Last Lecture:

—> Separable equations: \( y' = g(x)h(y) \)
—> Slope fields

Today:

—> First order linear equation
First order linear equations are equation in the form:

\[ y'(x) + p(x)y(x) = f(x) \]

where \( p(x), f(x) \) are known functions.
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**Integrating factors and solution formula**
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**Integrating factors and solution formula**

Let \( P(x) = \int p(x) dx \) be any antiderivative of \( p(x) \). We can write the left hand side of the equation as:

\[ e^{-P} \frac{d}{dx} e^{P} y \]
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This is the general solution.
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Note: Let's choose \( P(x) = \int_{x_0}^{x} p(s) ds \) and do the integral from \( x_0 \) to \( x \) then \( C = y(x_0) \) so the solution satisfying the initial condition.
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\[ y(x) = e^{-P(x)} \int e^{P(s)} f(s)ds + C e^{-P(x)} \]

\[ y(x) = e^{-\int_{x_0}^{x} p(s)ds} \int_{x_0}^{x} e^{\int_{x_0}^{s} p(t)dt} f(s)ds + y_0 e^{-\int_{x_0}^{x} p(s)ds} \]
Note: The function $r(x) = e^{P(x)}$ is called an integrating factor
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$$r(y' + py) = \frac{d}{dx} ry$$

e.i. multiplying the left hand side by $r$ turns it into a derivative.
Example: Solve $xy' + y = e^x$, $y(1) = 1$
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Then

\[ P(x) = \int_1^x \frac{1}{s} \, ds = \ln(x) \]

so

\[ e^{P(x)} = e^{\ln(x)} = x \]
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\[
y(x) = e^{-\int_{x_0}^x p(s) ds} \int_{x_0}^x e^{\int_{s_0}^s p(t) dt} f(s) ds + y_0 e^{-\int_{s_0}^{x_0} p(s) ds}
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\[
\begin{align*}
y(x) &= e^{-\int_{x_0}^{x} p(s) ds} \int_{x_0}^{x} e^{\int_{s}^{x} p(t) dt} f(s) ds + y_0 e^{-\int_{x_0}^{x} p(s) ds} \\
y(x) &= e^{-\int_{1}^{x} \frac{1}{s} ds} \int_{1}^{x} e^{\int_{s}^{x} \frac{1}{t} dt} f(s) ds + y_0 e^{-\int_{1}^{x} \frac{1}{s} ds} \\
y(x) &= \frac{1}{x} \int_{1}^{x} e^s \frac{1}{s} ds + \frac{1}{x} = \frac{1}{x} [e^x - e] + \frac{1}{x} = \frac{1}{x} [e^x - e + 1]
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\[
y(x) = e^{x_0} \int_{x_0}^x f(s) ds + y_0 e^{x_0} - \int_{x_0}^x p(s) ds
\]

\[
y(x) = \frac{1}{x} \int_1^x s e^s ds + \frac{1}{x} = \frac{1}{x} [e^x - e] + \frac{1}{x} = \frac{1}{x} [e^x - e + 1]
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Check: \( y' = -x^{-2} [e^x - e + 1] + \frac{1}{x} e^x \)
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\[
y(x) = e^{-\int_{x_0}^x p(s)ds} \int_{x_0}^x e^{\int_{t_0}^t f(s)ds} + y_0 e^{-\int p(s)ds} \]

\[
y(x) = \frac{1}{x} \int_1^x s \frac{e^s}{s} ds + \frac{1}{x} = \frac{1}{x} \left[ e^x - e \right] + \frac{1}{x} = \frac{1}{x} \left[ e^x - e + 1 \right]
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Check: \( y' = -x^{-2}[e^x - e + 1] + \frac{1}{x}e^x \)

