MATH 215/255

Lecture 11
Solutions to the homogeneous \( Ly = 0 \)

Subspace property: The set of solutions to \( Ly = 0 \) forms a subspace (the nullspace of \( L = N(L) \)).

If \( Ly_1 = 0 \) and \( Ly_2 = 0 \) then
\[
L(c_1y_1 + c_2y_2) = c_1Ly_1 + c_2Ly_2 = c_1 \cdot 0 + c_2 \cdot 0 = 0
\]

Example: If \( Ly = y'' - \omega^2 y \) (\( \omega \neq 0 \)) then \( y_1 = e^{\omega x} \) and \( y_2 = e^{-\omega x} \) are solutions to \( Ly = 0 \). (i.e. in \( N(L) \))

Thus \( c_1 e^{\omega x} + c_2 e^{-\omega x} \) is a solution of \( Ly = 0 \) \( \forall c_1, c_2 \)

Are there any other solutions?

Goal: Show that \( N(L) \) is 2-dimensional, i.e. if \( y_1(x) \) and \( y_2(x) \) are two linearly independent solutions to \( Ly = 0 \) then any solution is a linear combination
\[
c_1y_1(x) + c_2y_2(x)
\]
Linear dependence/independence

Definition: Two functions $y_1(x), y_2(x)$ are linearly dependent if $c_1y_1(x) + c_2y_2(x) = 0 \neq x$ for some $c_1, c_2$, not both zero.

If $c_1y_1(x) + c_2y_2(x) = 0$ implies $c_1 = c_2 = 0$ then $y_1, y_2$ are linearly independent.

How we can test two functions for dependence/independence?
If $y_1, y_2$ are solutions to $Ly = 0$ then $y_1(x), y_2(x)$ are linearly dependent (independent) as functions.

For some $x_0$ the vectors $\begin{bmatrix} y_1(x_0) \\ y'_1(x_0) \end{bmatrix}$ and $\begin{bmatrix} y_2(x_0) \\ y'_2(x_0) \end{bmatrix}$ are dependent (independent).

**Proof:** $y_1, y_2$ dependent $\Rightarrow c_1 y_1(x) + c_2 y_2(x) = 0, \forall x \text{ with } c_1, c_2 \text{ not both zero}$

$\Rightarrow c_1 y'_1 + c_2 y'_2 = 0, \forall x \Rightarrow c_1 \begin{bmatrix} y_1(x) \\ y'_1(x) \end{bmatrix} + c_2 \begin{bmatrix} y_2(x) \\ y'_2(x) \end{bmatrix} = 0, \forall x$

$\Rightarrow \begin{bmatrix} y_1(x_0) \\ y'_1(x_0) \end{bmatrix}$ and $\begin{bmatrix} y_2(x_0) \\ y'_2(x_0) \end{bmatrix}$ dependent for any $x_0$

$\Rightarrow$ dependent for some $x_0$

**Conversely:** if $c_1 \begin{bmatrix} y_1(x_0) \\ y'_1(x_0) \end{bmatrix} + c_2 \begin{bmatrix} y_2(x_0) \\ y'_2(x_0) \end{bmatrix} = 0$ for $c_1, c_2 \text{ not both zero}$ then $y(x) = c_1 y_1(x) + c_2 y_2(x)$ satisfies the initial condition $y(x_0) = 0, y'(x_0) = 0$. 
But the solution \( \ddot{y}(x) = 0 \) satisfies the same initial condition.

Thus by the uniqueness theorem \( y = \ddot{y} = 0 \)
i.e. \( c_1 y_1(x) + c_2 y_2(x) = 0 \)
Wronskian determinant

Definition: The wronskian of two solutions $y_1, y_2$ of $Ly = 0$ is defined as:

$$W(y_1, y_2)(x) = \det \begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix} = y_1 y_2' - y_2 y_1'$$

From above, we can see that:

$$W(y_1, y_2)(x_0) \neq 0 \implies y_1, y_2 \text{ are linearly independent.}$$

Notice: $W(y_1, y_2)(x_0)$ is either:

- zero for every $x_0$ (if $y_1, y_2$ are dependent)
- zero for no $x_0$ (if $y_1, y_2$ are independent)

Example: $W(e^x, e^{-w}e^x) = \det \begin{bmatrix} e^x & e^{-w}e^x \\ we^x & -we^x \end{bmatrix} = -we^x e^{-w} - we^x = -2we^x$
Formula for the Wronskian:

If \( L y_1 = L y_2 = 0 \), \( W(x) = y_1 y_2' - y_2 y_1' \), then

\[
W(x) = y_1 y_2' + y_1' y_2 - y_2 y_1' - y_1 y_2''
= y_1 (-p(x) y_2' - q(x) y_2) - y_2 (-p(x) y_1' - q(x) y_1)
= -p(x) (y_1 y_2' - y_2 y_1') = -p(x) W(x)
\]

\[-\int p(x) dx\]

\( \Rightarrow W(x) = C e \)

This is either zero for all \( x \) (if \( C = 0 \)) or never zero (if \( C \neq 0 \)).

Main result on solutions to \( Ly = 0 \):

If \( y_1, y_2 \) are two linearly independent solutions to \( Ly = 0 \), then any solution \( y \) is a linear combination \( y = C_1 y_1 + C_2 y_2 \).
Proof: Pick any $x_0$.

We know \[
\begin{bmatrix}
y_1(x_0) \\
y_1'(x_0)
\end{bmatrix}
\text{ and }
\begin{bmatrix}
y_2(x_0) \\
y_2'(x_0)
\end{bmatrix}
\]
are 2 linearly independent vectors in $\mathbb{R}^2$.

Thus \[
\begin{bmatrix}
y(x_0) \\
y'(x_0)
\end{bmatrix} = c_1 \begin{bmatrix}
y_1(x_0) \\
y_1'(x_0)
\end{bmatrix} + c_2 \begin{bmatrix}
y_2(x_0) \\
y_2'(x_0)
\end{bmatrix}
\]
for some $c_1, c_2$.

This implies $y(x)$ and $c_1 y_1(x) + c_2 y_2(x)$ satisfy the same initial condition at $x_0$.

By uniqueness, they must be equal. 

How to find 2 independent solutions?
(Next Lecture)