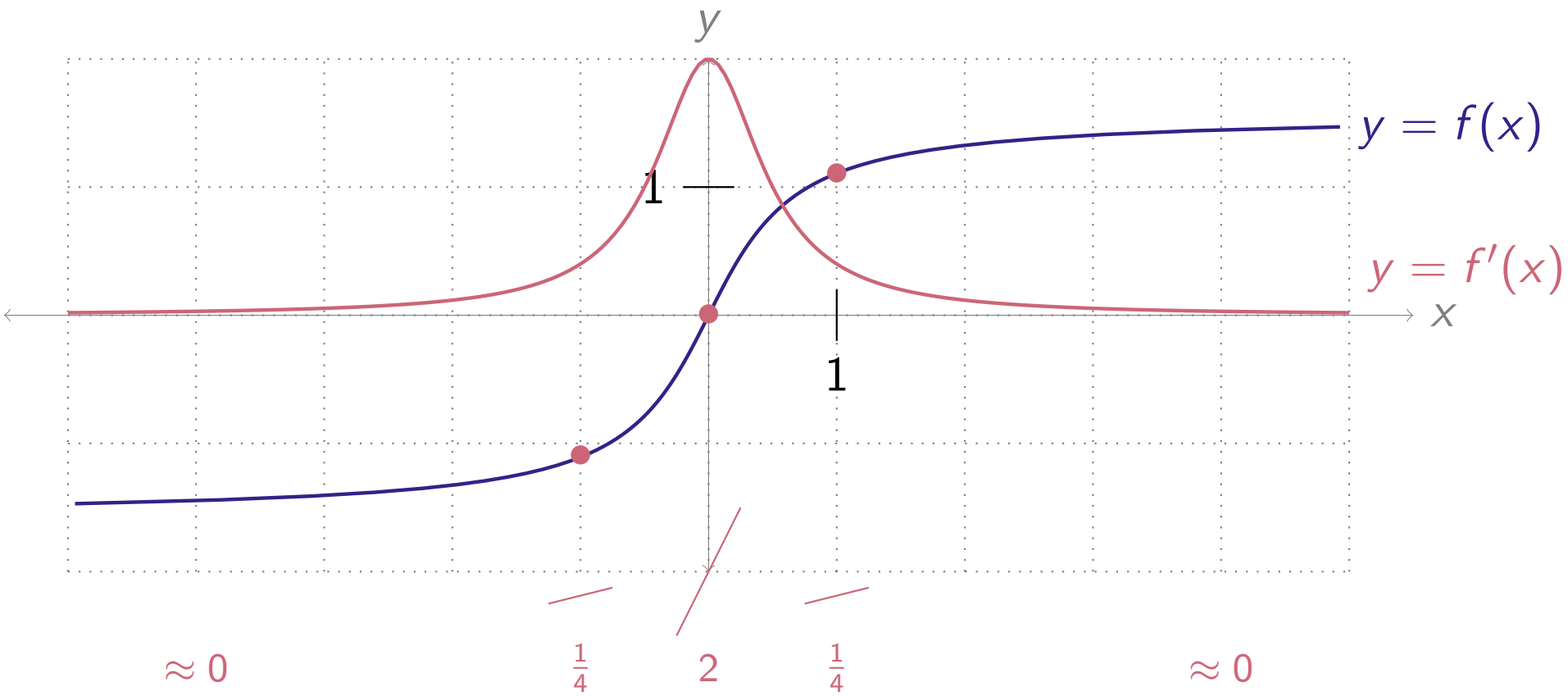
Sketch a derivative

Sketch the derivative of the function shown.



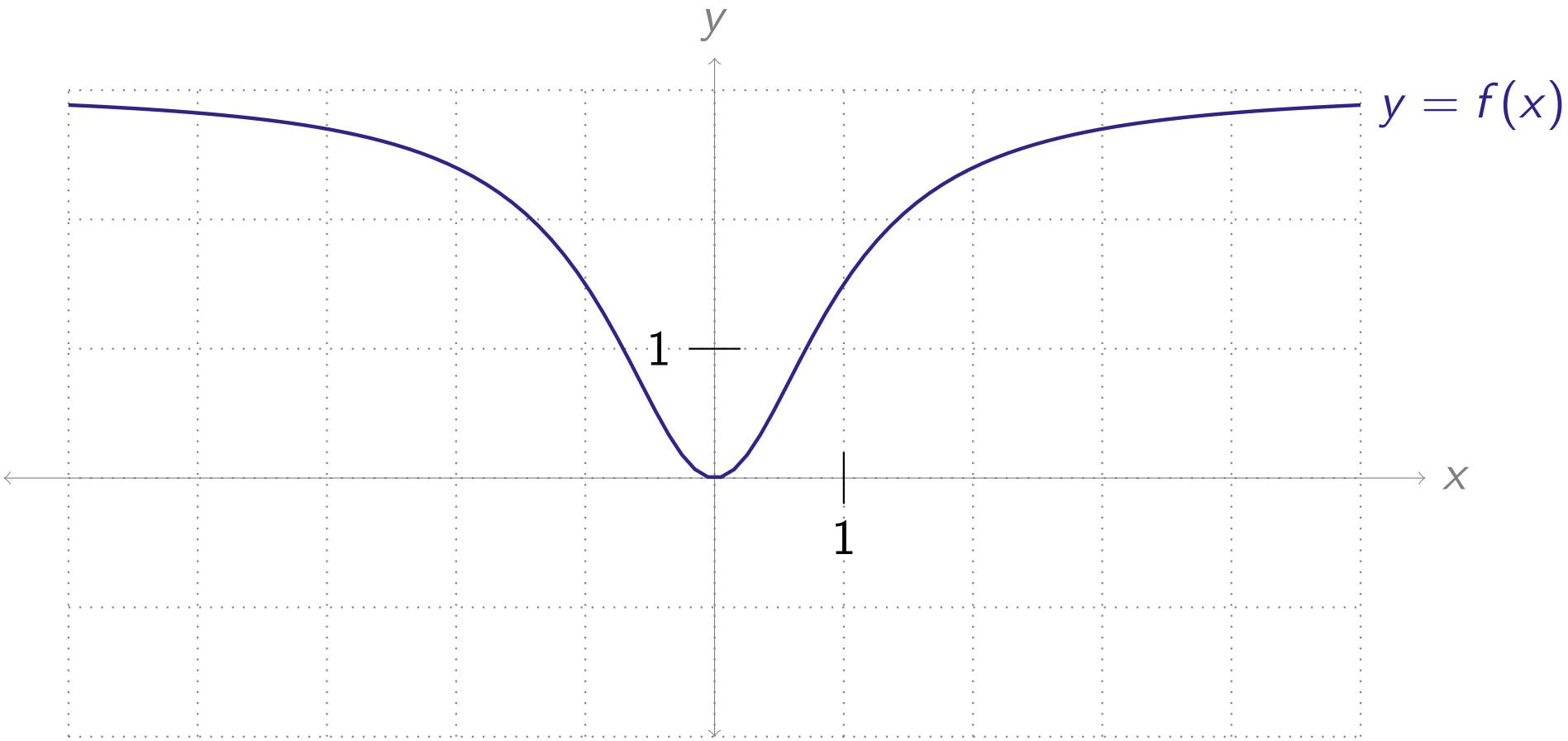
Sketch a derivative

Sketch the derivative of the function shown.



