

Math 100 /180 Final, Dec 2015
 Solutions hastily written by Elyse

1. (i) $\lim_{t \rightarrow 3} \frac{t-3}{(t+1)^2} = \frac{3-3}{(3+1)^2} = 0$

C

(ii) $\lim_{x \rightarrow \infty} \frac{x+1}{x^2+2x-8} \left(\frac{1/x^2}{1/x^2}\right) = \lim_{x \rightarrow \infty} \frac{\frac{1}{x} + \frac{1}{x^2}}{1 + 2/x - \frac{8}{x^2}} = \frac{0+0}{1+0-0} = 0$

A

(iii) $\lim_{x \rightarrow 4^+} x = 4 ; \lim_{x \rightarrow 4^+} (x-4)^2 = 0$

The function $\frac{x}{(x-4)^2}$ has an infinite discontinuity at $x=4$. When x is slightly larger than (but not equal to) 4, the function is positive. So,

$$\lim_{x \rightarrow 4^+} \frac{x}{(x-4)^2} = \infty$$

F

(iv) D

(i) $f(x) = \tan x ; f'(x) = \sec^2 x = \frac{1}{\cos^2 x}$

C

(ii) $f(x) = x \log x ; f'(x) = x\left(\frac{1}{x}\right) + \log x = 1 + \log x$

E

(iii) $f(x) = \frac{x^2}{x+1} ; f'(x) = \frac{(x+1)(2x) - x^2(1)}{(x+1)^2} = \frac{2x^2 + 2x - x^2}{(x+1)^2}$
 $= \frac{x^2 + 2x}{(x+1)^2} = \frac{x(x+2)}{(x+1)^2}$

D

(iv) $f(x) = \sin(x^2) ; f'(x) = \cos(x^2) \cdot 2x$

D

$$3. \quad (i) \quad f'(x) = \frac{1}{2\sqrt{x}} : \quad f(4) = 2$$

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

$$y - f(a) = f'(a)(x-a)$$

$$y - 2 = \frac{1}{4}(x-4)$$

B

$$(ii) \quad x \cdot y^2 + y \cdot x^2 = 2$$

$$\Rightarrow x(2y \cdot y') + y^2(1) + y(2x) + x^2 \cdot y' = 0$$

$$x=1, \quad y=1 : \quad 2y' + 1 + 2 + y' = 0$$

$$3y' + 3 = 0$$

A

$$y' = -1$$

$$(iii) \quad f(x) = x^x; \quad \log(f(x)) = \log(x^x)$$

$$\begin{aligned} & \log(f(x)) = x \log x \\ & \frac{f'(x)}{f(x)} = x \left(\frac{1}{x}\right) + \log x \end{aligned} \quad \left. \begin{array}{l} (\text{compare to}) \\ 2(ii) \end{array} \right)$$

$$\begin{aligned} f'(x) &= f(x) [1 + \log x] \\ &= x^x (1 + \log x) \end{aligned}$$

(iv) Solution 1:

$$\begin{aligned} T_3(x) &= f(3) + f'(3)(x-3) + \frac{f''(3)}{2}(x-3)^2 + \frac{f'''(3)}{3!}(x-3)^3 \\ &= 24 + 6(x-3) + 12(x-3)^2 + 4(x-3)^3 \end{aligned}$$

F

$$\text{So, } \frac{f''(3)}{2} = 12, \quad \text{so } f''(1) = 24$$

Solution 2:

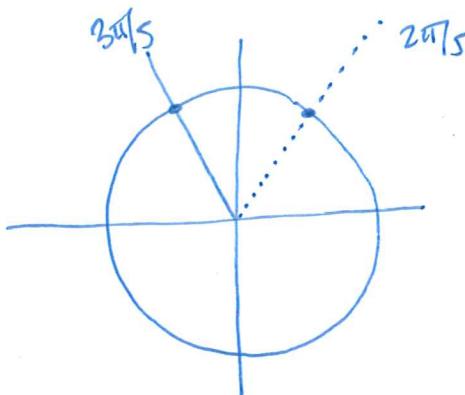
$$f''(3) = T_3''(3) :$$

$$T_3'(x) = 6 + 24(x-3) + 12(x-3)^2$$

$$T_3''(x) = 24 + 24(x-3)$$

$$T_3''(3) = 24 \quad \text{so } f''(3) = 24.$$

4. (a)



Note $\sin\left(\frac{3\pi}{5}\right) = \sin\left(\pi - \frac{3\pi}{5}\right) = \sin\left(\frac{2\pi}{5}\right)$,
and $\frac{2\pi}{5}$ is in the interval $(-\pi/2, \pi/2]$.

