Overview

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- **Geometric**: zooming into a line
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- **Geometric**: zooming into a line
- **Analytic**: continuity and rational functions
We already saw an algebraic way of thinking about a derivative.

- **Geometric**: zooming into a line
- **Analytic**: continuity and rational functions
- **Computational**: approximations with computers
Zooming In

For a smooth function, if we zoom in at a point, we see a line:

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\frac{\Delta y}{\Delta x} \approx -\frac{1}{2}
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Recall

The **secant line** to the curve \( y = f(x) \) through points \( R \) and \( Q \) is a line that passes through \( R \) and \( Q \).
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The **secant line** to the curve \( y = f(x) \) through points \( R \) and \( Q \) is a line that passes through \( R \) and \( Q \). We call the slope of the secant line the **average rate of change of** \( f(x) \) **from** \( R \) **to** \( Q \).
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**Definition**

The straight line that we see when we zoom into the graph of a smooth function at some point \( P \) is called the **tangent line** at \( P \).
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The **secant line** to the curve $y = f(x)$ through points $R$ and $Q$ is a line that passes through $R$ and $Q$.

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**Definition**

The straight line that we see when we zoom into the graph of a smooth function at some point $P$ is called the **tangent line** at $P$.

The slope of the tangent line is the **instantaneous rate of change** (derivative).
On the graph below, draw the secant line to the curve through points $P$ and $Q$.

On the graph below, draw the tangent line to the curve at point $P$. 
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Derivatives of Familiar Functions

1. The derivative of the function $f(x) = Ax$ is:
   (a) 0
   (b) 1
   (c) A
   (d) $x$
   (e) $Ax$

2. The derivative of the function $f(x) = A$ is:
   (a) 0
   (b) 1
   (c) A
   (d) $x$
   (e) $Ax$

3. The derivative of the function $f(x) = A + x$ is:
   (a) 0
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Derivatives of Familiar Functions

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3.1 The geometric view: zooming into the graph of a function

- A. secant line to $y = s(t)$ from $t = 8:00$ to $8:07$
- B. slope of the secant line to $y = s(t)$ from $t = 8:00$ to $8:07$
- C. tangent line to $y = s(t)$ at $8:07$
- D. slope of the tangent line to $y = s(t)$ at $8:07$
Chapter 3: Three faces of the derivative

3.1 The geometric view: zooming into the graph of a function

\[ y = s(t) \]

The graph shows the distance from home as a function of time. The function is defined as:

\[ y = s(t) \]

- **A.** secant line to \( y = s(t) \) from \( t = 8:00 \) to \( t = 8:07 \)

- **B.** slope of the secant line to \( y = s(t) \) from \( t = 8:00 \) to \( t = 8:07 \)

- **C.** tangent line to \( y = s(t) \) at \( t = 8:07 \)

- **D.** slope of the tangent line to \( y = s(t) \) at \( t = 8:07 \)
It takes me half an hour to bike 6 km. So, 12 kph represents the:

A. secant line to $y = s(t)$ from $t = 8 : 00$ to $t = 8 : 30$
B. slope of the secant line to $y = s(t)$ from $t = 8 : 00$ to $t = 8 : 30$
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When I pedal up Crescent Rd at 8:25, the speedometer on my bike tells me I’m going 5kph. This corresponds to the:

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Zooming in on functions that aren’t smooth

For a function with a cusp or a discontinuity, even though we zoom in very closely, we don’t see simply a single straight line.

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Discontinuity:
Sketch the derivative of the function shown.

\[ y = f(x) \]
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\[ y = f(x) \]

\[ y = f'(x) \]
Sketch a derivative

Sketch the derivative of the function shown.

![Graph of a function with its derivative](image_url)
Sketch a derivative

Sketch the derivative of the function shown.

$y = f(x)$
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Sketch the approximate behaviour of the function, then use your sketch of the function to sketch its derivative.

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Kinesin

Kinesin transports vesicles along *microtubules*, which have a plus and minus end.

- Kinesin only travels towards the plus end.
- Kinesin can hop on and off microtubules.
Kinesin

Kinesin: link

- Kinesin transports vesicles along microtubules, which have a plus and minus end
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More explanation of Kinesin’s walking: link
Kinesin

Below is a sketch of the position of a kinesin molecule over time, as it travels around a network of microtubules of fixed position.

1. Sketch the velocity of the kinesin molecule over the same time period.
2. Describe what happened to the kinesin molecule.
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Chapter 3: Three faces of the derivative

3.1 The geometric view: zooming into the graph of a function

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Below is a sketch of the position of a kinesin molecule over time, as it travels around a network of microtubules of fixed position.