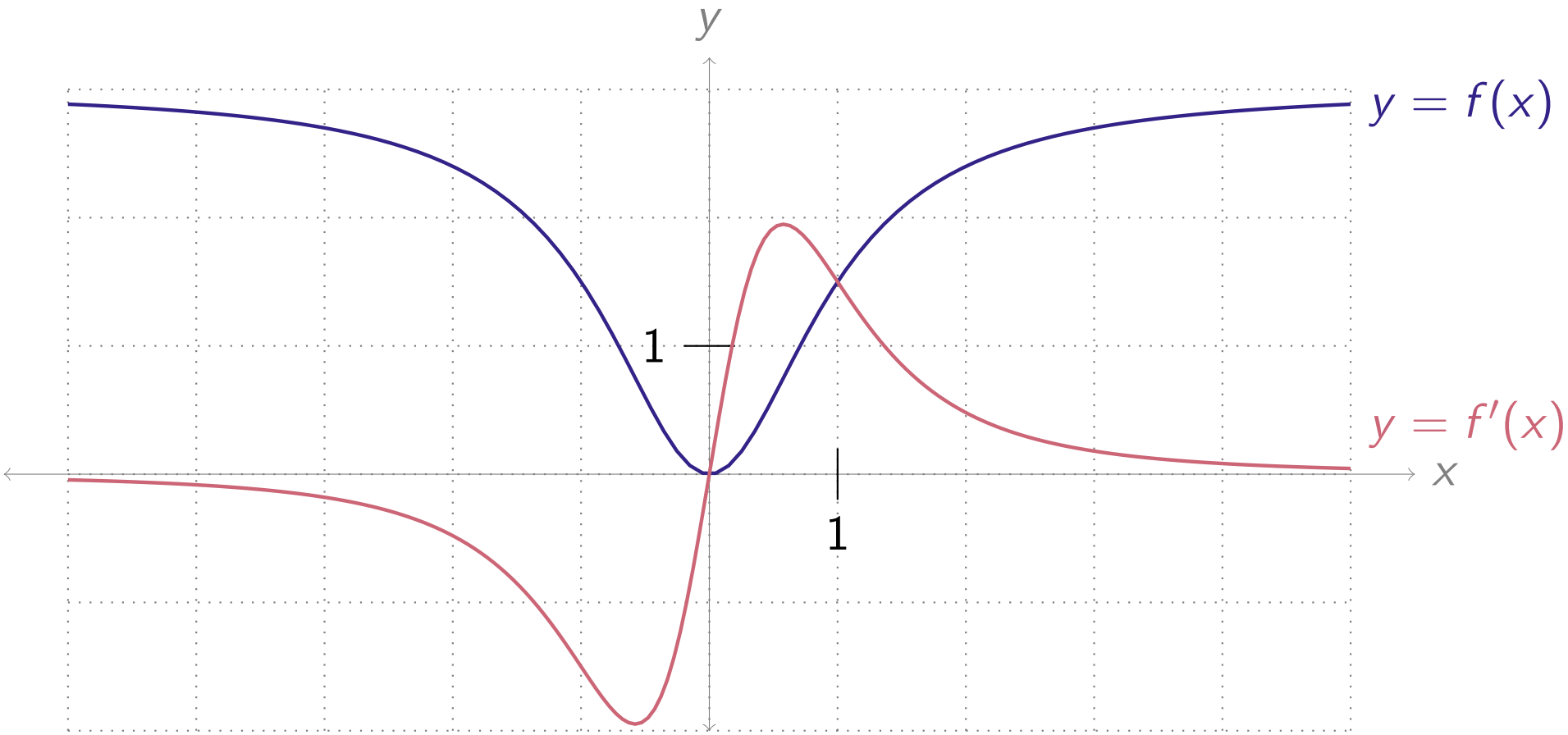
Sketch a derivative

Sketch the derivative of the function shown.



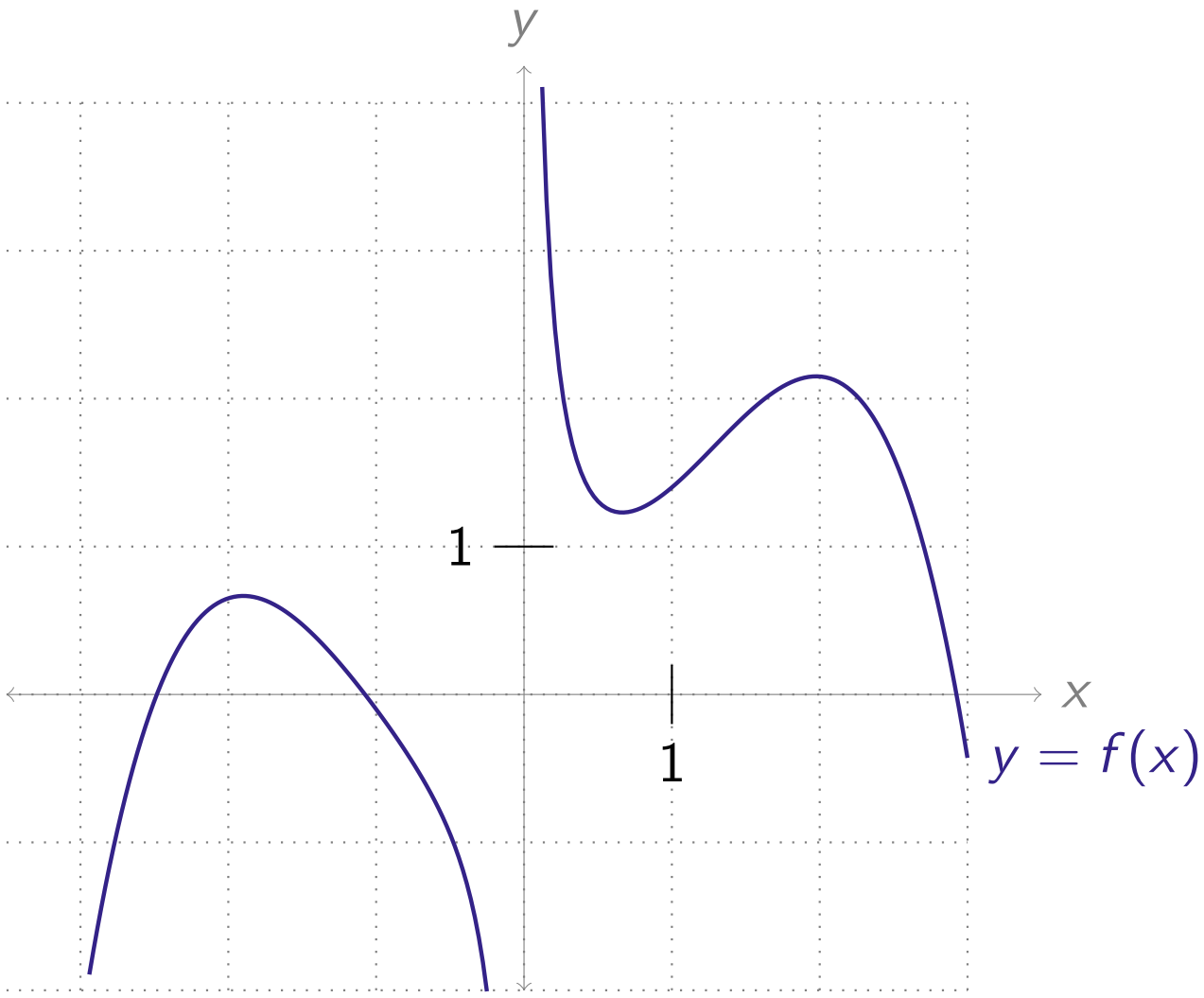
Sketch a derivative

Sketch the derivative of the function shown.



