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Then: \( y' = g(x) \cdot h(y) \)

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(we follow the process we had for \( y' = f(y) \))

\[
\frac{1}{h(y)} y' = g(x) \rightarrow \int \frac{1}{h(y)} \, dy = \int g(x) \, dx + C
\]

(implicit equation)

To satisfy \( y(x_0) = y_0 \) use:

\[
\int_{y_0}^y \frac{1}{h(y)} \, dt = \int_{x_0}^x g(s) \, ds
\]
Example
Example: Solve \( y' = y \sin(x) \), \( y(0) = 1 \)
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Solution: $(h(y) = y, g(x) = \sin(x))$
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$$\int_1^y \frac{1}{t} \, dt = \int_0^x \sin(s) \, ds$$

$$\Rightarrow \ln(y) = -\cos(x) + 1$$
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$$\int_{\gamma} \frac{1}{t} \, dt = \int_0^x \sin(s) \, ds$$

$\Rightarrow \ln(y) = -\cos(x) + 1$

$\Rightarrow y(x) = e^{-\cos(x)}$ (This solution is defined for all $x$)
Example:
Example: Solve $y' = x^2 e^{-y}$, $y(0) = 1$

Solution
Example: Solve \( y' = x^2 e^{-y}, \ y(0) = 1 \)

Solution: \( h(y) = e^{-y} \) and \( g(x) = x^2 \)

\[
\int_1^y \frac{1}{e^t} \, dt = \int_0^x s^2 \, ds
\]
Example: Solve \( y' = x^2 e^{-y}, \ y(0) = 1 \)

Solution: \( h(y) = e^{-y} \) and \( g(x) = x^2 \)

\[
\int_1^y e^t \, dt = \int_0^x s^2 \, ds
\]

\[
\Rightarrow e^y - e^1 = \frac{x^3}{3}
\]

\[
\Rightarrow y = \ln(e + \frac{x^3}{3})
\]

Defined for \( x > 3 \)
Example: Solve \( y' = x^2 e^{-y}, \ y(0) = 1 \)

Solution: \( (h(y) = e^{-y} \) and \( g(x) = x^2 \) \)

\[ \int_{y}^{x} e^{-t} \, dt = \int_{0}^{x} s^2 \, ds \]

\[ \Rightarrow e^{y} - e^{0} = \frac{x^3}{3} \]

\[ \Rightarrow e^{y} - e' = \frac{x^3}{3} \]

\[ \Rightarrow y = \ln(e + \frac{x^3}{3}) \]

Defined for \( \frac{x^3}{3} > -e \) \( \Rightarrow x > -(3e)^{1/3} \)
Slope fields
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A solution to \( y' = f(x,y) \) is a function \( y(x) \), whose slope as it passes through the point \((x,y)\) is \( f(x,y) \).
Slope fields

A solution to $y' = f(x,y)$ is a function $y(x)$, whose slope as it passes through the point $(x,y)$ is $f(x,y)$. By plotting the slopes we can get a good picture of how solutions behave. (Even if we can't solve)
Slope fields

A solution to \( y' = f(x, y) \) is a function \( y(x) \), whose slope as it passes through the point \((x, y)\) is \( f(x, y) \). By plotting the slopes we can get a good picture of how solutions behave. (Even if we can't solve)

**Example:** \( f(x, y) = x \), \( y(x) = \frac{x^2}{2} + C \)

```matlab
>> f = @(x,y) x
[X,Y]=meshgrid([-2:.3:2,-2:.3:2]);
DY=f(X,Y); DX=ones(size(DY));
yx = @(x) (x.^2)/2-1
YY=yx([-2:0.1:2])
quiver(X,Y,DX,DY);grid on;hold on;plot([-2:0.1:2],YY)
```
Slope fields

A solution to \( y' = f(x, y) \) is a function \( y(x) \), whose slope as it passes through the point \((x, y)\) is \( f(x, y) \). By plotting the slopes we can get a good picture of how solutions behave. (Even if we can't solve)

Example: \( f(x, y) = x \), \( y(x) = \frac{x^2}{2} + C \)
Example: \( f(x, y) = y \implies y(x) = ce^x \)
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Example: \( f(x, y) = y \implies y(x) = ce^x \)

Example: \( f(x, y) = \sqrt{1 - y^2} \)
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Example: $f(x, y) = \sqrt{1 - y^2}$
Picard's Theorem
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When can we be sure that a unique solution exists near \((x_0, y_0)\) ?
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**Picard's Theorem**: Suppose \(f(x, y)\) and \(\frac{df}{dy}(x, y)\) are continuous near \((x_0, y_0)\).
Picard's Theorem

When can we be sure that a unique solution exists near \((x_0, y_0)\)?

**Picard's Theorem**: Suppose \(f(x,y)\) and \(\frac{df}{dy}(x,y)\) are continuous near \((x_0, y_0)\). Then there is a unique solution to \(y' = f(x,y), \ y(x_0) = y_0\) defined for some (possibly small) interval of \(x\) values around \(x_0\).
Examples from last lecture.
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\[ f(x, y) = 1 + y^2 \rightarrow \frac{\partial f}{\partial y} = 2y \] are continuous everywhere, so \( y' = 1 + y^2 \), \( y(x_0) = y_0 \) always has a unique solution for \( x \) close to \( x_0 \).
Examples from last lecture:

- \( f(x, y) = 1 + y^2 \rightarrow \frac{\partial f}{\partial y} = 2y \) are continuous everywhere, so \( y' = 1 + y^2 \), \( y(x_0) = y_0 \) always has a unique solution for \( x \) close to \( x_0 \).

- \( f(x, y) = \sqrt{1 - y^2} \), \( \frac{\partial f}{\partial y} = \frac{-y}{\sqrt{1 - y^2}} \) are continuous near \((x_0, y_0)\), provided \( |y_0| < 1 \). But \( \frac{\partial f}{\partial y} \) is not continuous (or even defined) when \( y = \pm 1 \).
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- \( f(x, y) = \sqrt{1 - y^2} \), \( \frac{\partial f}{\partial y} = \frac{-y}{\sqrt{1 - y^2}} \) are continuous near \((x_0, y_0)\), provided \( |y_0| < 1 \). But \( \frac{\partial f}{\partial y} \) is not continuous (or even defined) when \( y = \pm 1 \).

Example: Does \( y' = 1 + 1x |y|^2 \), \( y(0) = 0 \) have a unique solution near \( x = 0 \)?
Examples from last lecture:

- \( f(x, y) = 1 + y^2 \rightarrow \frac{\partial f}{\partial y} = 2y \) are continuous everywhere, so \( y' = 1 + y^2, \ y(x_0) = y_0 \) always has a unique solution for \( x \) close to \( x_0 \).

- \( f(x, y) = \sqrt{1 - y^2} \), \( \frac{\partial f}{\partial y} = \frac{-y}{\sqrt{1 - y^2}} \) are continuous near \((x_0, y_0)\), provided \(|y_0| < 1\). But \( \frac{\partial f}{\partial y} \) is not continuous (or even defined) when \( y = \pm 1 \).

Example: Does \( y' = 1 + 1 \cdot 1 \cdot y^2, \ y(0) = 0 \) have a unique solution near \( x = 0 \)?

Solution: \( f(x, y) = 1 + 1 \cdot 1 \cdot y^2 \), \( \frac{\partial f}{\partial y} = 2 \cdot 1 \cdot 1 \cdot y \) are both cts near \( x_0 = 0 \) so yes!!!