(0, \(a\)) The molecule travels along a microtubule at a velocity of 1 in the positive direction.
Kinesin

Below is a sketch of the position of a kinesin molecule over time, as it travels around a network of microtubules of fixed position.

\[(0, a)\] The molecule travels along a microtubule at a velocity of 1 in the positive direction.

\[(a, b)\] Then, it falls off or gets stuck.
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Below is a sketch of the position of a kinesin molecule over time, as it travels around a network of microtubules of fixed position.

(0, a) The molecule travels along a microtubule at a velocity of 1 in the positive direction.

(a, b) Then, it falls off or gets stuck.

(b, c) Then it travels on along a microtubule with the opposite orientation, so it moves in the negative direction.
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Below is a sketch of the position of a kinesin molecule over time, as it travels around a network of microtubules of fixed position.

\[(0, a)\] The molecule travels along a microtubule at a velocity of 1 in the positive direction.

\[(a, b)\] Then, it falls off or gets stuck.

\[(b, c)\] Then it travels on along a mictorubule with the opposite orientation, so it moves in the negative direction.

\[t > c\] Finally, it hops onto another microtubule (possibly the same as the first) and travels in the positive direction.
Overview

- Explain the definition of a continuous function.
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- Explain the definition of a continuous function.
- Identify functions with various types of discontinuities.
Overview

- Explain the definition of a continuous function.
- Identify functions with various types of discontinuities.
- Evaluate simple limits of rational functions.
Overview

- Explain the definition of a continuous function.
- Identify functions with various types of discontinuities.
- Evaluate simple limits of rational functions.
- Calculate the derivative of a simple function using the definition of the derivative.
Intuitive Continuity

Intuitively, we think of a point on a function as **continuous** if it’s “connected” to the points around it.
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[Diagrams showing intuitive continuity and jump discontinuity]
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Removable Discontinuity
Intuitive Continuity

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Blow-Up (Infinite) Discontinuity
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Blow-Up (Infinite) Discontinuity

Raden Saleh, *Gunung Merapi, letusan pada malam hari*
Intuitive Continuity

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Blow-Up (Infinite) Discontinuity

If a graph has a discontinuity at a point, it has no tangent line at that point (and so no derivative at that point).
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Blow-Up (Infinite) Discontinuity

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Intuition Failing

\[ f(x) = \begin{cases} 
\sin \left(\frac{1}{x}\right) & \text{if } x \neq 0 \\
0 & \text{if } x = 0
\end{cases} \]
Intuition Failing

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Intuition Failing

$$f(x) = \begin{cases} \frac{x}{3} & \text{if } x \text{ is rational} \\ -\frac{x}{3} & \text{if } x \text{ is irrational} \end{cases}$$
A More Rigorous Definition

Definition

A function $f(x)$ is **continuous** at a point $a$ in its domain if

$$\lim_{{x \to a}} f(x) = f(a)$$
A More Rigorous Definition

**Definition**
A function $f(x)$ is **continuous** at a point $a$ in its domain if

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- $f(a)$ must exist
- As $x$ approaches $a$, there are no “surprises.”
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![Graphs showing continuity](image-url)
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**Jump:** Limit doesn’t exist (left and right don’t match)
A More Rigorous Definition

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Removable: Limit doesn’t match $f(a)$, or $f(a)$ doesn’t even exist
A More Rigorous Definition

Definition
A function \( f(x) \) is \textbf{continuous} at a point \( a \) in its domain if

\[
\lim_{x \to a} f(x) = f(a)
\]

- \( f(a) \) must exist
- As \( x \) approaches \( a \), there are no “surprises.”
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**Blow-up:** Limit doesn’t exist.
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$$f(x) = \begin{cases} 
\sin \left( \frac{1}{x} \right) & \text{if } x \neq 0 \\
0 & \text{if } x = 0 
\end{cases}$$
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Limit doesn’t exist (oscillates)
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$$f(x) = \begin{cases} 
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\end{cases}$$

Continuous!
A More Rigorous Definition

Definition

A function \( f(x) \) is **continuous** at a point \( a \) in its domain if

\[
\lim_{{x \to a}} f(x) = f(a)
\]

- \( f(a) \) must exist
- As \( x \) approaches \( a \), there are no “surprises.”

\[
f(x) = \begin{cases} 
\frac{x}{3} & \text{if } x \text{ is rational} \\
-\frac{x}{3} & \text{if } x \text{ is irrational}
\end{cases}
\]
A More Rigorous Definition

Definition
A function \( f(x) \) is **continuous** at a point \( a \) in its domain if

\[
\lim_{{x \to a}} f(x) = f(a)
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-\frac{x}{3} & \text{if } x \text{ is irrational} 
\end{cases}
\]

Continuous at \( x = 0 \); discontinuous elsewhere.