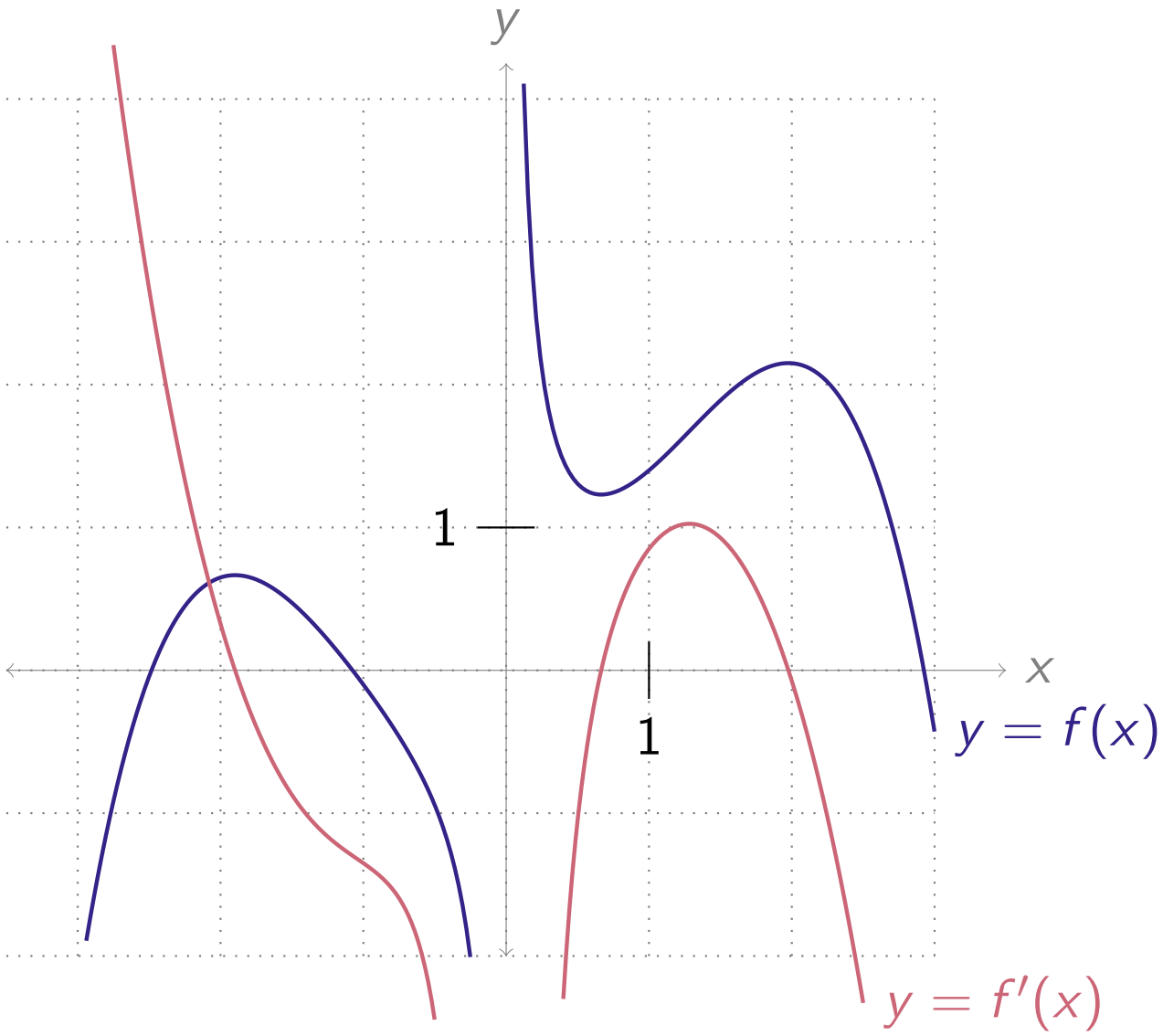
Sketch a derivative

Sketch the derivative of the function shown.



Sketch a derivative

Sketch the derivative of the function shown.



Sketch a derivative

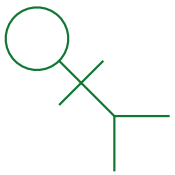
Sketch the approximate behaviour of the function, then use your sketch of the function to sketch its derivative.

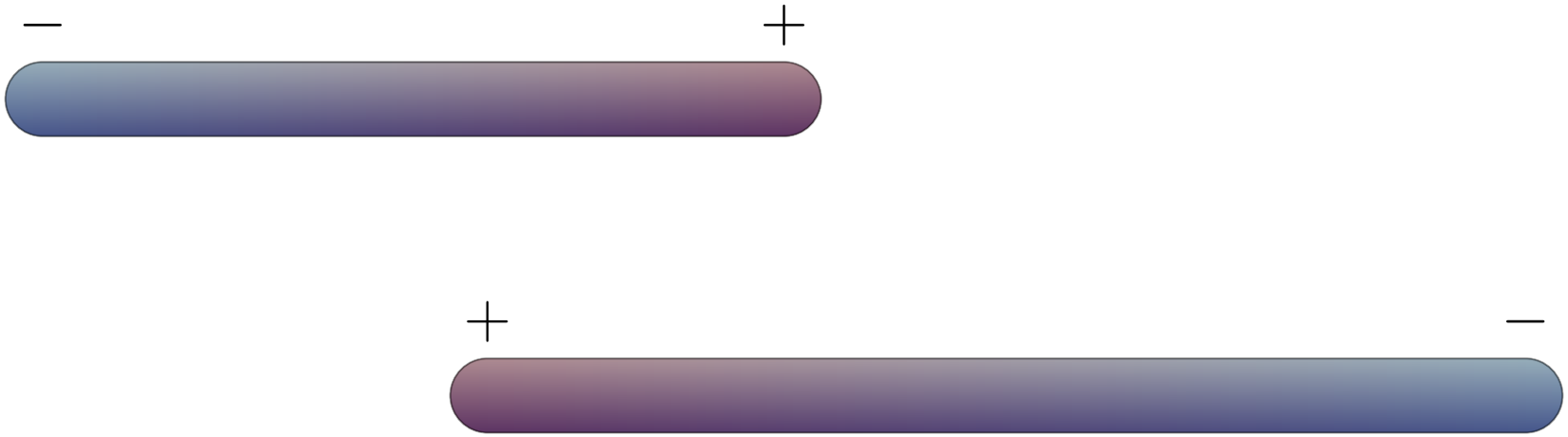
$$f(x) = \frac{-5x^2}{10 + x^2}$$

Kinesin

Kinesin: link

- Kinesin transports vesicles along *microtubules*, which have a plus and minus end
- Kinesin only travels towards the plus end
- Kinesin can hop on and off microtubules

