Overview

We already saw an algebraic way of thinking about a derivative.

- **Geometric**: zooming into a line
- **Analytic**: continuity and rational functions
- **Computational**: approximations with computers
For a smooth function, if we zoom in at a point, we see a line:

In this example, the slope of our zoomed-in line looks to be about:

\[
\frac{\Delta y}{\Delta x} \approx -\frac{1}{2}
\]
Recall
The secant line to the curve $y = f(x)$ through points $R$ and $Q$ is a line that passes through $R$ and $Q$. We call the slope of the secant line the average rate of change of $f(x)$ from $R$ to $Q$.

Definition
The straight line that we see when we zoom into the graph of a smooth function at some point $P$ is called the tangent line at $P$.

The slope of the tangent line is the instantaneous rate of change (derivative).
On the graph below, draw the secant line to the curve through points $P$ and $Q$.

On the graph below, draw the tangent line to the curve at point $P$. 
Derivatives of Familiar Functions

1. The derivative of the function $f(x) = Ax$ is:
   (a) 0
   (b) 1
   (c) A
   (d) x
   (e) Ax

2. The derivative of the function $f(x) = A$ is:
   (a) 0
   (b) 1
   (c) A
   (d) x
   (e) Ax

3. The derivative of the function $f(x) = A + x$ is:
   (a) 0
   (b) 1
   (c) A
   (d) x
   (e) Ax
It takes me half an hour to bike 6 km. So, 12 kph represents the:

A. secant line to $y = s(t)$ from $t = 8:00$ to $t = 8:30$
B. slope of the secant line to $y = s(t)$ from $t = 8:00$ to $t = 8:30$
C. tangent line to $y = s(t)$ at $t = 8:30$$t = 8:25$
D. slope of the tangent line to $y = s(t)$ at $t = 8:30$
When I pedal up Crescent Rd at 8:25, the speedometer on my bike tells me I’m going 5kph. This corresponds to the:

A. secant line to $y = s(t)$ from $t = 8:00$ to $t = 8:25$
B. slope of the secant line to $y = s(t)$ from $t = 8:00$ to $t = 8:25$
C. tangent line to $y = s(t)$ at $t = 8:25$
D. slope of the tangent line to $y = s(t)$ at $t = 8:25$
Zooming in on functions that aren’t smooth

For a function with a cusp or a discontinuity, even though we zoom in very closely, we don’t see simply a single straight line.

Cusp:

Discontinuity:
Sketch a derivative

Sketch the derivative of the function shown.
Sketch a derivative

Sketch the derivative of the function shown.
Sketch a derivative

Sketch the derivative of the function shown.
Sketch a derivative

Sketch the derivative of the function shown.
Sketch a derivative

Sketch the derivative of the function shown.
Sketch a derivative

Sketch the derivative of the function shown.

\[ y = f(x) \]

\[ y = f'(x) \]
Sketch a derivative

Sketch the approximate behaviour of the function, then use your sketch of the function to sketch its derivative.

\[ f(x) = \frac{-5x^2}{10 + x^2} \]
Kinesin

Kinesin transports vesicles along *microtubules*, which have a plus and minus end.
Kinesin only travels towards the plus end.
Kinesin can hop on and off microtubules.

More explanation of Kinesin’s walking: link
Kinesin

Below is a sketch of the position of a kinesin molecule over time, as it travels around a network of microtubules of fixed position.

1 Sketch the velocity of the kinesin molecule over the same time period.
2 Describe what happened to the kinesin molecule.
Kinesin

Below is a sketch of the position of a kinesin molecule over time, as it travels around a network of microtubules of fixed position.

(0, a) The molecule travels along a microtubule at a velocity of 1 in the positive direction.

(a, b) Then, it falls off or gets stuck.

(b, c) Then it travels on along a microtubule with the opposite orientation, so it moves in the negative direction.

(t > c) Finally, it hops onto another microtubule (possibly the same as the first) and travels in the positive direction.
Overview

- Explain the definition of a continuous function.
- Identify functions with various types of discontinuities.
- Evaluate simple limits of rational functions.
- Calculate the derivative of a simple function using the definition of the derivative.
Intuitive Continuity

Intuitively, we think of a point on a function as **continuous** if it’s “connected” to the points around it.

**Jump Discontinuity**
Intuitive Continuity

Intuitively, we think of a point on a function as **continuous** if it’s “connected” to the points around it.

Removable Discontinuity
Intuitive Continuity

Intuitively, we think of a point on a function as **continuous** if it’s “connected” to the points around it.

Blow-Up (Infinite) Discontinuity

If a graph has a discontinuity at a point, it has no tangent line at that point (and so no derivative at that point).
f(x) = \begin{cases} 
\sin \left( \frac{1}{x} \right) & \text{if } x \neq 0 \\
0 & \text{if } x = 0 
\end{cases}
Intuition Failing

\[ f(x) = \begin{cases} 
  x \sin \left( \frac{1}{x} \right) & \text{if } x \neq 0 \\
  0 & \text{if } x = 0 
\end{cases} \]
Intuition Failing

\[ f(x) = \begin{cases} 
\frac{x}{3} & \text{if } x \text{ is rational} \\
-\frac{x}{3} & \text{if } x \text{ is irrational}
\end{cases} \]
A More Rigorous Definition

**Definition**
A function $f(x)$ is **continuous** at a point $a$ in its domain if

$$\lim_{{x \to a}} f(x) = f(a)$$

- $f(a)$ must exist
- As $x$ approaches $a$, there are no “surprises.”