So, $\arcsin\left(\sin\left(\frac{3\pi}{5}\right)\right) = \arcsin\left(\sin\left(\frac{2\pi}{5}\right)\right) = \boxed{\frac{2\pi}{5}}$

(b) $\lim_{x \rightarrow 2} \frac{x^2 - 3x + 2}{x - 2} = \lim_{x \rightarrow 2} \frac{(x-1)(x-2)}{x-2} = \lim_{x \rightarrow 2} x-1 = \boxed{1}$

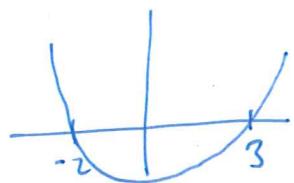
(c) $\lim_{x \rightarrow -\infty} \frac{x + \sqrt{4x^2 - x}}{6x} \left(\frac{1/x}{1/x} \right) = \lim_{x \rightarrow -\infty} \frac{1 + \frac{1}{x}\sqrt{4x^2 - x}}{6}$
 $\stackrel{(*)}{=} \lim_{x \rightarrow -\infty} \frac{1 + \frac{1}{-1/x}\sqrt{4x^2 - x}}{6} = \lim_{x \rightarrow -\infty} \frac{1 - \sqrt{4 - 1/x}}{6}$
 $= \frac{1 - \sqrt{4 - 0}}{6} = \frac{1 - 2}{6} = \boxed{-\frac{1}{6}}$

(*) Note: Since our limit is $x \rightarrow -\infty$, we are only concerned with negative values of x . Then $\sqrt{x^2} = |x| = \underbrace{-x}_{\text{positive}}$.

So, $\frac{1}{x} = \frac{-1}{\sqrt{x^2}}$.

(4, cont'd)

(d) $x^2 - x - 6 > 0 \Leftrightarrow (x-3)(x+2) > 0$



$$(-\infty, -2) \cup (3, \infty)$$

5. (a) $y^2 + 4xy - 2x^2 = 3 \Rightarrow 2y \cdot y' + 4x \cdot y' + 4y - 4x = 0$

If $y' = 0$: $4y - 4x = 0$, hence $y = x$

Returning to our original equation, if $y = x$, then:

$$x^2 + 4x \cdot x - 2x^2 = 3$$

$$\Rightarrow x^2 + 4x^2 - 2x^2 = 3$$

$$\Rightarrow 3x^2 = 3$$

$$\Rightarrow x = 1 \text{ or } x = -1$$

(b) Given information: $f'(3) = 2$, $g'(3) = 5$

Differentiate $g(x) = x \cdot f(x)$:

$$g'(x) = xf'(x) + f(x)$$

When $x = 3$: $g'(3) = 3f'(3) + f(3)$

$$5 = 3 \cdot 2 + f(3)$$

$$f(3) = -1$$

(5, cont'd)

(c) $h(x) = xe^x$; $h'(x) = xe^x + e^x$;

$$h''(x) = xe^x + e^x + e^x = e^x(x+2)$$

$$h''(x) < 0 \text{ when } x < -2 ;$$

$$h''(x) > 0 \text{ when } x > -2$$

$\boxed{x = -2}$ only inflection point

(d) $\lim_{x \rightarrow 0} \frac{\log(1+x) - \sin x}{x^2} = \lim_{x \rightarrow 0} \frac{\frac{1}{1+x} - \cos x}{2x} = \lim_{x \rightarrow 0} \frac{\frac{-1}{(1+x)^2} + \sin x}{2}$

$\underbrace{\hspace{10em}}$ $\underbrace{\hspace{10em}}$

NUM $\rightarrow 0$ NUM $\rightarrow 1-1=0$
DEN $\rightarrow 0$ DEN $\rightarrow 0$

$$= \frac{\frac{-1}{1} + 0}{2} = \boxed{\frac{-1}{2}}$$

6. (a) "A colony doubles every 4 hours"

$$\underline{Q(t) = Q(0)e^{kt}} ; Q(4) = 2Q(0) ; Q(6) = 2000$$

$$2Q(0) = Q(4) = Q(0)e^{4k} \Rightarrow 2 = e^{4k} \Rightarrow e^k = 2^{1/4}$$

$$2000 = Q(6) = Q(0)(e^k)^6 = Q(0) \cdot 2^{6/4} = Q(0) \cdot 2^{3/2}$$

$$\Rightarrow Q(0) = \frac{2000}{2\sqrt{2}} = \boxed{\frac{1000}{\sqrt{2}}} \text{ initial members}$$