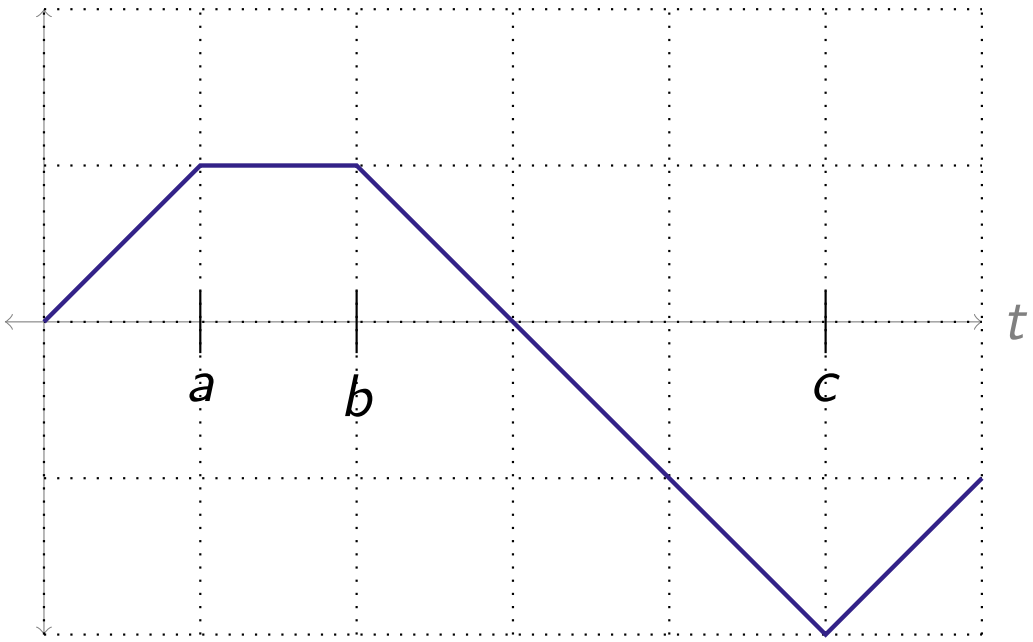




More explanation of Kinesin's walking: link

Kinesin

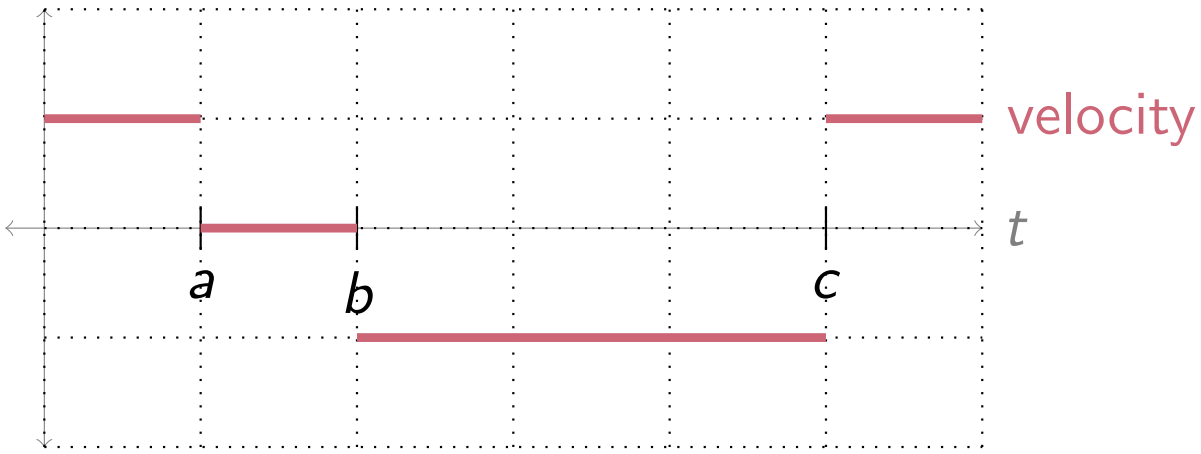
Below is a sketch of the position of a kinesin molecule over time, as it travels around a network of microtubules of fixed position.



1. Sketch the velocity of the kinesin molecule over the same time period.
2. Describe what happened to the kinesin molecule.

Kinesin

Below is a sketch of the position of a kinesin molecule over time, as it travels around a network of microtubules of fixed position.



$(0, a)$ The molecule travels along a microtubule at a velocity of 1 in the positive direction.

(a, b) Then, it falls off or gets stuck.

(b, c) Then it travels on along a mictorubule with the *opposite orientation*, so it moves in the negative direction.

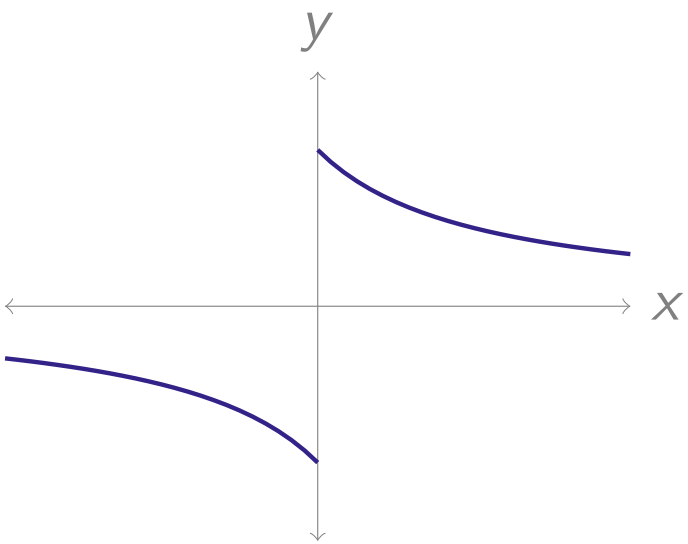
$t > c$ Finally, it hops onto another microtubule (possibly the same as the first) and travels in the positive direction.

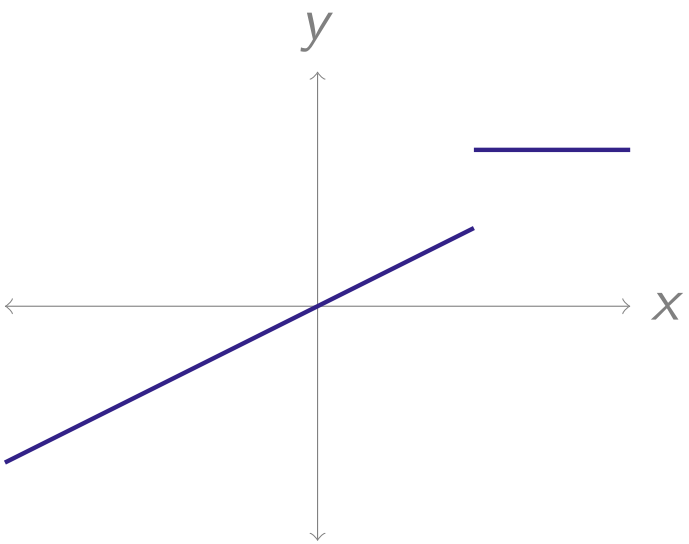
Overview

- Explain the definition of a continuous function.
- Identify functions with various types of discontinuities.
- Evaluate simple limits of rational functions.
- Calculate the derivative of a simple function using the definition of the derivative

Intuitive Continuity

Intuitively, we think of a point on a function as **continuous** if it's "connected" to the points around it.

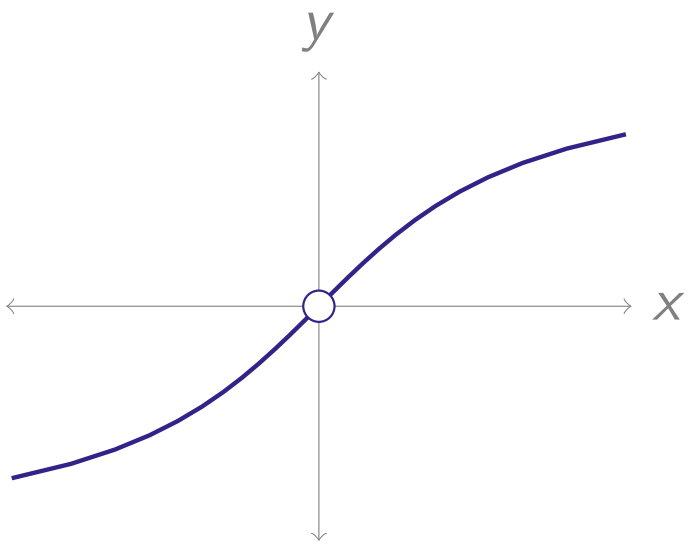


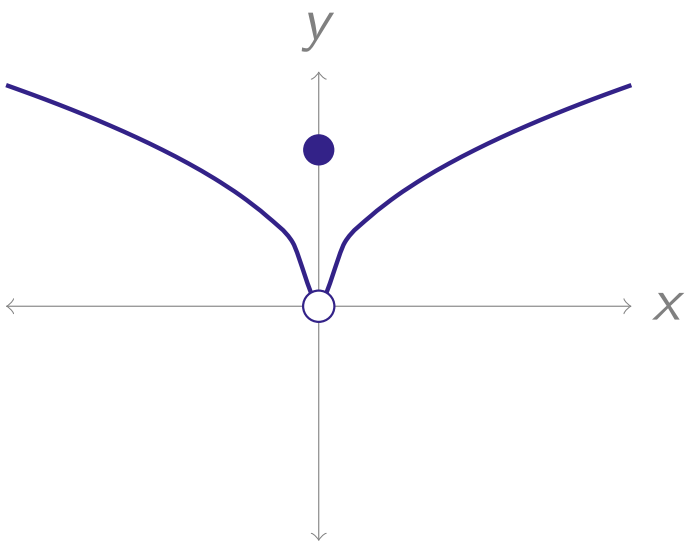


Jump Discontinuity

Intuitive Continuity

Intuitively, we think of a point on a function as **continuous** if it's "connected" to the points around it.