(A more rigorous limit definition is helpful here.)
Continuity Proofs

Decide whether $f(x) = \frac{x^2 - 4}{x - 2}$ is continuous or discontinuous at $x = 2$. Justify your result.
Decide whether \( f(x) = \frac{x^2 - 4}{x - 2} \) is continuous or discontinuous at \( x = 2 \).

Justify your result.

It is discontinuous, because 2 is not in the domain of \( f(x) \). That is, \( f(2) \) does not exist.
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**Example 1:**

For which value(s) of \( a \) is \( f(x) = \begin{cases} \frac{x^2 - 4}{x - 2} & \text{if } x \neq 2 \\ a & \text{if } x = 2 \end{cases} \) continuous at \( x = 2 \)?

Prove your result.
Decide whether \( f(x) = \frac{x^2 - 4}{x - 2} \) is continuous or discontinuous at \( x = 2 \).

Justify your result.

It is discontinuous, because 2 is not in the domain of \( f(x) \). That is, \( f(2) \) does not exist.

**Example 1:**

For which value(s) of \( a \) is \( f(x) = \begin{cases} 
  \frac{x^2 - 4}{x - 2} & \text{if } x \neq 2 \\
  a & \text{if } x = 2 
\end{cases} \) continuous at \( x = 2 \)?

Prove your result.

Only for \( a = 4 \).

If \( x \neq 2 \), then \( f(x) = \frac{x^2 - 4}{x - 2} = \frac{(x + 2)(x - 2)}{x - 2} = x + 2 \). So,

\[
\lim_{x \to 2} f(x) = \lim_{x \to 2} x + 2 = 4.
\]

In order for \( f(x) \) to be continuous at \( x = 2 \), we need:

\[
\lim_{x \to 2} f(x) = f(2)
\]

That is, we need \( 4 = a \).
Example 2: Let $f(x) = \frac{(x + 1)x(x - 1)(x - 2)}{(x - 1)(x - 2)}$

Note the domain of $f(x)$ does not include $x = 1$ or $x = 2$. Evaluate the following:

(a) $\lim_{{x \to 0}} f(x)$

(b) $\lim_{{x \to 1}} f(x)$

(c) $\lim_{{x \to 2}} f(x)$

(d) $\lim_{{x \to 3}} f(x)$
Rational Functions with Holes

Example 2: Let \( f(x) = \frac{(x + 1)x(x - 1)(x - 2)}{(x - 1)(x - 2)} \)

Note the domain of \( f(x) \) does not include \( x = 1 \) or \( x = 2 \). Evaluate the following:

(a) \( \lim_{x \to 0} f(x) = f(0) = 0 \)

(b) \( \lim_{x \to 1} f(x) = \lim_{x \to 1} \left[\frac{(x + 1)x(x - 1)(x - 2)}{(x - 1)(x - 2)}\right] = 2 \)

(c) \( \lim_{x \to 2} f(x) = \lim_{x \to 2} \left[\frac{(x + 1)x(x - 1)(x - 2)}{(x - 1)(x - 2)}\right] = 6 \)

(d) \( \lim_{x \to 3} f(x) = f(3) = \frac{4 \times 3 \times 2 \times 1}{2 \times 1} = 12 \)
Rational Functions with Holes

Example 2: Let \( f(x) = \frac{(x + 1)x(x - 1)(x - 2)}{(x - 1)(x - 2)} \)

Note the domain of \( f(x) \) does not include \( x = 1 \) or \( x = 2 \). Evaluate the following:

(a) \( \lim_{x \to 0} f(x) = f(0) = 0 \)

(b) \( \lim_{x \to 1} f(x) = \lim_{x \to 1} [(x + 1)x] = 2 \)

(c) \( \lim_{x \to 2} f(x) = \lim_{x \to 2} [(x + 1)x] = 6 \)

(d) \( \lim_{x \to 3} f(x) = f(3) = \frac{4 \times 3 \times 2 \times 1}{2 \times 1} = 12 \)
Using limits to evaluate derivatives

Evaluate the derivative of \( f(x) = \sqrt{x} \) using the definition of the derivative,

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
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Evaluate the derivative of \( f(x) = \sqrt{x} \) using the definition of the derivative, 