(6, cont'd)

$$a=9$$

(b) $f(x) = \sqrt{x}$ $f(9) = 3$

$$f'(x) = \frac{1}{2\sqrt{x}} \quad f'(9) = \frac{1}{2 \cdot 3} = \frac{1}{6}$$

$$L(x) = f(a) + f'(a)(x-a)$$

$$= 3 + \frac{1}{6}(x-9)$$

$$L(8) = 3 + \frac{1}{6}(-1) = 3 - \frac{1}{6} = \boxed{\frac{17}{6}} \approx \sqrt{8}$$

(c) $f'(x) = e^{x^2-9x^2+15x-1} \cdot (3x^2-18x+15)$

$$= e^{x^2-9x^2+15x-1} \cdot (3)(x^2-6x+5)$$

$$= e^{x^2-9x^2+15x-1} \cdot (3)(x-5)(x-1).$$

No power of e is ever 0, so CPs are

$$\boxed{x=5, \quad x=1}$$

(d) $f(x) = -\cos x + 4\sqrt{x} + C$,

$$0 = f(\pi) = -\cos(\pi) + 4\sqrt{\pi} + C = 1 + 4\sqrt{\pi} + C$$

$$\text{So, } C = -1 - 4\sqrt{\pi}$$

$$\boxed{f(x) = -\cos x + 4\sqrt{x} - 1 - 4\sqrt{\pi}}$$

$$7. f'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{f(h) - 0}{h} = \lim_{h \rightarrow 0} \frac{f(h)}{h}$$

if limit exists.

To evaluate limit, we consider one-sided limits.

$$\lim_{\substack{h \rightarrow 0^- \\ h < 0}} \frac{f(h)}{h} \underset{P}{=} \lim_{h \rightarrow 0^-} \frac{\sqrt{1+h^2} - 1}{h} \left(\frac{\sqrt{1+h^2} + 1}{\sqrt{1+h^2} + 1} \right)$$

$h < 0$

$$= \lim_{h \rightarrow 0^-} \frac{(1+h^2)^{-1/2} - 1}{h(\sqrt{1+h^2} + 1)} = \lim_{h \rightarrow 0^-} \frac{h^2}{h(\sqrt{1+h^2} + 1)}$$

$$= \lim_{h \rightarrow 0^-} \frac{h}{\sqrt{1+h^2} + 1} \underset{P}{\rightarrow} \frac{0}{\sqrt{1+0^2} + 1} = 0$$

$$\lim_{\substack{h \rightarrow 0^+ \\ h > 0}} \frac{f(h)}{h} \underset{P}{=} \lim_{h \rightarrow 0^+} \frac{h^2 \cos(1/h)}{h} = \lim_{h \rightarrow 0^+} h \underbrace{\cos(1/h)}$$

Note: this piece's limit
done

• Recall $-1 \leq \cos(1/h) \leq 1$, so

$$-h \leq h \cos(1/h) \leq h$$

$$\bullet \text{ Note } \lim_{h \rightarrow 0^+} (-h) = \lim_{h \rightarrow 0^+} (h) = 0$$

• So, by Squeeze Theorem,

$$\lim_{h \rightarrow 0^+} h \cos(1/h) = 0$$

$$\text{So } \lim_{h \rightarrow 0^+} \frac{f(h)}{h} = \lim_{h \rightarrow 0^+} h \cos(1/h) = 0.$$

Therefore $\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h}$ exists, s.

f(x) is differentiable at x=0.

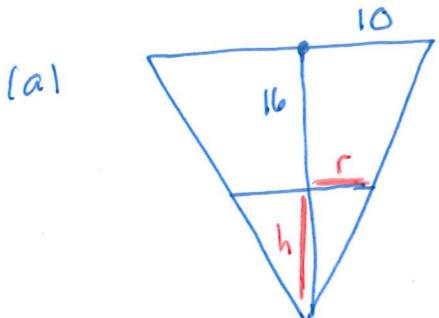
8. KNOW: $\frac{dV}{dt} = +2 \text{ m}^3/\text{min}$ (where V is volume of water in cone, t is time)

WANT TO FIND: $\frac{dh}{dt}$ when $h=10 \text{ m}$ (h is height of water in cone)

So, we need: an equation relating V and t .

