First-Order Linear Differential Equation

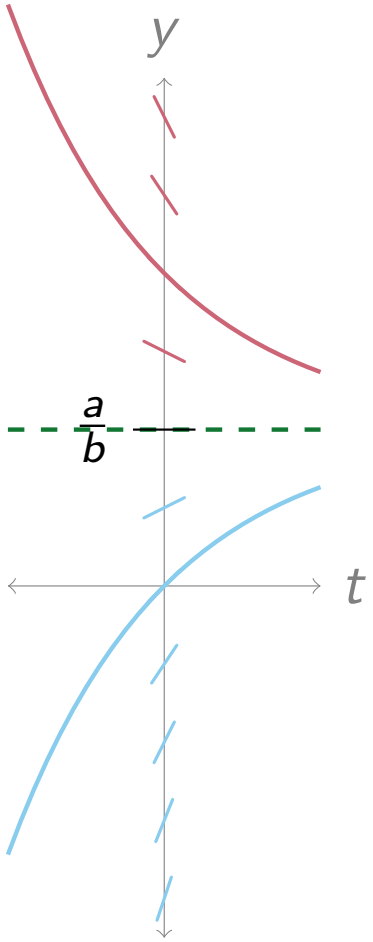
$$\frac{dy}{dt} = a - by$$

with $y(0) = y_0$ given.

Steady state solutions:

$$0 = \frac{dy}{dt} = a - by$$
$$y = \frac{a}{b}$$

$$\frac{dy}{dt} = a - by$$



Transformation

$$\frac{dy}{dt} = a - by$$

Let's change this to a different form:

$$\frac{dy}{dt} = -b\underbrace{\left(y - \frac{a}{b}\right)}_{\text{deviation}}$$

deviation: how far y is from its steady state

$$\frac{dz}{dt} = \frac{dy}{dt}$$

Let $z(t) = y(t) - \frac{a}{b}$

$$\frac{dz}{dt} = -bz$$

$$z(t) = Ce^{-bt}$$

$$y(t) - \frac{a}{b} = Ce^{-bt}$$

Transformation Interpretation

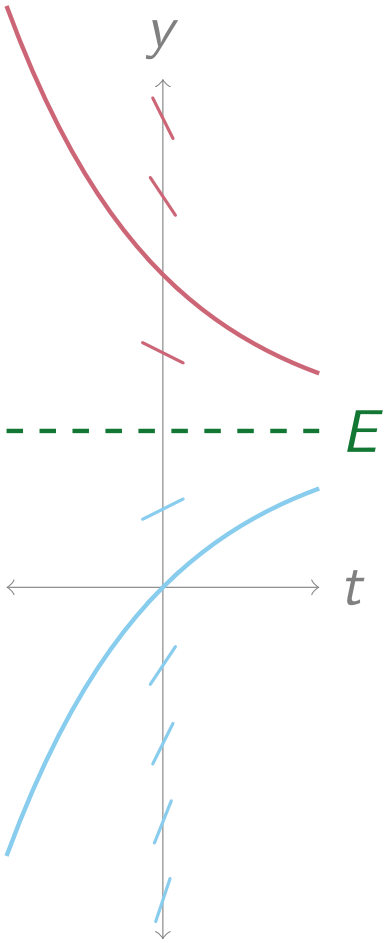
- $\frac{dy}{dt} = a - by$
- $y = \frac{a}{b}$: steady state solution
- $z = y(t) - \frac{a}{b}$: deviation from the steady state
- $z = Ce^{-bt}$: for $b > 0$, the deviation decays exponentially. That is, if y is far from b/a, it will approach it rapidly.
- $y(t) = \frac{a}{b} + Ce^{-bt}$: general solution
- $y(t) = \frac{a}{b} + \left(y_0 - \frac{a}{b}\right) e^{-bt}$
  C is the initial deviation

Example 1:

What is the solution to the differential equation $\frac{dy}{dt} = 3 - 2y$, $y(0) = \frac{1}{2}$?

Newton's Law of Cooling

The rate of change of temperature T of an object is proportional to the difference between its temperature and the ambient temperature, E.