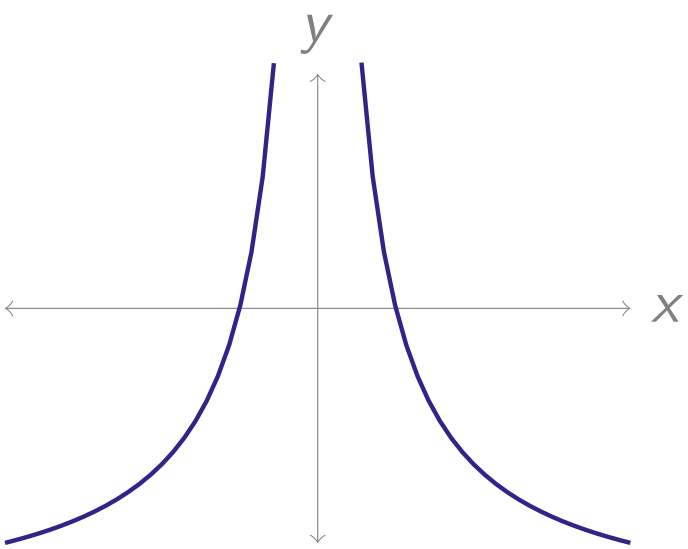


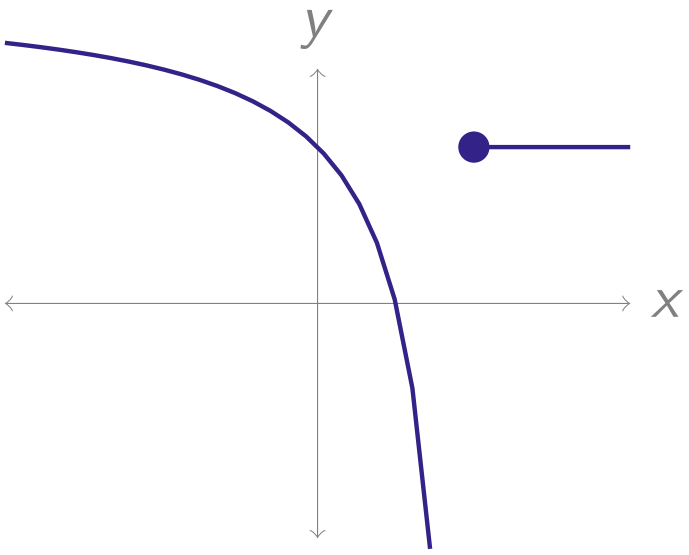


Removable Discontinuity

Intuitive Continuity

Intuitively, we think of a point on a function as **continuous** if it's "connected" to the points around it.

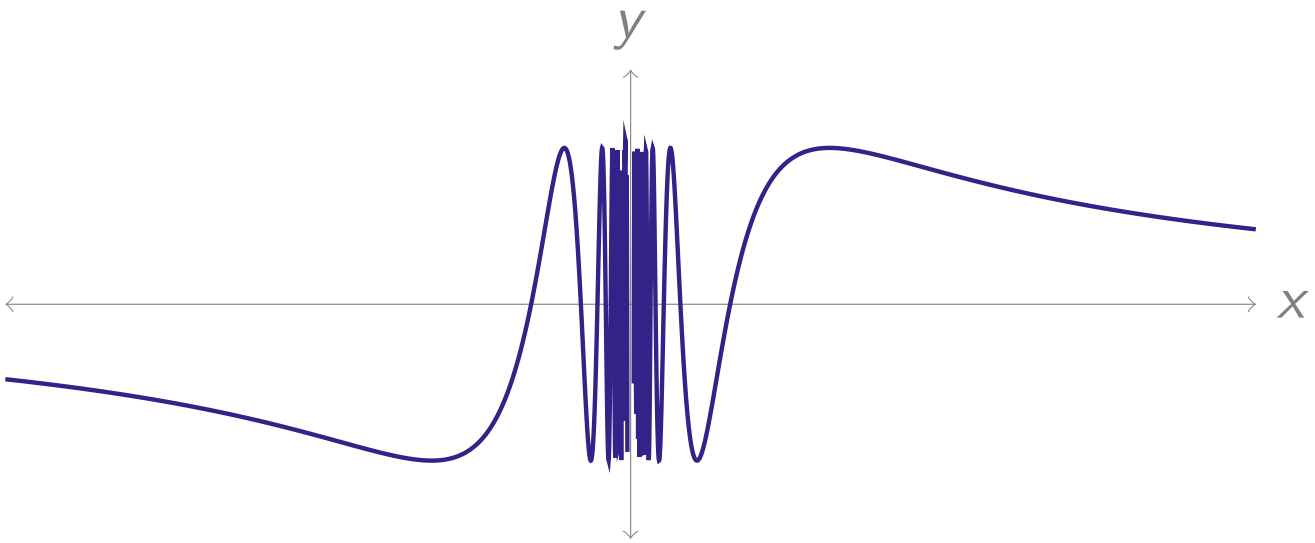




Blow-Up (Infinite) Discontinuity

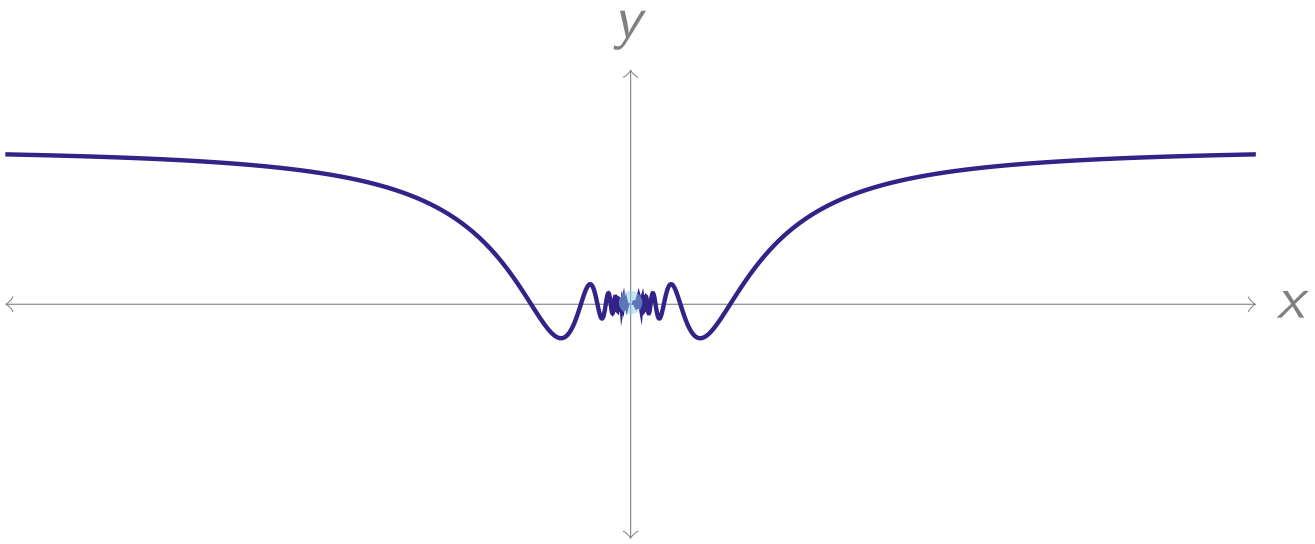
If a graph has a discontinuity at a point, it has no tangent line at that point (and so no derivative at that point).