Jump: Limit doesn’t exist (left and right don’t match)
Continuity Proofs

Decide whether \( f(x) = \frac{x^2 - 4}{x - 2} \) is continuous or discontinuous at \( x = 2 \). Justify your result.

It is discontinuous, because 2 is not in the domain of \( f(x) \). That is, \( f(2) \) does not exist.

Example 1:

For which value(s) of \( a \) is \( f(x) = \begin{cases} \frac{x^2 - 4}{x - 2} & \text{if } x \neq 2 \\ a & \text{if } x = 2 \end{cases} \) continuous at \( x = 2 \)?

Prove your result.
Rational Functions with Holes

Example 2: Let \( f(x) = \frac{(x + 1)x(x - 1)(x - 2)}{(x - 1)(x - 2)} \)

Note the domain of \( f(x) \) does not include \( x = 1 \) or \( x = 2 \). Evaluate the following:

(a) \( \lim_{x \to 0} f(x) \)

(b) \( \lim_{x \to 1} f(x) \)

(c) \( \lim_{x \to 2} f(x) \)

(d) \( \lim_{x \to 3} f(x) \)
Using limits to evaluate derivatives

Evaluate the derivative of \( f(x) = \sqrt{x} \) using the definition of the derivative,

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

\[
f'(x) = \lim_{h \to 0} \frac{\sqrt{x + h} - \sqrt{x}}{h}
\]

\[
= \lim_{h \to 0} \frac{\sqrt{x + h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x + h} + \sqrt{x}}{\sqrt{x + h} + \sqrt{x}}
\]

\[
= \lim_{h \to 0} \frac{(x + h) - x}{h(\sqrt{x + h} + \sqrt{x})}
\]

\[
= \lim_{h \to 0} \frac{1}{\sqrt{x + h} + \sqrt{x}}\]

\[
= \frac{1}{\sqrt{x} + 0 + \sqrt{x}} = \frac{1}{2\sqrt{x}}
\]
Using limits to evaluate derivatives

Example 3:

Find the derivatives of the following functions using the definition of a derivative:

(a) \( f(x) = \frac{1}{x} \)

(b) \( g(x) = \frac{1}{\sqrt{x}} \)

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]
Concept Check

True or False: If \( f(x) \) is not defined at \( x = 1 \), then \( \lim_{x \to 1} f(x) \) does not exist.

True or False: If \( \lim_{x \to 1} f(x) \) exists, then it’s equal to \( f(1) \).

True or False: If \( f(x) \) is continuous at \( x = 1 \), then \( f'(1) \) exists.

True or False: If \( f'(1) \) exists, then \( f(x) \) is continuous at \( x = 1 \).
Example 4: Approximate \( \lim_{{x \to 0}} \frac{\sin x}{x} \) using the graph below.
Overview

1. Use software to numerically compute an approximation to the derivative.
2. Explain that the approximation replaces a (true) tangent line with an (approximating) secant line.
3. Explain using words how the derivative shape is connected with the shape of the original function.
Idea: Approximate Tangent Line with Secant Line

Derivative (slope of tangent line) \[
\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

Approximate derivative Slope of secant line \[
\frac{f(x + h) - f(x)}{h}
\]
Approximate Derivative

\[ f(x) = \frac{2x^5}{2^5 + x^5} \]

\[ f'(1) = \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} = \frac{320}{33^2} \approx 0.29385... \]

<table>
<thead>
<tr>
<th>( h )</th>
<th>( \frac{f(1 + h) - f(1)}{h} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.93939...</td>
</tr>
<tr>
<td>0.1</td>
<td>0.35228...</td>
</tr>
<tr>
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<tr>
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<tr>
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<td>0.29390...</td>
</tr>
<tr>
<td>0.00001</td>
<td>0.29385...</td>
</tr>
</tbody>
</table>

spreadsheet link
Approximate Derivative

\[ f(x) = \frac{2x^5}{2^5 + x^5} \quad f'(1) \approx \frac{f(1 + h0.1) - f(1)}{h0.1} \]

Using very small values of \( h \) can get us an approximation of the derivative at a particular point.
We can also use a spreadsheet to guess the derivative at a number of different points, using tiny secant lines.
Approximate derivative at many points

\[ f(x) = \frac{2x^5}{2^5 + x^5} \]

Interval: \([-1, 4]\]
Small \( h \): say \( h = 0.1 \)
Predator-Prey Interactions: Holling Predator Response

Type III: I can hardly find the prey when the prey density is low, but I also get satiated at high prey density.

(a) Label the asymptote on the $y$-axis.
(b) Label the half-max activation level on the $x$-axis. (c) At what prey density is the predation rate changing the fastest? That is, at which density is each additional prey animal responsible for the most increase in feeding by the predators?

Type III: \[ P(x) = \frac{3x^3}{1 + x^3} \]