First-Order Linear Differential Equation

\[ \frac{dy}{dt} = a - by \]

with \( y(0) = y_0 \) given.

Steady state solutions:

\[ 0 = \frac{dy}{dt} = a - by \]

\[ y = \frac{a}{b} \]

Let’s change this to a different form:

\[ \frac{dy}{dt} = -b \left( y - \frac{a}{b} \right) \]

**deviation**: how far \( y \) is from its steady state

Let \( z(t) = y(t) - \frac{a}{b} \)

\[ \frac{dz}{dt} = \frac{dy}{dt} \]

\[ \frac{dz}{dt} = -bz \]

\[ z(t) = Ce^{-bt} \]

\[ y(t) - \frac{a}{b} = Ce^{-bt} \]
Transformation Interpretation

- \( \frac{dy}{dt} = a - by \)
- \( y = \frac{a}{b} \): steady state solution
- \( z = y(t) - \frac{a}{b} \): deviation from the steady state
- \( z = Ce^{-bt} \): for \( b > 0 \), the deviation decays exponentially.
  That is, if \( y \) is far from \( b/a \), it will approach it rapidly.
- \( y(t) = \frac{a}{b} + Ce^{-bt} \): general solution
- \( y(t) = \frac{a}{b} + (y_0 - \frac{a}{b})e^{-bt} \)
  \( C \) is the initial deviation

Example 1

What is the solution to the differential equation \( \frac{dy}{dt} = 3 - 2y \), \( y(0) = \frac{1}{2} \)?

Newton’s Law of Cooling

The rate of change of temperature \( T \) of an object is proportional to the difference between its temperature and the ambient temperature, \( E \).

- \( E \): steady state
- \( \frac{dT}{dt} \): proportional to the deviation from the steady state,
  \( z(t) = T(t) - E \)
- \( \frac{dz}{dt} = -\alpha z \)
  Following our earlier calculation, \( T(t) = E + (T_0 - E)e^{-\alpha t} \)

Example 2: A farrier works a horseshoe heated to 400° C, then dunks it in a pool of room-temperature (25° C) water. The water near the horseshoe boils for 30 seconds, but the temperature of the pool as a whole hasn’t changed appreciably. The horseshoe is safe for the horse when it’s 40° C. When can the farrier put on the horseshoe? You may assume the water stops boiling when the horseshoe dips below 100° C.
Suppose a body is discovered at 3:45 pm, in a room held at 20\(^\circ\); and the body’s temperature is 27\(^\circ\), not the normal 37\(^\circ\). At 5:45 pm, the temperature of the body has dropped to 25.3\(^\circ\). When did the owner of the body die?

You may assume \( T(t) = E + (T_0 - E)e^{-\alpha t} \).

A glass of just-boiled tea is put on a porch outside. After ten minutes, the tea is 40\(^\circ\), and after 20 minutes, the tea is 25\(^\circ\). What is the temperature outside? You may assume \( T(t) = E + (T_0 - E)e^{-\alpha t} \).

### Numerical solutions versus analytic solutions

**Definition**

- **Analytic solution** to a DE: explicit formula for the solution
- **Numerical solution** to a DE: approximation of the solution

We saw something similar with Newton’s Method:

\[ f(x) = x^2 - 3 \]

**Roots:** \( \pm \sqrt{3} \) (analytic)

**Roots:** approximately \( \pm 1.73 \) (numeric)

\[ f(x) = x^3 + x^3 + 1 \]

**Root:** ??? (analytic)

**Root:** approximately 0.838 (numeric)
Numerical solutions to diff eqs: Euler’s method

Concept
Euler’s method turns slope fields into equations.

\[ \frac{dy}{dt} = y^2, \quad y(0) = \frac{1}{2} \]

Euler’s method: computation

- Start with a differential equation with initial values, e.g. \( \frac{dy}{dt} = y^2 \) and \( y_0 = \frac{1}{2} \)
- Choose a step, usually called \( \Delta t \), preferably small
- Make a linear approximation of \( y(t_k) \) to approximate \( y(t_{k+1}) \)
- Iterate.

\[ y_1 = y_0 + \Delta y \]

Example 3:
Use Euler’s method, with \( \Delta t = 0.1 \), to approximate \( y(0.5) \), given the following:

\[ \frac{dy}{dt} = y + 2e^{y-1}, \quad y(0.3) = -1 \]
Euler’s method:  
\[ y_{k+1} \approx y_k + y'_k \cdot \Delta t \]

**Example 4:**
Suppose  
\[ \frac{dy}{dt} = \frac{1}{y} \]
Use Euler’s method, with \( \Delta t = 0.05 \), to approximate \( y(0.1) \), given these initial conditions:
(a) \( y(0) = 1 \)
(b) \( y(0) = -\frac{1}{e} \)

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**Disease Dynamics**

Setup:
- \( S(t) \): susceptible individuals
- \( I(t) \): infected (and infectious) individuals
- Everybody mixes
- Fixed probability of catching illness from contact
- Fixed time to get better

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**Disease Dynamics**

Infected population:
- According to the law of mass action, rate of transmission should be proportional to the product of the two populations, \( S \cdot I \)
- Let \( \beta \) be that constant of proportionality, so the rate of new infections is \( \beta(SI) \).
- Let \( \mu \) be the rate of recovery per person, so the rate of recovery over the infected population is \( \mu I \).

\[ \frac{dI}{dt} = \text{[rate of new infections]} - \text{[rate of recovery]} \]
\[ = \beta IS - \mu I \]
\[ = \beta I(N - I) - \mu I \]

where \( N \) is the total (constant) size of the population, \( S + I \).
Disease dynamics

\[ \frac{dI}{dt} = \beta I (N - I) - \mu I = \beta I (K - I) \]

Where \( K \) is some constant, \( N \) is the size of the population, \( \mu \) is the rate of recovery, and \( \beta \) is a measure of ease of transmission.

(a) What is \( K \), in terms of the other constants?

(b) Which leads to a larger \( K \): a disease with difficult transmission and quick recovery, or a disease with easy transmission and slow recovery?

The sign of \( K \) is important!

\[ K = N - \frac{\mu}{\beta} \]

- \( \frac{N\beta}{\mu} > 1 \): disease becomes endemic, stabilizes at some percent of the population being sick all the time
- \( \frac{N\beta}{\mu} < 1 \): disease is eradicated, because people recover faster than they get sick