Intuition Failing



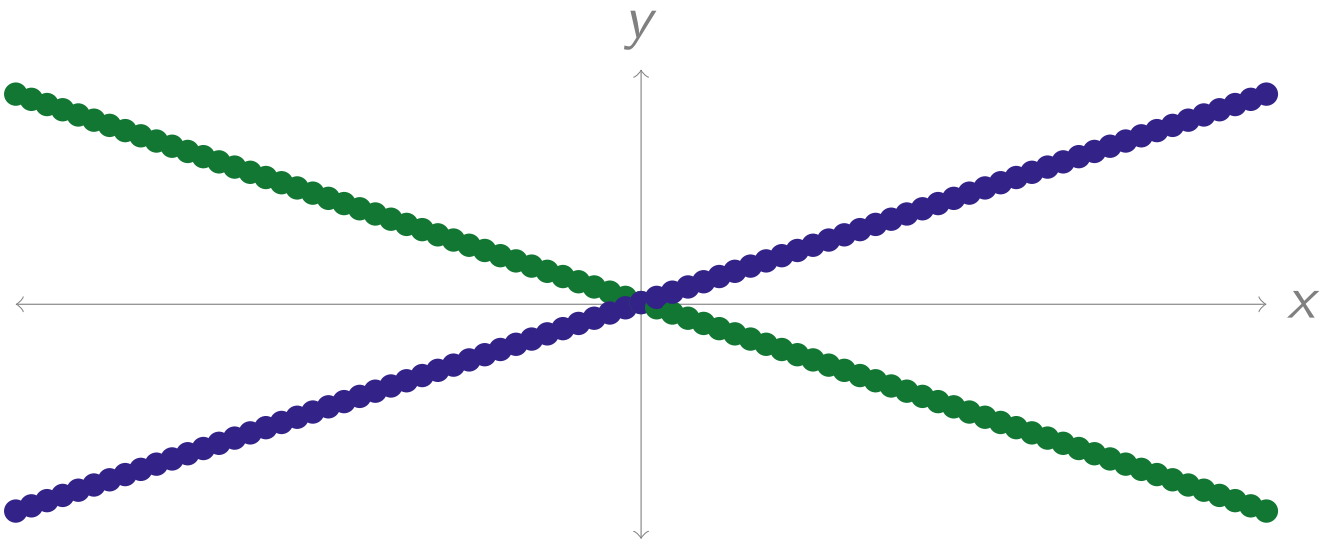
$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 01 & \text{if } x = 0 \end{cases}$$

Intuition Failing



$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Intuition Failing



$$f(x) = \begin{cases} \frac{x}{3} & \text{if } x \text{ is rational} \\ -\frac{x}{3} & \text{if } x \text{ is irrational} \end{cases}$$

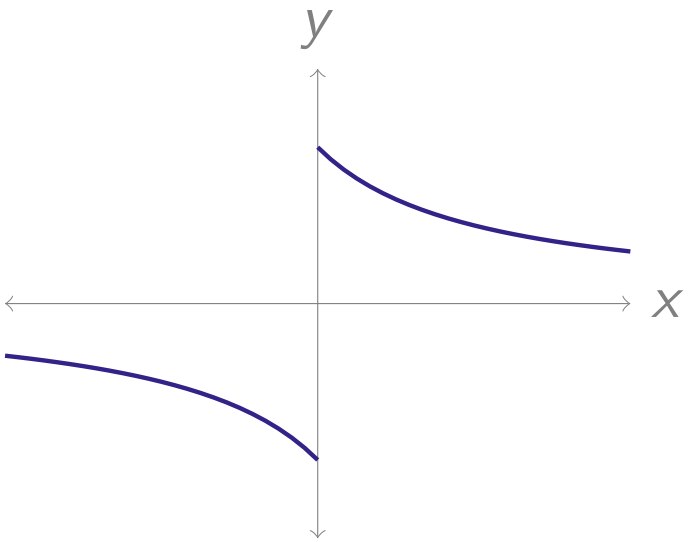
A More Rigorous Definition

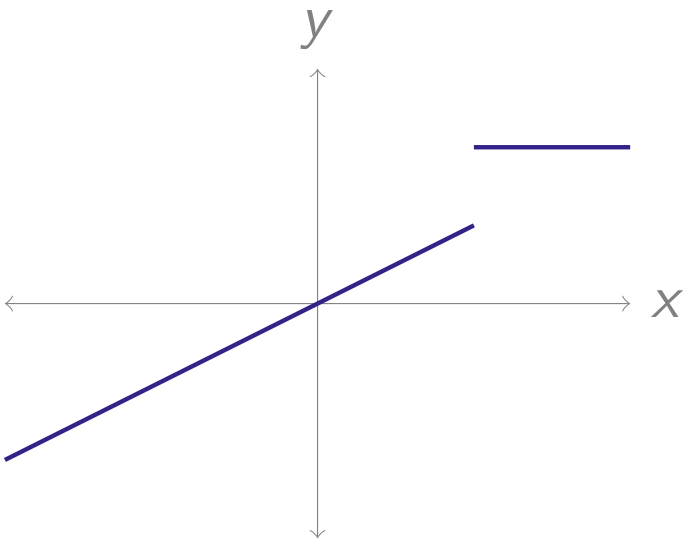
Definition

A function $f(x)$ is **continuous** at a point a in its domain if

$$\lim_{x \to a} f(x) = f(a)$$

- $f(a)$ must exist
- As x approaches a, there are no "surprises."





Jump: Limit doesn't exist (left and right don't match)

y y

Continuity Proofs

Decide whether $f(x) = \dfrac{x^2 - 4}{x - 2}$ is continuous or discontinuous at $x = 2$. Justify your result.

It is discontinuous, because 2 is not in the domain of $f(x)$. That is, $f(2)$ does not exist.

Example 1:

For which value(s) of a is $f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{if } x \neq 2 \\ a & \text{if } x = 2 \end{cases}$ continuous at $x = 2$?

Prove your result.

Rational Functions with Holes

Example 2: Let $f(x) = \dfrac{(x+1)x(x-1)(x-2)}{(x-1)(x-2)}$

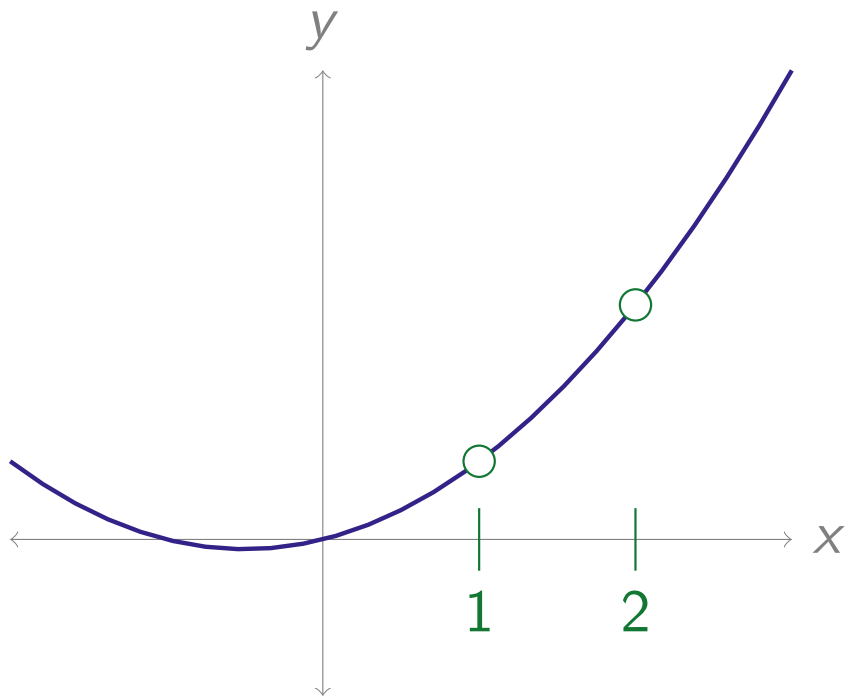
Note the domain of $f(x)$ does not include $x = 1$ or $x = 2$. Evaluate the following:

