Differential Equations

Definition

A **differential equation** is an equation involving the derivative of an unknown function.

Examples: $\frac{dy}{dx} = 2x$; $\quad x\frac{dy}{dx} = 7xy + y$

Definition

If a function makes a differential equation true, we say it **satisfies** the differential equation.

Example: $y = x^2$ and $y = x^2 + 1$ both satisfy the first differential equation;

the function $y = xe^{7x+9}$ satisfies the second.

Differential Equations

Consider the differential equation

$$\frac{dy}{dx} + x\frac{dy}{dx} = 2y(x + x^2)$$

Which of the following equations satisfy this differential equation?

$$(a)\ y = e^x \qquad (b)\ y = e^{x^2} \qquad (c)\ y = e^{x^2} + 1 \qquad (d)\ y = \frac{e^{x^2}}{2}$$

Suppose we **also** stipulate that $y(0) = 5$. Then what equation(s) do you think will satisfy the differential equation?

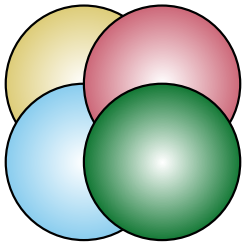
An Important Class of Differential Equations

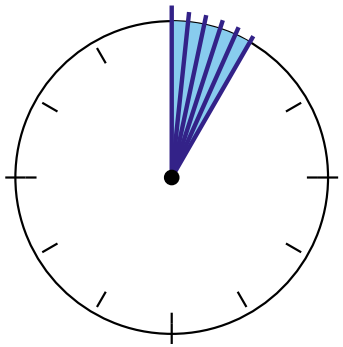
$$\frac{dy}{dx} = ky$$

where k is a constant.
Interpretation: the rate of change of y is proportional to the value of y.

Consider the following:
$y = e^x$, $y = e^x + C$, $y = Ce^x$, $y = Ce^{kx}$, $y = Ce^{x+k}$

Radioactive Decay





Suppose over a 5-second interval, the probability an atom decays is $\frac{1}{100}$
With only one atom, it's hard to know what will happen.
What if we had 100 atoms?

Radioactive decay model

Suppose in a **small** time interval h, the probability of an atom decaying is kh, where k is a positive constant.

Let $Q(t)$ be the quantity of the radioactive isotope at time t.

We predict:

$$Q(t+h) = Q(t) - khQ(t)$$

So,

$$\begin{aligned} \frac{Q(t+h) - Q(t)}{h} &= -kQ(t) \\ \lim_{h \to 0} \frac{Q(t+h) - Q(t)}{h} &= \lim_{h \to 0} -kQ(t) \\ Q'(t) &= -kQ(t) \end{aligned}$$

Radioactive Decay

The number of atoms in a sample that decay in a given time interval is proportional to the number of atoms in the sample.

Differential Equation

Let $Q = Q(t)$ be the amount of a radioactive substance at time t. Then for some positive constant k:

$$\frac{dQ}{dt} = -kQ$$

Solution

Let $\boxed{Q(t) = Ce^{-kt}}$, where k and C are constants. Then:

$$\frac{dQ}{dt} = C \cdot e^{-kt} \cdot (-k) = -kCe^{-kt} = -kQ$$

Radioactive Decay

Quantity of a Radioactive Isotope

$$Q(t) = Ce^{-kt}$$

$Q(t)$: quantity at time t

What is the sign of $Q(t)$?

A. positive or zero
B. negative
C. could be anything

What is the sign of C?

A. positive or zero
B. negative
C. could be anything

Seaborgium-266

Carbon Decay

The amount of ^{266}Sg (Seaborgium-266) in a sample at time t (measured in seconds) is given by

$$Q(t) = Ce^{-kt}$$

According to Wikipedia, the half life of ^{266}Sg is 30 seconds. That is, every 30 seconds, the size of the sample halves.

What are C and k?

Radioactive Decay

A sample of radioactive matter is stored in a lab in 2000. In the year 2002, it is tested and found to contain 10 units of a particular radioactive isotope. In the year 2005, it is tested and found to contain only 2 units of that same isotope. How many units of the isotope were present in the year 2000?

$Q'(t) = kQ(t)$

The number of atoms in a sample that decay in a given time interval is proportional to the number of atoms in the sample.
The rate of growth of a population in a given time interval is propotional to the number of individuals in the population, when the population has ample resources
The amount of interest a bank account accrues in a given time interval is proportional to the balance in that bank account.

Exponential Growth

Let $Q = Q(t)$ satisfy:

$$\frac{dQ}{dt} = kQ$$

for some constant k. Then

$$Q(t) = Ce^{kt}$$

where $C = Q(0)$ is a constant.

Population Growth

Suppose a petri dish starts with a culture of 100 bacteria cells and a limited amount of food and space. The population of the culture at different times is given in the table below. At approximately what time did the culture start to show signs of limited resources?

time	population
0	100
1	1000
3	100000
5	1000000

Flu Season

The CDC keeps records (link) on the number of flu cases in the US by week. At the start of the flu season, the 40th week of 2014, there are 100 cases of a particular strain. Five weeks later (at week 45), there are 506 cases. What do you think was the first week to have 5,000 cases? What about 10,000 cases?

