**Constant Coefficients**

Consider the operator

\[ Ly = ay'' + by' + cy \] where \( a, b, c \in \mathbb{R}, a \neq 0 \)

and the homogeneous problem:

\[ Ly = 0 \]

**Example:** Find the general solution to \( y'' - y' - 2y = 0 \)

**Solution:** Try \( y(x) = e^{rx} \). Can we find \( r \) that makes this a solution?

\[
\begin{align*}
y &= e^{rx} \\
y' &= re^{rx} \\
y'' &= r^2e^{rx}
\end{align*}
\]

\[ y'' - y' - 2y = 0 \]

\[ r^2e^{rx} - re^{rx} - 2e^{rx} = 0 \]

\[ e^{rx}(r^2 - r - 2) = 0 \]

\[ r^2 - r - 2 = 0 \]

\[ (r+1)(r-2) = 0 \]

\[ r = -1 \quad \text{or} \quad r = 2 \]
So we have found 2 solutions:
\[ y = e^{-x} \text{ and } y = e^{2x} \]

Are they independent?

\[ W(e^{-x}, e^{2x}) = \det \begin{bmatrix} e^{-x} & e^{2x} \\ -e^{-x} & 2e^{2x} \end{bmatrix} = 3e^x \neq 0 \]

so \( y(x) = e^{-x} \) and \( y(x) = e^{2x} \).

Hence general solution is \( y(x) = c_1 e^{-x} + c_2 e^{2x} \).

Example: Find the solution to the equation above that satisfies \( y(0) = 1 \), \( y'(0) = 1 \).

Solution: \( y(0) = 1 \Rightarrow c_1 + c_2 = 1 \) \( \text{solve} \)
\[ c_1 = \frac{1}{3} \]
\[ c_2 = \frac{2}{3} \]

\( y'(0) = 1 \Rightarrow -c_1 + 2c_2 = 1 \)

so the solution is:
\[ y(x) = \frac{1}{3} e^{-x} + \frac{2}{3} e^{2x} \]
Does this work for \( Ly = ay'' + by' + cy = 0 \)?

**Methodology:** Try \( e^{rx} \) as above and we get:

\[
L[e^{rx}] = (ar^2 + br + c)e^{rx}
\]

Since \( L[y] = 0 \) \( \Rightarrow \) \((ar^2 + br + c) = 0\) \( \circ \)

\( \Leftrightarrow \) solve for \( r \).

**Case 1:** "distinct roots."

If \( b^2 - 4ac > 0 \) then \( \circ \) has 2 distinct roots.

solutions \( r_+ \), \( r_- \) given by quadratic formula:

\[
r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

Thus \( e^{r+} \), \( e^{r-} \) solves \( Ly = 0 \). They are also independent.

since:

\[
w(e^{r+}, e^{r-}) = \begin{bmatrix} e^{r+} & e^{r-} \\ e^{r-} & e^{r+} \end{bmatrix} = (r_+ - r_-) e^{(r_+ + r_-)x} \neq 0
\]

\( \text{Since } r_+ \neq r_- \)

General solution:

\[
y(x) = c_1 e^{r+} + c_2 e^{r-}
\]
What happens when the quadratic have one (double) root?

Example: \( y'' + 2y' + y = 0 \)
Try \( e^x \), as before this is a solution, provided

\[
(r^2 + 2r + 1) = 0 \quad \Rightarrow \quad (r + 1)^2 = 0
\]

This have one solution \( r = -1 \) so \( e^{-x} \) solves the equation.

Q: How can we get a second solution?

A: Another solution is \( xe^{-x} \).

Q: How did we get this?
A: Given a solution to a 2nd order linear homogeneous equation, we can cook up another solution using reduction of order.
Reduction of order

Suppose $y'' + p(x)y' + q(x)y = 0$. Let's try to find another solution $y_1(x) = v(x)y(x)$.

We have: $y_1' = v'y + vy' \Rightarrow y_1'' = v''y + 2v'y' + vy''$

so

$y'' + py' + qy = v''y + 2v'y' + pv'y + qvy$

$= v''y + 2v'y' + pv'y + v(y'' + py' + qy)$

$= 0$

$= v''y + 2v'y' + pv'y$

Now $y_1$ is a solution if $v''y + (2y' + py)v' = 0$

$\Rightarrow v'' + (2\frac{y'}{y} + p)v' = 0 \Rightarrow$ First order for $v'$ and

by solving:
\[ v' = e^{\ln y} - Spdx = e^{-2 \ln |y|} - Spdx = e^{-Spdx} e^{\ln y^2} \]

and finally: \[ v = \int \frac{e^{-Spdx}}{y^2} \, dx \]

In our example, \( y = e^{-x}, \ p = 9 \) so \( Spdx = 9x, \ v = \int e^{-2x} \, dx = x \).

\[ v' = e^{\ln y} + by' + cy = 0 \]

Case 2: \( b^2 = 4ac, \ ar^2 + br + c \) has root \( r = -\frac{b}{2a} \), so \( y(x) = e^{-\frac{b}{2a} x} \) is a solution.

Second solution is \( vy \), where

\[ v = \int \frac{e^{-Spdx}}{(e^{-\frac{b}{2a} x})^2} = \int 1 \, dx = x \]
Now $xe^{-\frac{b}{2a}x}$ and $e^{-\frac{b}{2a}x}$ are two independent solutions:

$$W = \det \begin{bmatrix} xe^x & e^x \\ e^x + xe^x & re^x \end{bmatrix} = e^r (xr - (1+xr))$$

$$= -e^r \neq 0$$

So solutions are independent.

Moreover the general solution is:

$$y(x) = C_1 e^x + C_2 xe^x$$

Example: $2y'' + 8y' + 8y = 0$ , $y(0) = 0$ , $y'(0) = 1$

Solution:

Note: $b^2 - 4ac = 64 - 4 \cdot 2 \cdot 8 = 0$

So $2r^2 + 8r + 8 = 0 \Rightarrow 2(r+2)^2 = 0 \Rightarrow r = -2$
That means that $e^{2x}$, $xe^{-2x}$ are independent solutions and

$$y(x) = c_1 e^{-2x} + c_2 xe^{-2x}$$

is the general solution.

We impose the initial condition:

$$y(0) = c_1 = 0$$

$$y'(0) = -2c_1 + c_2 = 1 = 0$$

$$c_2 = 1$$

thus $y = xe^{-2x}$