(a) $\lim_{x \to 0} f(x)$

(b) $\lim_{x \to 1} f(x)$

(c) $\lim_{x \to 2} f(x)$

(d) $\lim_{x \to 3} f(x)$



Using limits to evaluate derivatives

Evaluate the derivative of $f(x) = \sqrt{x}$ using the definition of the derivative,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) \\
&= \lim_{h \to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h \to 0} \frac{1}{(\sqrt{x+h} + \sqrt{x})} \\
&= \frac{1}{\sqrt{x+0} + \sqrt{x}} = \frac{1}{2\sqrt{x}}
\end{aligned}$$

Using limits to evaluate derivatives

Example 3:

Find the derivatives of the following functions using the definition of a derivative:

(a) $f(x) = \dfrac{1}{x}$

(b) $g(x) = \dfrac{1}{\sqrt{x}}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Concept Check

True or False: If $f(x)$ is not defined at $x = 1$, then $\lim_{x \to 1} f(x)$ does not exist.

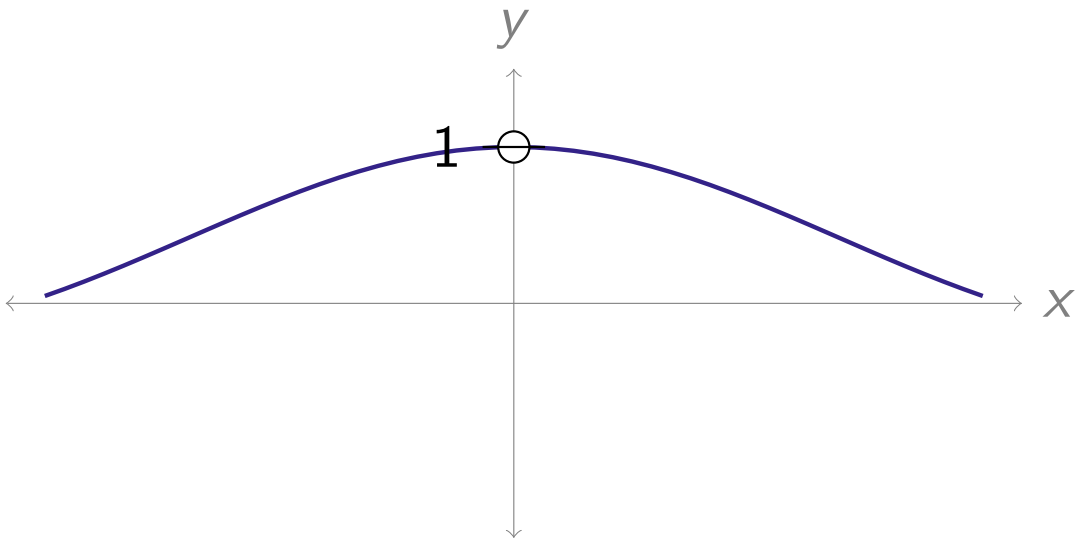
True or False: If $\lim_{x \to 1} f(x)$ exists, then it's equal to $f(1)$.

True or False: If $f(x)$ is continuous at $x = 1$, then $f'(1)$ exists.

True or False: If $f'(1)$ exists, then $f(x)$ is continuous at $x = 1$.

Using Graphs

Example 4: Approximate $\lim_{x \to 0} \frac{\sin x}{x}$ using the graph below.



Overview

1. Use software to numerically compute an **approximation** to the derivative.
2. Explain that the approximation replaces a (true) tangent line with an (approximating) secant line.
3. Explain using words how the derivative shape is connected with the shape of the original function.

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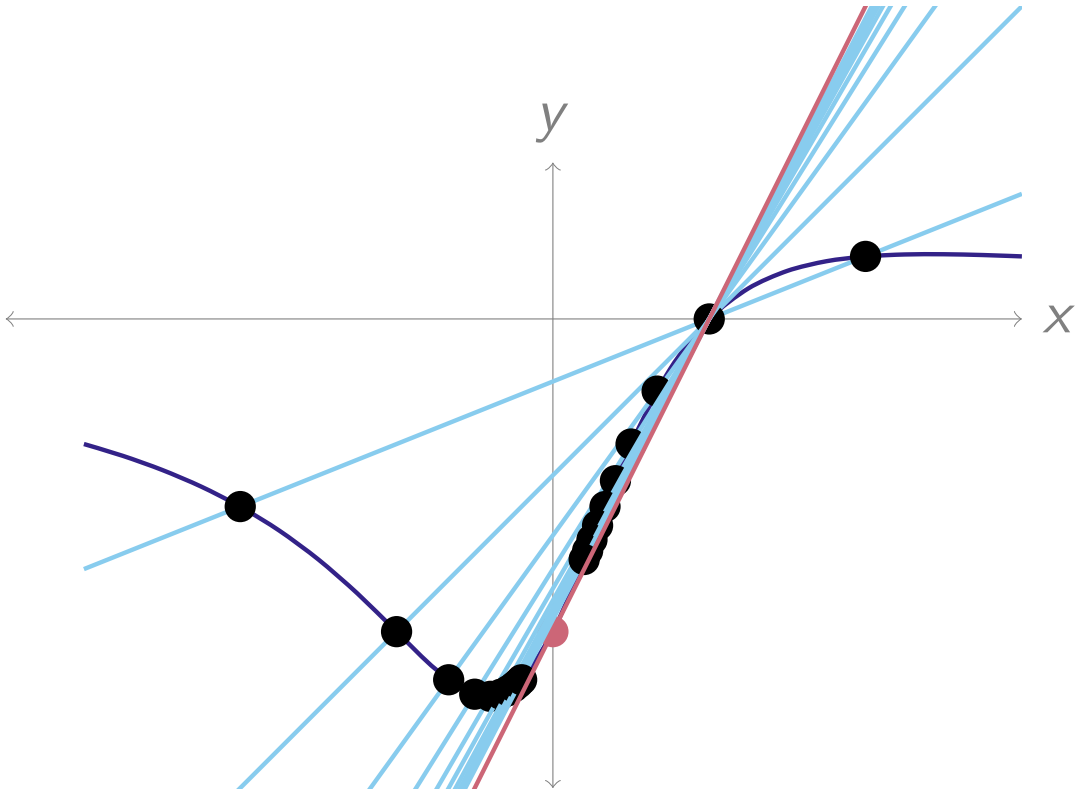
Idea: Approximate Tangent Line with Secant Line



Derivative	(slope of tangent line)	$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
Approximate derivative	Slope of secant line	$\frac{f(x+h) - f(x)}{h}$

Approximate Derivative

$$f(x) = \frac{2x^5}{2^5 + x^5}$$

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \frac{320}{33^2} \approx 0.29385...$$

$\mathbf{h}$	$\dfrac{\mathbf{f(1+h) - f(1)}}{\mathbf{h}}$
1	0.93939...
0.1	0.35228...
0.01	0.29932...
0.001	0.29439...
0.0001	0.29390...
0.00001	0.29385...