- E: steady state

Newton's law of cooling: $T(t) = E + (T_0 - E)\, e^{-\alpha t}$

Example 2: A farrier works a horseshoe heated to 400° C,

then dunks it in a pool of room-temperature (25° C) water. The water near the horseshoe boils for 30 seconds, but the temperature of the pool as a whole hasn't changed appreciably. The horseshoe is safe for the horse when it's 40° C. When can the farrier put on the horseshoe? You may assume the water stops boiling when the horseshoe dips below 100°C.

source

Suppose a body is discovered at 3:45 pm, in a room held at 20°, and the body's temperature is 27°: not the normal 37°. At 5:45 pm, the temperature of the body has dropped to 25.3°. When did the owner of the body die?

You may assume $T(t) = E + (T_0 - E)e^{-\alpha t}$.

A glass of just-boiled tea is put on a porch outside. After ten minutes, the tea is 40°, and after 20 minutes, the tea is 25°. What is the temperature outside? You may assume $T(t) = E + (T_0 - E)e^{-\alpha t}$.

Numerical solutions verses analytic solutions

Definition

Analytic solution to a DE: explicit formula for the solution
Numerical solution to a DE: approximation of the solution

We saw something similar with Newton's Method:

$$f(x) = x^2 - 3$$

Roots: $\pm\sqrt{3}$ (analytic) Roots: approximately ± 1.73 (numeric)

$$f(x) = x^5 + x^3 + 1$$

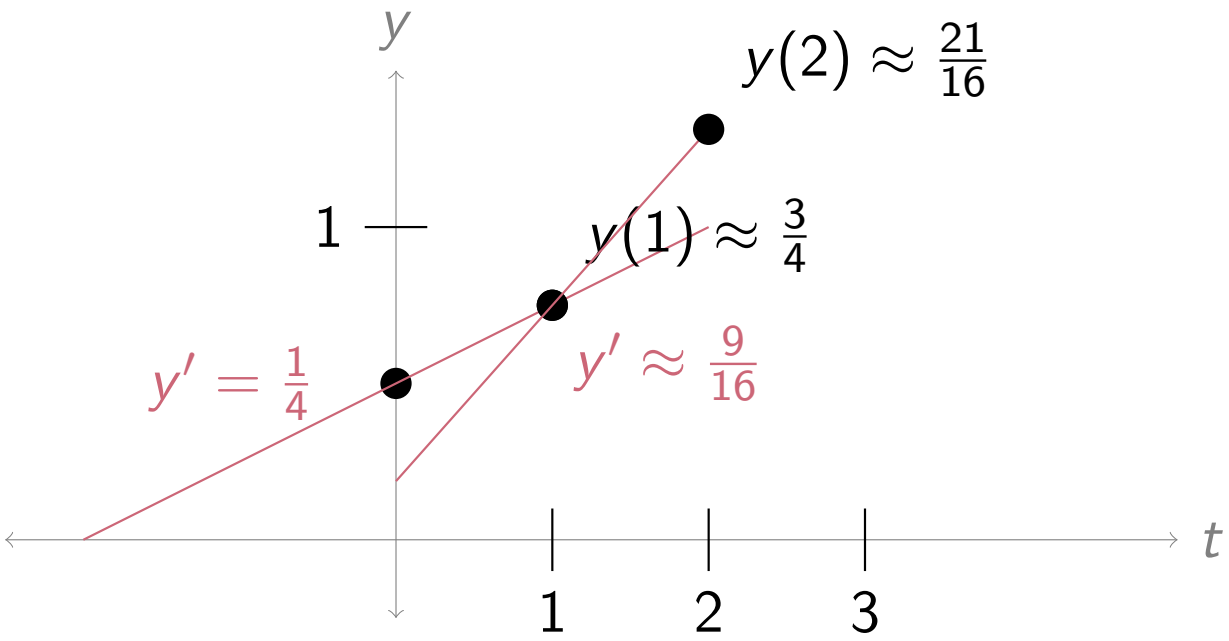
Root: ????? (analytic) Root: approximately 0.838 (numeric)