Volume of cone w/ height h & radius r :

$$V = \frac{\pi}{3} r^2 h$$



$$\text{So: } \frac{r}{h} = \frac{10}{16} = \frac{5}{8}$$

(by similar triangles)

$$\text{So } r = \frac{5}{8}h$$

$$V = \frac{\pi}{3} \underbrace{\left(\frac{5}{8}h\right)^2}_r h = \frac{\pi \cdot 5^2}{3 \cdot 8^2} \cdot h^3$$

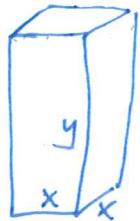
Differentiate with respect to time:

$$\frac{dV}{dt} = \frac{\pi \cdot 5^2}{8^2} h^2 \cdot \frac{dh}{dt}$$

$$\Rightarrow 2 = \frac{\pi \cdot 5^2}{8^2} \cdot 10^2 \cdot \frac{dh}{dt}$$

$$\boxed{\frac{2 \cdot 8^2}{\pi \cdot 5^2 \cdot 10^2} = \frac{dh}{dt}}$$

9. Cost : $C = 2(4)(\text{area of upright side}) + 8(2)(\text{area of top})$



$$= 2 \cdot 4 \cdot xy + 8 \cdot 2 \cdot x^2 \\ = 8xy + 16x^2 \quad \} \text{ This is the function to minimize}$$

Volume : $32 = x^2y$

$$\Rightarrow y = \frac{32}{x^2}$$

Cost : $C = 8 \times \underbrace{\left(\frac{32}{x^2}\right)}_y + 16x^2$

$$= 8 \cdot 32 \cdot x^{-2} + 16x^2$$

$$C'(x) = -8 \cdot 32 \cdot x^{-3} + 32x$$

$$CP: 32x = \frac{8 \cdot 32}{x^2}$$

$$x^3 = 8$$

$$x = 2$$

Note: $\lim_{x \rightarrow 0^+} C(x) = \lim_{x \rightarrow \infty} C(x) = \infty$,

so CP is global min over $(0, \infty)$.

When $x = 2$, $y = \frac{32}{x^2} = \frac{32}{4} = 8$

Dimensions: base is 2×2 m
8 m high

10. (a) all real numbers

(b) $\boxed{x = -2, x = -1}$ x -intercepts

$$f(0) = 2(1)e^0 = 2 \quad \boxed{y=2} \quad y\text{-intercept}$$

(c) $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{x^2 + 3x + 2}{e^x} = \lim_{x \rightarrow \infty} \frac{2x + 3}{e^x} = \lim_{x \rightarrow \infty} \frac{2}{e^x} = 0$

$\underbrace{\text{NUM} \rightarrow \infty}_{\text{DEN} \rightarrow \infty}$ $\underbrace{\text{NUM} \rightarrow \infty}_{\text{DEN} \rightarrow \infty}$ $\underbrace{\text{NUM} \rightarrow 0}_{\text{DEN} \rightarrow \infty}$

$\boxed{y=0}$ horiz. asympt.

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} \frac{(x+2)(x+1)e^{-x}}{\rightarrow \infty} = \infty$$

no second horiz. asympt.

(d) no vertical asymptotes

(A v.asympt is an infinite discontinuity. Since $f(x)$ has no discontinuities of any type, it has no VA.)

(e) $f(x) = (x^2 + 3x + 2)e^{-x}$

$$\begin{aligned} f'(x) &= (x^2 + 3x + 2)(-1)e^{-x} + (2x + 3)e^{-x} \\ &= e^{-x}(-x^2 - 3x - 2 + 2x + 3) = e^{-x}(-x^2 - x + 1) \end{aligned}$$

$\boxed{-e^{-x}(x^2 + x - 1)}$

(f) $x^2 + x - 1 = 0$ when $x = \frac{-1 \pm \sqrt{1 - 4(1)(-1)}}{2} = \frac{-1 \pm \sqrt{5}}{2}$

$\boxed{\text{CP: } x = \frac{-1 - \sqrt{5}}{2}, x = \frac{-1 + \sqrt{5}}{2}}$

SP: none

(g) $f(x)$ increasing on $\left(\frac{-1 - \sqrt{5}}{2}, \frac{-1 + \sqrt{5}}{2}\right)$

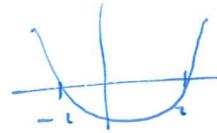
$f(x)$ decreasing on $(-\infty, \frac{-1 - \sqrt{5}}{2})$ and on $(\frac{-1 + \sqrt{5}}{2}, \infty)$