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\]

\[
f'(x) = \lim_{h \to 0} \frac{\sqrt{x + h} - \sqrt{x}}{h} = \lim_{h \to 0} \frac{\sqrt{x + h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x + h} + \sqrt{x}}{\sqrt{x + h} + \sqrt{x}}
\]

\[
= \lim_{h \to 0} \frac{(x + h) - x}{h(\sqrt{x + h} + \sqrt{x})} = \lim_{h \to 0} \frac{h}{h(\sqrt{x + h} + \sqrt{x})}
\]

\[
= \lim_{h \to 0} \frac{1}{\sqrt{x + h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}
\]
Chapter 3: Three faces of the derivative

3.2 The analytic view: calculating the derivative

Using limits to evaluate derivatives

Example 3:

Find the derivatives of the following functions using the definition of a derivative:

(a) \( f(x) = \frac{1}{x} \)

(b) \( g(x) = \frac{1}{\sqrt{x}} \)

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]
Using limits to evaluate derivatives

Example 3:

Find the derivatives of the following functions using the definition of a derivative:

(a) \( f(x) = \frac{1}{x} \)

\[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \to 0} \frac{\frac{x - (x+h)}{x(x+h)}}{h} = \lim_{h \to 0} \frac{x - (x + h)}{x(x + 0)(h)} = \frac{-1}{x(x + 0)} = -\frac{1}{x^2} \]

(b) \( g(x) = \frac{1}{\sqrt{x}} \)

\[ g'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} = \lim_{h \to 0} \frac{\frac{x - (x+h)}{x(x+h)^{3/2}}}{h} = \lim_{h \to 0} \frac{x - (x + h)}{x(x + 0)h^{3/2}} = \frac{-1}{x(x + 0)h^{3/2}} \]
Using limits to evaluate derivatives

Example 3:

Find the derivatives of the following functions using the definition of a derivative:

(b) \( g(x) = \frac{1}{\sqrt{x}} \)

\[
g'(x) = \lim_{h \to 0} \frac{g(x + h) - g(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} = \lim_{h \to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h \sqrt{x+h} \sqrt{x}} \left( \frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}} \right)
\]

\[
= \lim_{h \to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h \sqrt{x+x+h} \sqrt{x}} \frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}} = \lim_{x \to 0} \frac{-h}{h \sqrt{x} + h \sqrt{x + h} + \sqrt{x}}
\]

\[
= \lim_{x \to 0} \frac{-1}{\sqrt{x} + \sqrt{x + h} + \sqrt{x}} = \frac{1}{\sqrt{x} + 0 \sqrt{x + 0 + \sqrt{x}}}
\]

\[
= \frac{-1}{x(2\sqrt{x})} = -\frac{1}{2x^{3/2}}
\]
Concept Check

True or False: If \( f(x) \) is not defined at \( x = 1 \), then \( \lim_{x \to 1} f(x) \) does not exist.

True or False: If \( \lim_{x \to 1} f(x) \) exists, then it’s equal to \( f(1) \).

True or False: If \( f(x) \) is continuous at \( x = 1 \), then \( f'(1) \) exists.

True or False: If \( f'(1) \) exists, then \( f(x) \) is continuous at \( x = 1 \).
Concept Check

True or False: If $f(x)$ is not defined at $x = 1$, then 
$$\lim_{x \to 1} f(x)$$
does not exist.
In general, false. The limit completely ignores what happens at $x = 1$. The limit may or may not exist. In most examples we've seen of limits, we are exclusively interested in the limit of a function where the function does not exist.

True or False: If $\lim_{x \to 1} f(x)$ exists, then it’s equal to $f(1)$.
In general, false. This is true if the function is continuous at $x = 1$, and false otherwise.

True or False: If $f(x)$ is continuous at $x = 1$, then $f'(1)$ exists.
In general, false. The derivative may or may not exist. For example, there may be a cusp, with no derivative; or there may be a smooth line, with a derivative.

True or False: If $f'(1)$ exists, then $f(x)$ is continuous at $x = 1$.
True. In order for a function to be differentiable, it must be continuous.
Example 4: Approximate \( \lim_{x \to 0} \frac{\sin x}{x} \) using the graph below.
Example 4: Approximate \( \lim_{{x \to 0}} \frac{\sin x}{x} \) using the graph below.

\[ \lim_{{x \to 0}} \frac{\sin x}{x} = 1 \]
Overview

1. Use software to numerically compute an approximation to the derivative.
2. Explain that the approximation replaces a (true) tangent line with an (approximating) secant line.
Overview

1. Use software to numerically compute an approximation to the derivative.
2. Explain that the approximation replaces a (true) tangent line with an (approximating) secant line.
3. Explain using words how the derivative shape is connected with the shape of the original function.
Idea: Approximate Tangent Line with Secant Line
Idea: Approximate Tangent Line with Secant Line