Numerical solutions to diff eqs: Euler's method

Concept

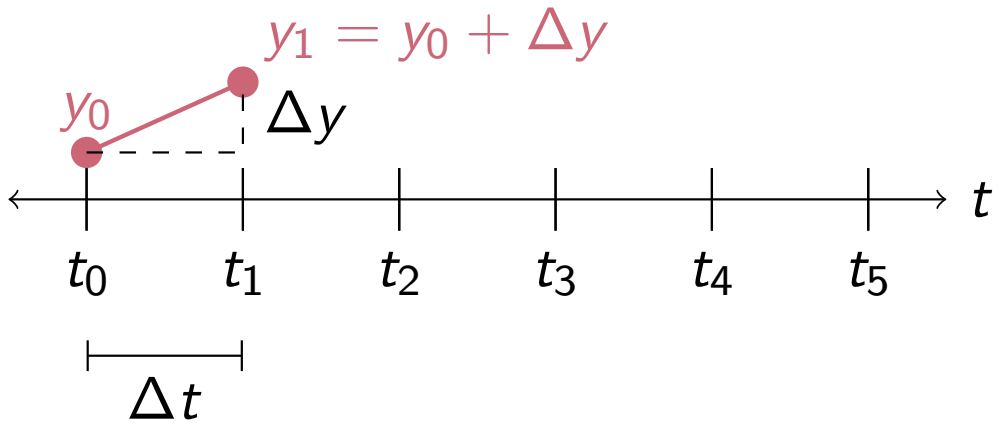
Euler's method turns slope fields into equations.

$$\frac{dy}{dt} = y^2 \qquad y(0) = \frac{1}{2}$$



Euler's method: computation

- Start with a differential equation with initial values, e.g. $\frac{dy}{dt} = y^2$ and $y_0 = \frac{1}{2}$
- Choose a step, usually called Δt, preferably small
- Make a linear approximation of $y(t_k)$ to approximate $y(t_{k+1})$.
- Iterate.



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Euler's method: $y_{k+1} \approx y_k + y_k' \cdot \Delta t$

Example 3:

Use Euler's method, with $\Delta t = 0.1$, to approximate $y(0.5)$, given the following:

$$\frac{dy}{dt} = y + 2e^{y+1}, \quad y(0.3) = -1$$

Euler's method: $y_{k+1} \approx y_k + y_k' \cdot \Delta t$

Example 4:

Suppose

$$\frac{dy}{dt} = \frac{1}{y}.$$

Use Euler's method, with $\Delta t = 0.05$, to approximate $y(0.1)$, given these initial conditions:

(a) $y(0) = 1$

(b) $y(0) = -\frac{1}{4}$

Disease Dynamics

Setup:

- $S(t)$: susceptible individuals
- $I(t)$: infected (and infectious) individuals
- Everybody mixes
- Fixed probability of catching illness from contact
- Fixed time to get better

Disease dynamics

Infected population:

- According to the law of mass action, rate of transmission should be proportional to the product of the two populations, $S \cdot I$
- Let β be that constant of proportionality, so the rate of new infections is $\beta(SI)$.
- Let μ be the rate of recovery per person, so the rate of recovery over the infected population is μI.
- $$\begin{aligned} \frac{dI}{dt} &= [\text{rate of new infections}] - [\text{rate of recovery}] \\ &= \beta IS - \mu I \\ &= \beta I(N - I) - \mu I \end{aligned}$$

  where N is the total (constant) size of the population, $S + I$.

Disease dynamics

$$\frac{dI}{dt} = \beta I(N - I) - \mu I = \beta I(K - I)$$

Where K is some constant, N is the size of the population, μ is the rate of recovery, and β is a measure of ease of transmission.

(a) What is K, in terms of the other constants?

(b) Which leads to a larger K: a disease with difficult transmission and quick recovery, or a disease with easy transmission and slow recovery?

Disease dynamics

$$\frac{dI}{dt} = \beta I(N - I) - \mu I = \beta I(K - I)$$

Where $K = N - \frac{\mu}{\beta}$, N is the size of the population, μ is the rate of recovery, and β is a measure of ease of transmission.

(a) What are the steady states of this differential equation?

(b) Draw two state diagrams: one for the case $K > 0$, one for the case $K < 0$.

(c) Which steady states are stable and which not?

(d) Give a biological interpretation for the steady states.

Disease Dynamics

The sign of K is important!

$K = N - \frac{\mu}{\beta}$

- $\dfrac{N\beta}{\mu} > 1$: disease becomes endemic, stabilizes at some percent of the population being sick all the time
- $\dfrac{N\beta}{\mu} < 1$: disease is eradicated, because people recover faster than they get sick