First-Order Linear Differential Equation

\[ \frac{dy}{dt} = a - by \]

with \( y(0) = y_0 \) given.

Steady state solutions:

\[ 0 = \frac{dy}{dt} = a - by \]

\[ y = \frac{a}{b} \]
\[ \frac{dy}{dt} = a - by \]
Transformation

\[ \frac{dy}{dt} = a - by \]

Let's change this to a different form:

\[ \frac{dy}{dt} = -b \left( y - \frac{a}{b} \right) \quad \text{\textit{deviation}}: \text{ how far } y \text{ is from its steady state} \]

\[ \frac{dz}{dt} = \frac{dy}{dt} \]
\[ \frac{dz}{dt} = -bz \]

\[ z(t) = Ce^{-bt} \]

\[ y(t) - \frac{a}{b} = Ce^{-bt} \]
Transformation Interpretation

- $\frac{dy}{dt} = a - by$
- $y = \frac{a}{b}$: steady state solution
- $z = y(t) - \frac{a}{b}$: deviation from the steady state
- $z = Ce^{-bt}$: for $b > 0$, the deviation decays exponentially. That is, if $y$ is far from $b/a$, it will approach it rapidly.
- $y(t) = \frac{a}{b} + Ce^{-bt}$: general solution
- $y(t) = \frac{a}{b} + (y_0 - \frac{a}{b}) e^{-bt}$
  $C$ is the initial deviation

Example 1:

What is the solution to the differential equation $\frac{dy}{dt} = 3 - 2y$, $y(0) = \frac{1}{2}$?
Newton’s Law of Cooling

The rate of change of temperature $T$ of an object is proportional to the difference between its temperature and the ambient temperature, $E$.

- $E$: steady state
Newton’s law of cooling: \( T(t) = E + (T_0 - E) e^{-\alpha t} \)

Example 2: A farrier works a horseshoe heated to 400° C, then dunks it in a pool of room-temperature (25° C) water. The water near the horseshoe boils for 30 seconds, but the temperature of the pool as a whole hasn’t changed appreciably. The horseshoe is safe for the horse when it’s 40° C. When can the farrier put on the horseshoe? You may assume the water stops boiling when the horseshoe dips below 100° C.
Suppose a body is discovered at 3:45 pm, in a room held at 20°, and the body’s temperature is 27°: not the normal 37°. At 5:45 pm, the temperature of the body has dropped to 25.3°. When did the owner of the body die?

You may assume $T(t) = E + (T_0 - E)e^{-\alpha t}$. 
A glass of just-boiled tea is put on a porch outside. After ten minutes, the tea is 40°, and after 20 minutes, the tea is 25°. What is the temperature outside? You may assume \( T(t) = E + (T_0 - E)e^{-\alpha t} \).
Definition

**Analytic solution** to a DE: explicit formula for the solution

**Numerical solution** to a DE: approximation of the solution

We saw something similar with Newton’s Method:

\[ f(x) = x^2 - 3 \]

Roots: \( \pm \sqrt{3} \) (analytic)  
Roots: approximately \( \pm 1.73 \) (numeric)

\[ f(x) = x^5 + x^3 + 1 \]

Root: ????? (analytic)  
Root: approximately 0.838 (numeric)
Numerical solutions to diff eqs: Euler’s method

Concept

Euler’s method turns slope fields into equations.

\[
\frac{dy}{dt} = y^2 \quad y(0) = \frac{1}{2}
\]

\[
y(1) \approx \frac{3}{4} \quad y'(1) \approx \frac{9}{16}
\]

\[
y(2) \approx \frac{21}{16}
\]
Euler’s method: computation

- Start with a differential equation with initial values, e.g. \( \frac{dy}{dt} = y^2 \) and \( y_0 = \frac{1}{2} \)
- Choose a step, usually called \( \Delta t \), preferably small
- Make a linear approximation of \( y(t_k) \) to approximate \( y(t_{k+1}) \).
- Iterate.

\[
\begin{align*}
\Delta y &= y_1 - y_0 \\
y_1 &= y_0 + \Delta y
\end{align*}
\]
Euler’s method: \( y_{k+1} \approx y_k + y_k' \cdot \Delta t \)

**Example 3:**

Use Euler’s method, with \( \Delta t = 0.1 \), to approximate \( y(0.5) \), given the following:

\[
\frac{dy}{dt} = y + 2e^{y+1}, \quad y(0.3) = -1
\]
Euler’s method: \( y_{k+1} \approx y_k + y'_k \cdot \Delta t \)

Example 4:

Suppose

\[ \frac{dy}{dt} = \frac{1}{y}. \]

Use Euler’s method, with \( \Delta t = 0.05 \), to approximate \( y(0.1) \), given these initial conditions:

(a) \( y(0) = 1 \)

(b) \( y(0) = -\frac{1}{4} \)
Disease Dynamics

Setup:
- \( S(t) \): susceptible individuals
- \( I(t) \): infected (and infectious) individuals
- Everybody mixes
- Fixed probability of catching illness from contact
- Fixed time to get better
Disease dynamics

Infected population:
- According to the law of mass action, rate of transmission should be proportional to the product of the two populations, $S \cdot I$.
- Let $\beta$ be that constant of proportionality, so the rate of new infections is $\beta(SI)$.
- Let $\mu$ be the rate of recovery per person, so the rate of recovery over the infected population is $\mu I$.

\[
\frac{dl}{dt} = \text{[rate of new infections]} - \text{[rate of recovery]}
\]

\[
= \beta IS - \mu I
\]

\[
= \beta I(N - I) - \mu I
\]

where $N$ is the total (constant) size of the population, $S + I$. 
Disease dynamics

\[
\frac{dl}{dt} = \beta l (N - l) - \mu l = \beta l (K - l)
\]

Where \(K\) is some constant, \(N\) is the size of the population, \(\mu\) is the rate of recovery, and \(\beta\) is a measure of ease of transmission.

(a) What is \(K\), in terms of the other constants?

(b) Which leads to a larger \(K\): a disease with difficult transmission and quick recovery, or a disease with easy transmission and slow recovery?
Disease dynamics

\[ \frac{dI}{dt} = \beta I(N - I) - \mu I = \beta I(K - I) \]

Where \( K = N - \frac{\mu}{\beta} \), \( N \) is the size of the population, \( \mu \) is the rate of recovery, and \( \beta \) is a measure of ease of transmission.

(a) What are the steady states of this differential equation?
(b) Draw two state diagrams: one for the case \( K > 0 \), one for the case \( K < 0 \).
(c) Which steady states are stable and which not?
(d) Give a biological interpretation for the steady states.
Disease Dynamics

The sign of $K$ is important!

$$K = N - \frac{\mu}{\beta}$$

- $\frac{N\beta}{\mu} > 1$: disease becomes endemic, stabilizes at some percent of the population being sick all the time

- $\frac{N\beta}{\mu} < 1$: disease is eradicated, because people recover faster than they get sick