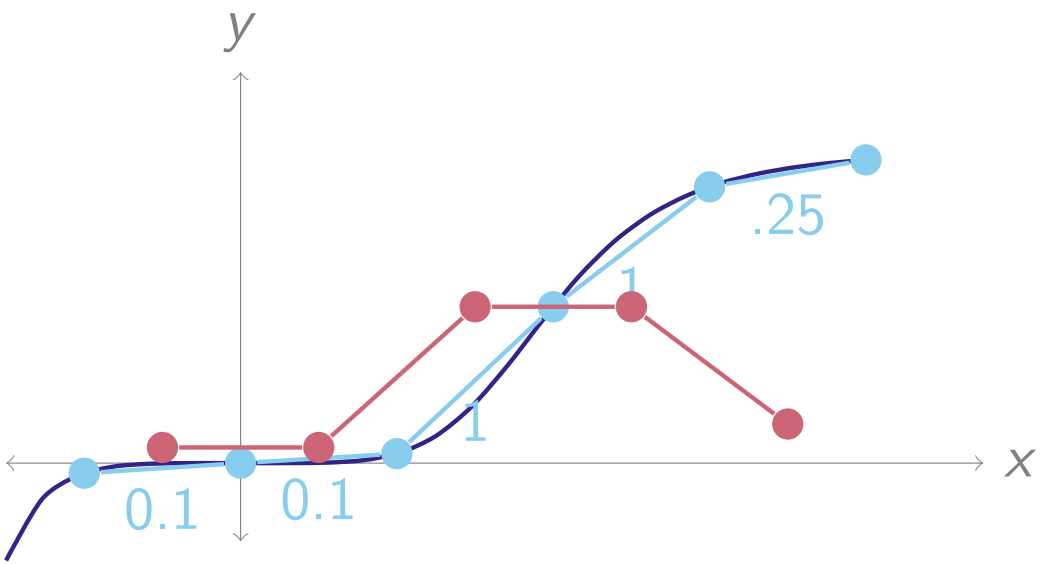
spreadsheet link

Approximate Derivative

$$f(x) = \frac{2x^5}{2^5 + x^5} \qquad f'(1) \approx \frac{f(1 + h0.1) - f(1)}{h0.1}$$

Using very small values of h can get us an approximation of the derivative at a particular point.

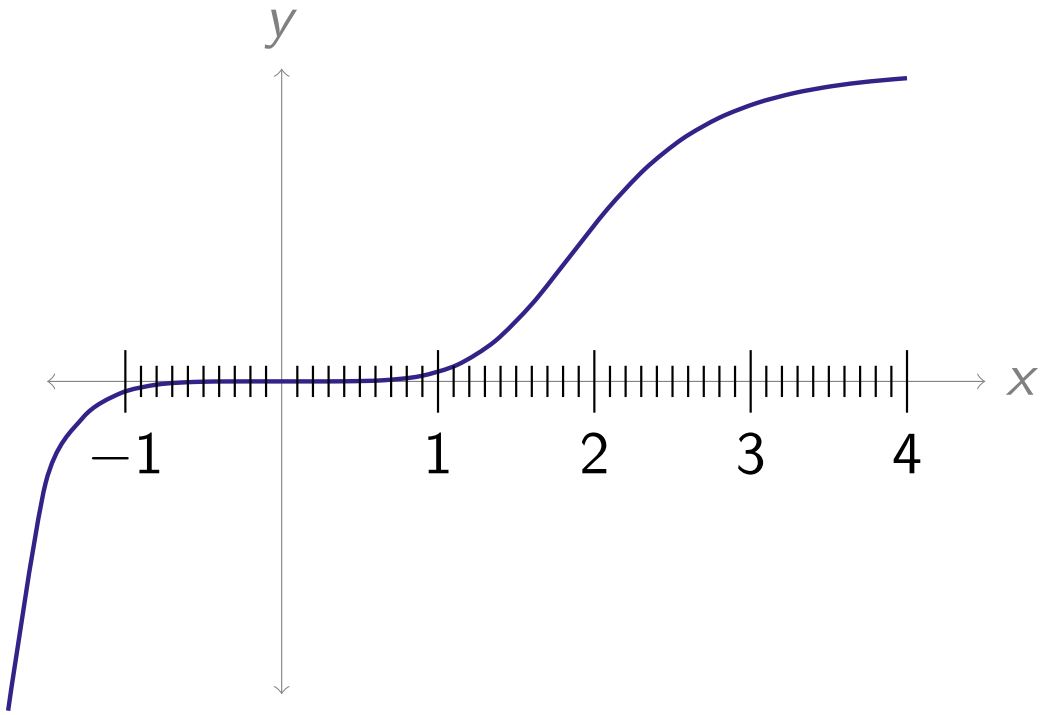
We can also use a spreadsheet to guess the derivative at a number of different points, using tiny secant lines.



Approximate derivative at many points

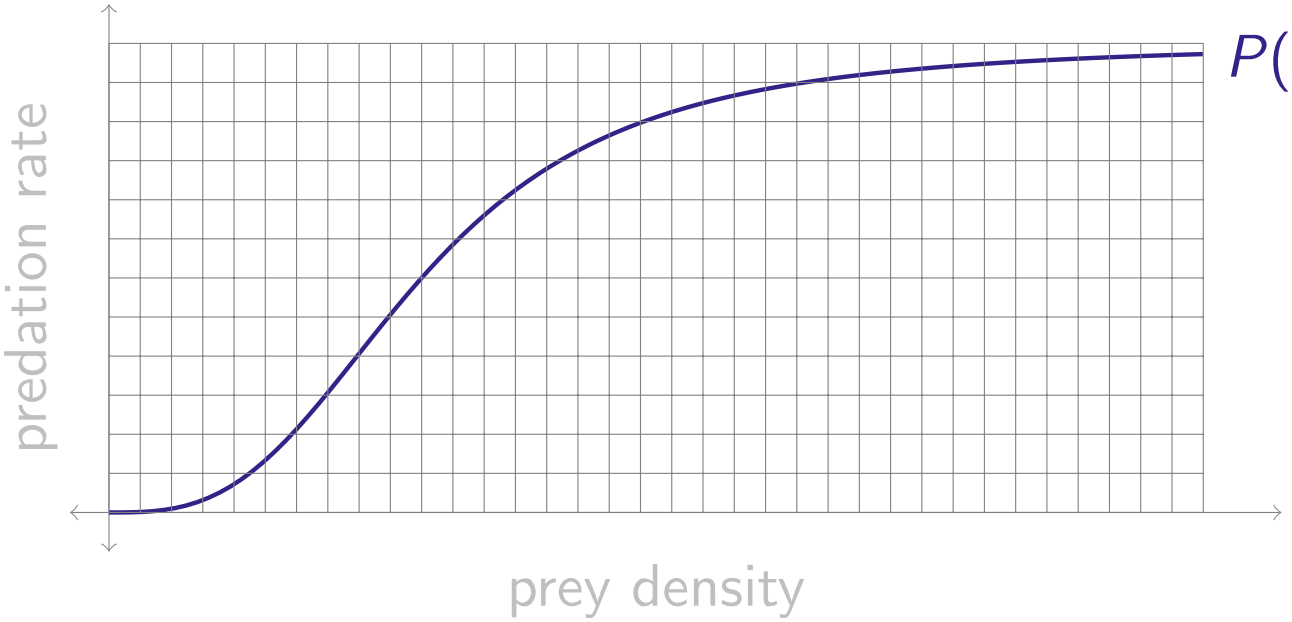
$$f(x) = \frac{2x^5}{2^5 + x^5}$$

Interval: $[-1, 4]$
Small h: say $h = 0.1$



spreadsheet link

Predator-Prey Interactions: Holling Predator Response



$$P(x) = \frac{3x^3}{1 + x^3}$$

Type III: I can hardly find the prey when the prey density is low, but I also get satiated at high prey density.

(a) Label the asymptote on the y-axis.

(b) Label the half-max activation level on the x-axis. (c) At what prey density is the predation rate changing the fastest?

That is, at which density is each additional prey animal responsible for the