Derivative (slope of tangent line) \[ \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]

Approximate derivative (slope of secant line) \[ \frac{f(x+h) - f(x)}{h} \]
Idea: Approximate Tangent Line with Secant Line

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Approximate derivative Slope of secant line
\[ \frac{f(x + h) - f(x)}{h} \]
Approximate Derivative

\[ f(x) = \frac{2x^5}{2^5 + x^5} \]
Approximate Derivative

\[ f(x) = \frac{2x^5}{2^5 + x^5} \]

\[ f'(1) = \lim_{{h \to 0}} \frac{f(1 + h) - f(1)}{h} = \frac{320}{33^2} \approx 0.29385... \]
### Approximate Derivative

Given the function

\[ f(x) = \frac{2x^5}{2^5 + x^5} \]

we can approximate its derivative at \( x = 1 \) using the limit:

\[
f'(1) = \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} = \frac{320}{33^2} \approx 0.29385\ldots
\]

<table>
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<th>( \frac{f(1 + h) - f(1)}{h} )</th>
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<td></td>
</tr>
<tr>
<td>0.00001</td>
<td></td>
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</tbody>
</table>

[spreadsheet link]
Approximate Derivative

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<table>
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<th>( \frac{f(1 + h) - f(1)}{h} )</th>
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</tr>
<tr>
<td>0.00001</td>
<td>0.29385\ldots</td>
</tr>
</tbody>
</table>

spreadsheet link
Approximate Derivative

\[ f(x) = \frac{2x^5}{2^5 + x^5} \]

\[ f'(1) \approx \frac{f(1 + h) - f(1)}{h} \]

Using very small values of \( h \) can get us an approximation of the derivative at a particular point.
Approximate Derivative

\[ f(x) = \frac{2x^5}{2^5 + x^5}, \quad f'(1) \approx \frac{f(1 + 0.1) - f(1)}{0.1} \]

Using very small values of \( h \) can get us an approximation of the derivative at a particular point.
Approximate Derivative

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Using very small values of \( h \) can get us an approximation of the derivative at a particular point.

We can also use a spreadsheet to guess the derivative at a number of different points, using tiny secant lines.
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Using very small values of \( h \) can get us an approximation of the derivative at a particular point.
We can also use a spreadsheet to guess the derivative at a number of different points, using tiny secant lines.

![Graph showing the approximate derivative using secant lines at different points.](image)
Approximate derivative at many points

\[ f(x) = \frac{2x^5}{2^5 + x^5} \]

Interval:
Small \( h \):
Chapter 3: Three faces of the derivative  
3.3 The computational view: software to the rescue!

Approximate derivative at many points

\[ f(x) = \frac{2x^5}{2^5 + x^5} \]

Interval: \([-1, 4]\)
Small \(h\):
Approximate derivative at many points

\[ f(x) = \frac{2x^5}{2^5 + x^5} \]

Interval: \([-1, 4]\)
Small \(h\): say \(h = 0.1\)
Approximate derivative at many points

\[ f(x) = \frac{2x^5}{2^5 + x^5} \]

**Interval:** \([-1, 4]\)

**Small \( h \):** say \( h = 0.1 \)
Predator-Prey Interactions: Holling Predator Response

\[ P(x) = \frac{3x^3}{1 + x^3} \]

**Type III:** I can hardly find the prey when the prey density is low, but I also get satiated at high prey density.
Predator-Prey Interactions: Holling Predator Response

\[ P(x) = \frac{3x^3}{1 + x^3} \]

**Type III:** I can hardly find the prey when the prey density is low, but I also get satiated at high prey density.

(a) Label the asymptote on the y-axis.
(b) Label the half-max activation level on the x-axis.
Predator-Prey Interactions: Holling Predator Response

Type III: I can hardly find the prey when the prey density is low, but I also get satiated at high prey density.

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Predator-Prey Interactions: Holling Predator Response

Type III: I can hardly find the prey when the prey density is low, but I also get satiated at high prey density.

(a) Label the asymptote on the $y$-axis.
(b) Label the half-max activation level on the $x$-axis.
(c) At what prey density is the predation rate changing the fastest? 
That is, at which density is each additional prey animal responsible for the most increase in feeding by the predators?

Make a prediction, then verify with a spreadsheet.
Predator-Prey Interactions: Holling Predator Response

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(c) At what prey density is the predation rate changing the fastest? That is, at which density is each additional prey animal responsible for the most increase in feeding by the predators?

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