Welcome to Math 102, Calculus for the Life Sciences

Hi! I’m Dr. Elyse Yeager.
You can call me Elyse, or Dr. Yeager.

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229F Mathematics building
Office hours: Mon 11:30am-1:00pm; Wed 1:30-3:00pm;
other times by appointment.
Office hours start next week.

Basically all of information about this course is on our webpage:
wiki.math.ubc.ca/mathbook/M102/

You can also find it by Googling something like “UBC Math 102 wiki”

Our text is a free, online pdf.

What will we learn?

Differential calculus: study of rates of change.
How fast is a population growing; how fast is the speed of your car changing; more
abstract uses.

Emphasis on life sciences interpretations.
Our book is tailored towards biological applications, but the underlying mathematics is
very similar to other flavours of calculus at UBC and other universities.

Science Communication
You’ll have assignments where you will have to clearly explain your reasoning.

Technology
You will also have assignments that involve using spreadsheet programs, like Excel or
Google Sheet (which is free).

Problem Solving
!!!!!!!!!!!!!!!!!!!!

Administrative Info

Once more, with feeling:

wiki.math.ubc.ca/mathbook/M102/

Grade breakdown:
50% Final exam (don’t make travel plans until it’s scheduled, in October)
15% Midterm (Oct 26, evening)
15% 3 in-class quizzes (register with Access and Diversity if appropriate)
10% WebWork (online homework–first one due next Monday)
10% OSH (old-school homework, 6 assignments, presentation matters)

Course calendar on website.

More info later.
In Class

These notes are on the wiki. Frame-by-frame and print versions available on our section’s webpage.

Some exercises I do, some you do. Bring something to write with and on. When you’re working, talking with your neighbours is encouraged.

I’ll wander around while you’re working. Sit on the aisle and I’ll peek at your work; sit in the middle and I won’t bother you.

Asking questions is encouraged, but they should be relevant to the topic at hand.

Chapter 1: Power Functions as Building Blocks

Building Blocks

We’re going to be using a lot of examples, and in order to start off simply, most of these examples will use power functions.

Let’s get to know power functions:

- What they are
- Shapes, relative to one another
- Behaviour near the origin and far away
- Where two power functions intersect

(None of this is calculus—quick review.)

Power Functions

Power functions:

\[ f(x) = Kx^n, \quad \text{where } n \text{ is a positive integer and } K \text{ is a nonzero constant} \]

Which are power functions?

\[ f(x) = x^2 \]
\[ f(x) = \pi x^{102} \]
\[ f(x) = \sqrt{x} = x^{1/2} \]
\[ f(x) = \frac{1}{x} = x^{-1} \]
Chapter 1: Power Functions as Building Blocks

1.1: Power Functions

Relative Shapes of Power Functions

Familiar: \( y = x^2 \)

What about \( y = x^3 \)?

- Flatter: \( y = x^1 \)
- Steeper: \( y = x^2 \)
- Lower power dominates
- Higher power dominates

Example 1: Where do \( f(x) = 3x^4 \) and \( g(x) = 5x^7 \) intersect?

Notes

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Intersections of Power Functions: solution

**Solution 1:** Where do \( f(x) = 3x^4 \) and \( g(x) = 5x^7 \) intersect?

\[
5x^7 = 3x^4 \\
x^4(5x^3 - 3) = 0 \\
x^4 = 0 \quad \text{OR} \quad 5x^3 - 3 = 0 \\
x = 0 \quad \text{OR} \quad 5x^3 = 3 \\
x = 0 \quad \text{OR} \quad x^3 = \frac{3}{5} \\
x = 0 \quad \text{OR} \quad x = \sqrt[3]{\frac{3}{5}}
\]

Remark: finding a decimal approximation for \( \sqrt[3]{\frac{3}{5}} \) is pretty technical. We'll learn methods later on.

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**Example 2:** Where do \( f(x) = ax^n \) and \( g(x) = bx^m \) intersect?

