Lecture 31

Non-linear, Autonomous systems

We consider systems
\[ x' = f(x, y) \quad \text{with} \quad x(0) = x_0 \]
\[ y' = g(x, y) \quad \text{with} \quad y(0) = y_0 \]

Autonomous because the RHS does not depend on \( t \).
Solution is a path \((x(t), y(t))\), RHS represents a vector field.
Solution is a path whose velocity \( \mathbf{v}(x, y) \) is \((f(x, y), g(x, y))\) when it passes through \((x, y)\)

Equilibrium solutions: If \( f(x, y) = g(x, y) = 0 \) then \( x(t) = x_1, y(t) = y_1 \) is called an equilibrium solution and it satisfies the system.
What is the qualitative behaviour of solutions:
- large times $t \to \infty$
- local behaviour near equilibrium points.

**Population dynamic models**

In this models, $x(t), y(t)$ represent the population at time $t$ of two species. We have $x > 0, y > 0$

1. **Competing species**:

\[
x' = (a - ax + by)x = f(x, y), \quad a, b, a, b, c, d > 0
\]
\[
y' = (c - cx + dy)y = g(x, y)
\]

When $b = c = 0$ there is no interaction. Setting $b, c > 0$ means that presence of the other population reduces the reproduction rate.

2. **Predator–prey model**

\[
x' = (-a + ax + by)x \quad \text{← predator}
\]
\[
y' = (c - cx - dy)y \quad \text{← prey}
\]

If $y = 0$ the population $x$ will die out
If $x = 0$ then the population of $y$ will obey a logistic equation
3. Co-operation

\[ x' = (x - ax + by) x \]
\[ y' = (b + cx - dy) y \]

If \( y = 0 \) the population \( x \) will follow a logistic growth.
If \( x = 0 \) the population \( y \) will follow a logistic growth.

**Example:**

\[ x' = (1 - x - y) x \]
\[ y' = (\frac{3}{4} - \frac{1}{2}x - y) y \]

**Step 1** Find equilibrium solutions.

\[ x = 0 \Rightarrow x = 0 \quad \text{or} \quad (1 - x - y) = 0 \quad (y = 1 - x) \]

\[ y' = 0 \Rightarrow y = 0 \quad \text{or} \quad \left( \frac{3}{4} - \frac{1}{2}x - y \right) = 0 \quad (y = \frac{3}{4} - \frac{1}{2}x) \]

\[ \begin{align*}
y & = 1 - x \\
y & = \frac{3}{4} - \frac{1}{2}x
\end{align*} \]

\[ \begin{align*}
x & = \frac{1}{2} \\
y & = \frac{1}{2}
\end{align*} \]

Points: \((0,0), (0,3/4), (1,0), (\frac{1}{2}, \frac{1}{2})\)
step 2: large time behaviour.

- If \((x_0, y_0)\) is an equilibrium point the solution stays there.
- If \(x_0 = 0, y_0 > 0\), solution converges to \((0, \frac{3}{4})\).
- If \(y_0 = 0, x_0 > 0\), solution converges to \((1, 0)\).

Now assume \(x_0 > 0, y_0 > 0\)

For points above the line \(1 - x - y = (x\text{-nullcline})\), \(x' < 0\)

For points on the \(x\text{-nullcline} \) \(x' = 0\)

For points below \(x\text{-nullcline} \) \(x' > 0\)

and

For points above the line \(\frac{3}{2} - \frac{1}{2} y = (y\text{-nullcline})\), \(y' < 0\)

For points on the \(y\text{-nullcline}\) \(y' = 0\)

For points below the \(y\text{-nullcline}\) \(y' > 0\)

\(\Rightarrow\) In zone I \(x' > 0, y' > 0\)

zone II \(x' < 0, y' > 0\)

zone IV \(x' < 0, y' < 0\)

zone IV \(x' > 0, y' < 0\)

Conclusion: all trajectories with \(x_0 > 0, y_0 > 0\) must converge to \((\frac{1}{2}, \frac{1}{2})\).