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xy' = -\frac{1}{x}[e^x - e + 1] + e^x = -y + e^x
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y(x) = e^{-\int_{x_0}^x p(s) ds} \int_{x_0}^x e^{\int_{t_0}^t p(t) dt} f(s) ds + y_0 e^{-\int_{x_0}^x p(s) ds}
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y(x) = \frac{1}{x} \int_1^x s e^s \frac{ds}{s} + \frac{1}{x} = \frac{1}{x} [e^x - e] + \frac{1}{x} = \frac{1}{x} [e^x - e + 1]
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$$\begin{align*}
y(x) &= e^{-\int_0^x p(s)ds} \left( \int_0^x e^{\int_0^x f(t)dt} f(s)ds + y_0 e^{-\int_0^x p(s)ds} \right) \\
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Notice that the solution exists for $x > 0$ i.e. up to the point where $p(x) = \frac{1}{x}$ and $f(x) = \frac{e^x}{x}$ stop being continues.
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y(x) = e^{-\int_0^x p(s) ds} \int_0^x \left( \int_0^t p(t) dt \right) f(s) ds + y_0 e^{-\int_0^x p(s) ds}
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Notice that the solution exists for \( x > 0 \) i.e. up to the point where \( p(x) = \frac{1}{x} \) and \( f(x) = \frac{e^x}{x} \) stop being continues.

Sometimes solutions exists even though \( p(x) \) or \( f(x) \) have a singularity.
Example: \( y' - \frac{1}{x} y = 0, \quad y(1) = 1 \)
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Solution is only guaranteed to exist for \( x > 0 \) but in fact \( y(x) = x \) is the solution.
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Sometimes a solution extends past a singularity of \( p(x) \) or \( f(x) \) for some initial condition but not all.
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Sometimes a solution extends past a singularity of \( p(x) \) or \( f(x) \) for some initial condition but not all.

Example: \[ y' - \frac{1}{x}y = 2, \quad y(1) = y_0, \quad P(x) = \ldots = x \]

\[ y(x) = e^{-P(x)} = \int_1^x e^P(s)2ds + e^{-P(x)}y_0 = \frac{1}{x}2 \int_1^x sds + \frac{1}{x}y_0 \]
Example: \[
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\[
= \frac{2}{x}(\frac{x^2}{2} - \frac{1}{2}) + \frac{y_0}{x} = x + \frac{y_0 - 1}{x}
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Solution here: \( \longrightarrow \) exists for all \( x \) if \( y_0 = 1 \)

\( \longrightarrow \) blows up at \( x = 0 \) if \( y_0 \neq 1 \)
Mixing problem

Tank holds 120 liters of fresh water. Saltwater with concentration of \( \frac{g}{l} \) flows in at the rate of \( 2l/min \). Well stirred mixture flows out at the rate of \( 2l/min \). How much salt is in the tank at time \( t \)?
Mixing problem

Tank holds 120 liters of fresh water. Saltwater with concentration of $\gamma \frac{g}{l}$ flows in at the rate of $2l/min$. Well stirred mixture flows out at the rate of $2l/min$. How much salt is in the tank at time $t$?

Solution:

Let $x(t)$ be the amount of salt at time $t$ (in grams), then:
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Solution:

Let \( x(t) \) be the amount of salt at time \( t \) (in grams), then:

\[
x'(t) = \gamma \frac{g}{l} \times 2 \frac{l}{min} - \frac{x(t) \frac{g}{l} \times 2 \frac{l}{min}}{120}
\]
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Solution:

Let $x(t)$ be the amount of salt at time $t$ (in grams), then:

$$x'(t) = \gamma \left[ \frac{g}{l} \right] 2 \left[ \frac{l}{min} \right] - \frac{x(t)}{120} \left[ \frac{g}{l} \right] 2 \left[ \frac{l}{min} \right]$$

which give us the D.E.:

$$x'(t) + \frac{2}{120} x = 2\gamma, \quad x(0) = 0$$
The solution is:

\[ x(t) = e^{-\frac{1}{60}t} \int_0^t e^{\frac{1}{60}s} 2\gamma ds = 2\gamma e^{-\frac{1}{60}t} 60[ e^{\frac{1}{60}t} - 1 ] = 120\gamma[ 1 - e^{-\frac{1}{60}t} ] \]
The solution is:

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Notice: \( \lim_{t \to \infty} x(t) = 120\gamma \)
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\[ x(t) = e^{-\frac{1}{60}t} \int_0^t e^{\frac{1}{60}s} 2\gamma ds = 2\gamma e^{-\frac{1}{60}t} 60[e^{\frac{1}{60}t} - 1] \]

\[ = 120\gamma[1 - e^{-\frac{1}{60}t}] \]

Notice: \[ \lim_{t \to \infty} x(t) = 120\gamma \]

As an exercise try to solve the problem when the input consecration varies periodically:

\[ \gamma(t) = \gamma_0 \sin(\omega t) \]