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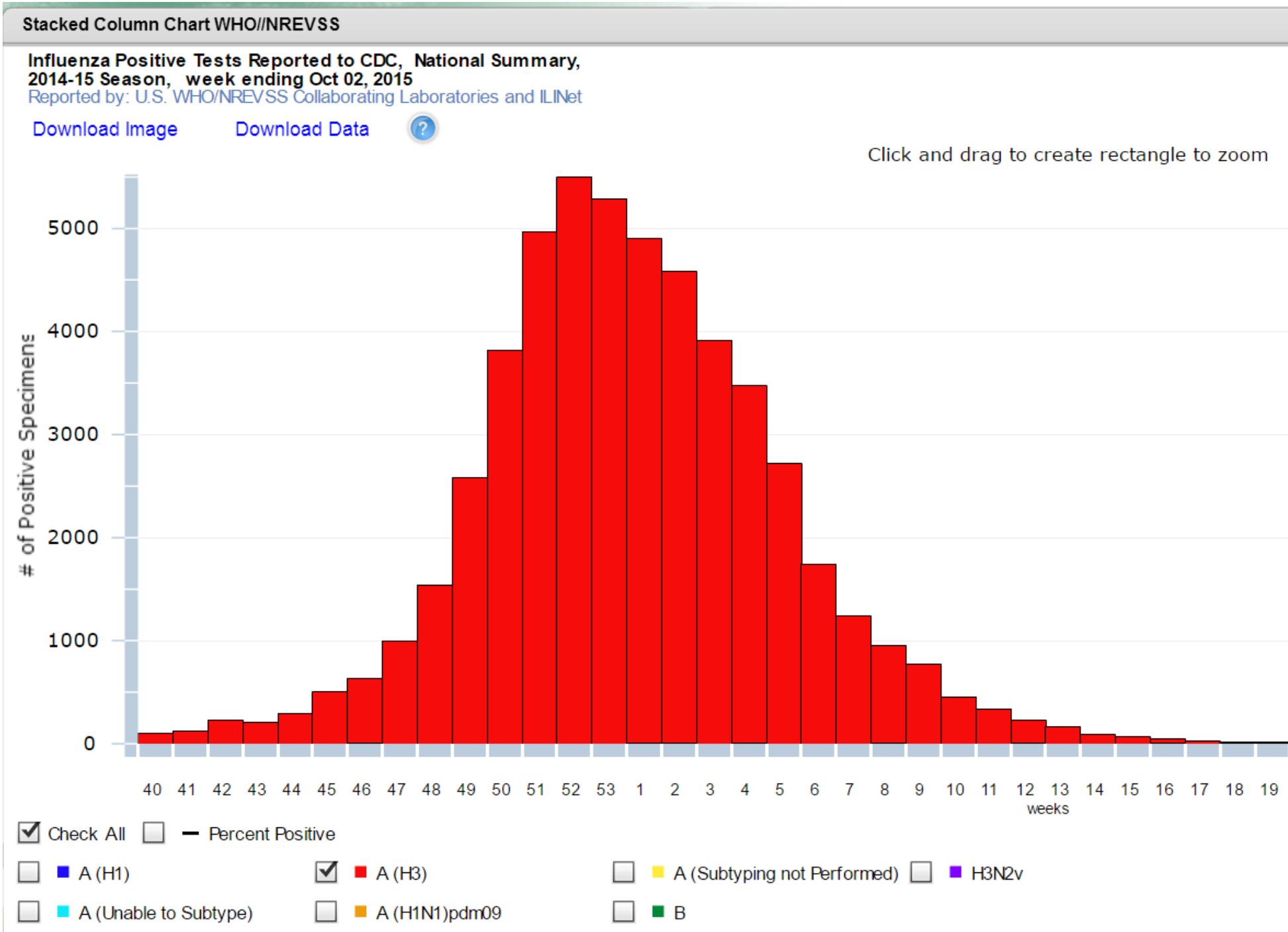
Stacked Column Chart WHO//NREVSS
Influenza Positive Tests Reported to CDC, National Summary, 2014-15 Season, week ending Oct 02, 2015
Reported by: U.S. WHO/NREVSS Collaborating Laboratories and ILINet
Download Image
Download Data
Click and drag to create rectangle to zoom
# of Positive Specimens
5000
4000
3000
2000
1000
0
40 41 42 43 44 45 46 47 48 49 50 51 52 53 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19
weeks
Check All
Percent Positive
A (H1)
A (H3)
A (Subtyping not Performed)
H3N2v
A (Unable to Subtype)
A (H1N1)pdm09
B

Radiocarbon Dating

Researchers at Charlie Lake in BC have found evidence[1] of habitation dating back to around 8500 BCE. For instance, a butchered bison bone was radiocarbon dated to about 10,500 years ago.

Suppose a comparable bone of a bison alive today contains 1mg of ^{14}C. If the half-life of ^{14}C is about 5730 years, how much ^{14}C do you think the researchers found in the sample?

Make a rough estimate first.

A. About $\frac{1}{10{,}500}$ mg
B. About $\frac{1}{4}$ mg
C. About $\frac{1}{2}$ mg
D. About 1mg

[1] http://pubs.aina.ucalgary.ca/arctic/Arctic49-3-265.pdf