You may assume that \( a \) and \( b \) are positive, and \( n \) and \( m \) are positive integers with \( n < m \).

\[
ax^n = bx^m \\
x^n(a - bx^{m-n}) = 0 \\
x^n = 0 \quad \text{OR} \quad a - bx^{m-n} = 0 \\
x = 0 \quad \text{OR} \quad a = bx^{m-n} \\
x = 0 \quad \text{OR} \quad \frac{a}{b} = x^{m-n} \\
x = 0 \quad \text{OR} \quad \left( \frac{a}{b} \right)^{\frac{1}{m-n}} = x
\]
Learning goals

- Follow and understand the derivation of a mathematical model (a simplified representation of a real situation) for cell nutrient absorption and consumption.
- Develop the skill of using parameters \((k_1, k_2)\) rather than specific numbers in mathematical expressions.
- Understand the link between power functions in Section 1.1 and cell nutrient balance in the model.
- Be able to verbally interpret the results of the model.

**Spherical Cells - Setup**

A cell absorbs nutrients through its wall, and metabolizes them inside.

**Assumptions:**

1. The cell is a smooth sphere. (Many cells are not even close to spherical—this model does not describe them well.)
2. The cell absorbs oxygen and nutrients through its surface. The larger the surface area, \(S\), the faster the total rate of absorption. We will assume that the rate of absorption is proportional to the surface area of the cell.
3. The rate at which nutrients are consumed (i.e., used up) in metabolism is proportional the volume, \(V\), of the cell. The bigger the volume, the more nutrients are needed to keep the cell alive.

**Constants:**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(S)</td>
<td>surface area of cell</td>
</tr>
<tr>
<td>(A)</td>
<td>rate of absorption</td>
</tr>
<tr>
<td>(k_S)</td>
<td>proportionality constant between surface area and absorption rate</td>
</tr>
<tr>
<td>(V)</td>
<td>volume of cell</td>
</tr>
<tr>
<td>(C)</td>
<td>rate of consumption</td>
</tr>
<tr>
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<td>proportionality constant between volume and consumption rate</td>
</tr>
</tbody>
</table>
Chapter 1: Power Functions as Building Blocks

1.2: How big can a cell be? A model for nutrient balance

Spherical Cells - Setup

\[ A = k_S \cdot S \quad \text{for some positive constant } k_S \]
\[ C = k_V \cdot V \quad \text{for some positive constant } k_V \]

Constants:

\[ S = 4\pi r^2 \quad \text{surface area of cell} \]
\[ A \quad \text{rate of absorption} \]
\[ k_S \quad \text{proportionality constant between surface area and absorption rate} \]
\[ r \quad \text{radius of cell} \]
\[ V = \frac{4}{3}\pi r^3 \quad \text{volume of cell} \]
\[ C \quad \text{rate of consumption} \]
\[ k_V \quad \text{proportionality constant between volume and consumption rate} \]

In order for the cell to not "starve," it needs to take in nutrients at least as fast as it can use them. That is, it must have:

\[
A \geq C \\
k_S \cdot S \geq k_V \cdot V \\
\frac{4\pi r^2}{3} \geq \frac{4\pi r^3}{3} \\
r \leq 3 \left( \frac{k_S}{k_V} \right)
\]

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\]

How Applicable is the Model?

Conclusion

\[ r \leq 3 \left( \frac{k_S}{k_V} \right) \]

If a cell fits our model, it can’t be too big, or it will starve.

- Most cells are not too big; less than 1 mm in diameter.
- Notable exceptions don’t fit the model:
  - Neurons are not spherical; they can get huge.

- Large plant cells may have a lot of water, and a vacuole to store nutrients

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Related Results: Bergmann’s Rule

Roughly speaking, warm-blooded animals in cold climates are bigger than their relatives in warm climates.

↑ colder

- White-tailed Deer, Canada 45-125 kg
- Wood Bison, avg 840 kg
- Polar bear, 350-700 kg
- Snowman, 10-200 kg
- Arctic Hare, 2.5-7 kg

↓ warmer

- Florida Keys and Tropics 35-50 kg
- Plains bison, avg 730 kg
- Brown bear, 80-600 kg
- Tropical snowman, 0-15 kg
- Rabbit, 0.5-2 kg

Obviously, it’s more complicated than this:

one of several factors, many exceptions, possibly not true at all.
Graph Sketching: Learning Goals

- Identify even and odd functions graphically/algebraically
- Sketch simply polynomials like $y = ax^n + bx^m$
- Sketch simple rational functions like $y = \frac{Ax^n}{b + x^m}$
- Gain intuition about the behaviours of simple functions

Even and Odd Power Functions

Power functions with even powers:

Symmetry: mirrored about y-axis

Power functions with odd powers:

Symmetry: Mirrored about both axes.
Even and Odd Power Functions

Example 3: Even, Odd, or Neither?