(10) cont'd

(h) $f''(x) = 0$ when $x=2, x=-1$

Note $e^{-x} > 0$ for all x

and $(x-2)(x+1)$ is a parabola:



positive when $x < -1$ and when $x > 2$

negative when $-1 < x < 2$

so:

$x = -1$ is an IP b/c $f''(x)$ changes from positive to negative at $x = -1$

$x = 2$ is an IP b/c $f''(x)$ changes from negative to positive at $x = 2$

II. (a) $T_2(x) = 5 - \frac{1}{3}x + 2x^2$

$$T_2'(x) = -\frac{1}{3} + 4x$$

$$T_2''(x) = 4$$

$$g(0) = T_2(0) = 5$$

$$g'(0) = T_2'(0) = -\frac{1}{3}$$

$$g''(0) = T_2''(0) = 4$$

$$h(x) = e^x g(x)$$

$$h'(x) = e^x g'(x) + e^x g(x)$$

$$= e^x (g'(x) + g(x))$$

$$\begin{aligned} h''(x) &= e^x g''(x) + e^x g'(x) + e^x g'(x) + e^x g(x) \\ &= e^x (g''(x) + 2g'(x) + g(x)) \end{aligned}$$

$$h(0) = e^0 g(0) = 1 \cdot 5 = 5$$

$$\begin{aligned} h'(0) &= e^0 (g'(0) + g(0)) = 1 \left(-\frac{1}{3} + 5\right) \\ &= \frac{14}{3} \end{aligned}$$

$$\begin{aligned} h''(0) &= e^0 (g''(0) + 2g'(0) + g(0)) \\ &= 4 + 2\left(-\frac{1}{3}\right) + 5 \end{aligned}$$

$$= 9 - \frac{2}{3} = \frac{25}{3}$$

Maclaurin Polynomial:

$$h(0) + h'(0)x + \frac{h''(0)}{2}x^2$$

$$= \boxed{5 + \frac{14}{3}x + \frac{25}{6}x^2}$$

(11 cont'd)

(b) Note $| \sin x |, |\cos x| \leq 1$ for all values of x .

Then $|x \sin x + x^2 \cos x| \leq |x| + |x^2| \leq 2$
when $0 \leq x \leq 1$.

Also $|10 - x^2| \geq 9$ when $0 \leq x \leq 1$

so $|f^{(3)}(c)| = \left| \frac{c \sin c + c^2 \cos c}{10 - c^2} \right| \leq \frac{2}{9}$
when $0 \leq c \leq 1$.

Using Taylor's Theorem: For some c b/w 0 & 1,

$$\begin{aligned} |\text{error}| &= \left| \frac{f^{(3)}(c)}{3!} (1-c)^3 \right| = \frac{1}{6} |f^{(3)}(c)| \\ &\leq \frac{1}{6} \left(\frac{2}{9} \right) = \frac{1}{27} < \frac{1}{25} \end{aligned}$$

12. (a) • Since $g'(x)$ exists, $g(x)$ is continuous & differentiable everywhere.
- By Rolle's Theorem, if there are $a < b < c$ such that $g(a)=g(b)=g(c)=0$, then there are d_1, d_2 such that $a < d_1 < b < d_2 < c$ ($\text{so } d_1 \neq d_2$) and $g'(d_1)=g'(d_2)=0$.

| So $g'(x)$ has at least 2 zeros |

- Since $g''(x)$ exists, $g'(x)$ is continuous & differentiable everywhere. Since $g'(d_1)=g'(d_2)$, by Rolle's Theorem, there exists some e b/w d_1 & d_2 such that $g''(e)=0$.

| So $g''(x)$ has at least one zero. |

(b) Let $f(x)=2x^3-3+\sin x+\cos x$.

Note $f(x)$ is continuous everywhere and:

- $f(-10) > 0$
- $f(0) < 0$
- $f(10) > 0$

By FVT, $f(a)=0$ for some a in $(-10, 0)$

and $f(b)=0$ for some b in $(0, 10)$.

(12, cont'd)

(c) $f(x) = 2x^2 - 3 + \sin x + \cos x$

$$f'(x) = 4x + \cos x - \sin x$$

$$f''(x) = 4 - \sin x - \cos x$$

Since $4 - \sin x - \cos x \geq 4 - 2 > 0$ for all x ,

$f''(x)$ has no zeroes.

If $f(x)$ were to have at least 3 zeroes, then

$f''(x)$ would have to have at least one zero. (part b)

$f''(x)$ does not, so we conclude that

$f(x)$ does not have at least 3 zeroes.

That is, $f(x)$ has at most 2 zeroes.