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
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<tbody>
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<td>A</td>
<td>B</td>
</tr>
<tr>
<td>C</td>
<td>D</td>
</tr>
</tbody>
</table>

Even function:

\[ y = f(x) \]

If \( f(3) = 7 \), then \( f(-3) = 7 \)

**Definition**

A function \( f(x) \) is **even** if \( f(x) = f(-x) \) for all \( x \) in its domain.

Odd function:

\[ y = f(x) \]

If \( f(3) = 7 \), then \( f(-3) = -7 \)

**Definition**

A function \( f(x) \) is **odd** if \( -f(x) = f(-x) \) for all \( x \) in its domain.
Even and Odd Functions

Definition
A function \( f(x) \) is even if \( f(x) = f(-x) \) for all \( x \) in its domain.
A function \( f(x) \) is odd if \( -f(x) = f(-x) \) for all \( x \) in its domain.

Example 4: Show that the following functions are even, odd, or neither.

- \( f(x) = x^4 - x^2 \)
- \( g(x) = x^3 + \frac{1}{x} \)
- \( h(x) = x + x^2 \)

Notes

A Quick Note about Approximations

There are about 7 billion people on the planet. One person is born. What is a good approximation for the number of people on the planet now?

Example 5: Partner A has way more money in the bank than Partner B.

They combine their accounts. What is the more reasonable approximation for their combined account balance?

(a) Partner A’s original balance
(b) Partner B’s original balance

Notes

Using Approximations to sketch Two-Term Polynomials

Notes
Using Approximations to Sketch Two-Term Polynomials

- \( f(x) = 3x + x^2 \)
- \( f(x) = 3x + x^2 \)
  - \( y = 3x \): straight line, pointing up
  - \( x^2 \): parabola, pointing up
  - Smaller power dominates near origin
  - Larger power dominates elsewhere

\( y \)
\( x \)

Using Approximations to Sketch Two-Term Polynomials

- \( f(x) = x^2 - x^4 \)
- \( f(x) = x^2 - x^4 \)
  - \( y = x^2 \): parabola, pointing up
  - \( -x^4 \): pointing down
  - Smaller power dominates near origin
  - Larger power dominates elsewhere

\( y \)
\( x \)

Now You!

- \( f(x) = x^3 - x^2 \)

\( y \)
\( x \)
Example 6: You buy one million grains of rice, and each grain of rice costs $1,000,000. (That is, each grain costs one-tenth of a cent.)

What is a good approximation for the amount of money you spent?
A. About $1,000,000
B. About $1
C. Both of these approximations are terrible.

The method we're using for approximating ("choose the bigger one") works pretty well for addition and subtraction, but not so much for multiplication.

Simple Rational Functions

Definition
A rational function is the quotient of two polynomials, \( \frac{\text{poly}}{\text{poly}} \),

\[ f(x) = \frac{Ax^n}{B + x^m} \]

We can use our approximation method on \( B + x^n \).

* \( f(0) = \)

* When \( x \) is very close to zero, \( B + x^n \approx B \), so
  \[ f(x) \approx \]

* When \( x \) is very far from zero, \( B + x^n \approx x^n \), so
  \[ f(x) \approx \]

Example 7: Sketch \( f(x) = \frac{5x^2}{1 + x^3} \)

\[ f(0) = 0 \]

* Near the origin, \( f(x) = \frac{5x^2}{1} = 5x^2 \)
  
* Away from origin, \( f(x) \approx \frac{5x^2}{x^3} = 5 \)
Enzyme-Catalyzed Reactions

The rate of reaction depends on the concentration of substrate, and properties of enzyme:

\[ y = \frac{A}{\frac{B}{x} + 1} \]

Michaelis-Menten kinetics

Hill Functions (Archibald Hill, 2010)

Definition

A Hill function has the form

\[ H(x) = \frac{Ax^n}{a^n + x^n}, \quad x \geq 0 \]

for some constants \( A, a \). We call \( a \) the half-maximal activation level.

Predator-Prey Interactions: Holling Predator Response

Type I: The more prey there is, the more I can eat
Type II: I get satiated and cannot keep eating more and more prey
Type III: I can hardly find the prey when the prey density is low, but I also get satiated at high prey density:

\[ P(x) = Kx \quad P(x) = \frac{Kx}{\frac{a}{x} + x} \quad P(x) = \frac{Kx^n}{a^n + x^n}, \quad n \geq 2 \]