RESEARCH STATEMENT

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Introduction

I am interested in algebraic geometry, specifically: moduli spaces, geometric invariant theory, stability in abelian and triangulated categories, and geometry inspired by string theory. Conveniently, these topics are all closely related by the following narrative. String theory motivates me to try to "count" stable objects in the derived category of a Calabi-Yau threefold. Very roughly, this is done by chosing a stability condition, constructing a moduli spaces of stable objects using geometric invariant theory, and either proving that the moduli space constructed has a perfect obstruction theory, or using Behrend's microlocal function to define invariants. Once invariants exist, it is desirable to explicitly compute them or to relate them to other invariants; this is done by wall-crossing techniques [11], by use of the integration map on the Hall algebra [6] [7], or by localization [9]. Let's take a more detailed look at a few of these topics.

The word "count" above is in quotations because the counts I am most interested in are virtual counts, numbers that represent how many objects should ideally appear. By analogy, "the number of complex roots of a degree n polynomial is equal to n" is a false statement: it becomes true if you understand that the counting process should include weighting roots by their multiplicity. This is what is meant by virtual count. To analogize another aspect of counting, the Euler characteristic of a finite discrete topological space is equal to its cardinality, so it is natural to think of the Euler characteristic of any space as some kind of count of the space. Thus, combining analogies, a virtual count should be thought of as a weighted Euler characteristic. Precisely this result was proved by Behrend [3]. Different choices of stability condition lead to different moduli spaces, and different numbers or invariants. The invariants I am most interested in are Donaldson-Thomas (DT) invariants and Pandharipande-Thomas (PT) invariants, which count ideal sheaves and stable pairs, respectively. They are thought to encode the BPS states of string theory.

Geometric invariant theory is a method for producing a scheme that is a quotient of a scheme by a group action. The indefinite article is appropriate here because there are many possible quotients, according to the fact that there are potentially many stability conditions. The idea behind GIT is that if one wants a well-behaved quotient, then one should restrict the orbits under consideration (namely: throw away the "unstable" orbits).

Stability appears in many guises. It is fundamental to geometric invariant theory. Pandharipande and Thomas conjecture [16] a relationship between PT and DT invariants that ought to be proven by varying stability conditions. This was later proved by Bridgeland in [7]. In general, one hopes to use so-called wall-crossing techniques to understand how invariants change as the stability condition varies.

Recent research

I am currently focussed on crepant resolutions Y of singular three-dimensional Calabi-Yau varieties X. There are two ways one might try to produce deformation invariants from X: by resolving X, and producing invariants from its resolution, or by constructing a Deligne-Mumford stack $\mathcal{X}$ whose coarse space is equal to X, and defining the invariants on the stack. The crepant resolution conjecture [9] states that (in the hard Lefschetz case) these two approaches are equivalent. A short-term research goal of mine is to prove the crepant resolution conjecture for PT invariants.

My thesis makes significant progress in this goal: I create invariants that act as a bridge between the PT invariants of the resolution Y and the PT invariants of the stack $\mathcal{X}$. I call these invariants *perverse pairs* or PP theory. In [7] it is made clear that both DT and PT theory give virtual counts of epimorphisms in an abelian category. In DT, the abelian category is the category of coherent sheaves. In PT, we tilt this category with respect to a particular torsion pair, and the result is the abelian category whose epimorphisms we are interested in. In PP, we tilt the abelian cateogry of coherent sheaves using a different torsion pair. More generally, tilting is a method for producing a new abelian category from a given abelian category that has two distinguished classes of objects: these two classes form a *torsion pair.* In DT, objects of your moduli space are epimorphisms of the form $\mathcal{O} \to F$. After the tilt in PT theory, the objects become maps $\mathcal{O} \to F$ that are no longer epimorphisms in the category of coherent sheaves, but they are epimorphisms in a tilt of the category of coherent sheaves. The result is that — instead of any surjection, $\mathcal{O} \to F$— restrictions on F are made stronger, but those on the cokernal are made weaker: instead of trivial cokernal and any sheaf F, the cokernal may be zero-dimensional, but the sheaf F must be pure on its support. In PP theory, we push this a little further: even stronger restrictions are put on F, and an even more general cokernal is allowed.

The next step to creating PP invariants is to construct a stability condition in the sense of Joyce [10]. The stability condition gives rise to a (generalized) Harder-Narasimhan filtration: a recipe for building any given sheaf out of sheaves that are semistable with respect to this new stability condition. Next, we want to have the sheaf F of the perverse pair $\mathcal{O} \to F$ to be semistable with respect to the stability condition we defined. This is achieved by deft choice of stability condition and a fluency in language of Harder-Narasimhan filtrations.

The final ingredient is the motivic Hall algebra, an algebra generated by symbols $[A \to M]$, where M is the stack of coherent sheaves, and $A \to M$ is a map of stacks, subject to the scissor relations and others. DT and PT generating series may be obtained from the Hall algebra by applying the so-called integration map to the appropriate element. Following Bridgeland, I have produced equations in the Hall algebra, and applied the integration map to the equations, yielding equations of generating series. Essential to this process is the ability to describe PP objects in terms of a stability condition, since the integration map is not defined on the whole Hall algebra [6].

The conclusion is the existence of new invariants, defined on resolutions of singular Calabi-Yau threefolds, and a strong relationship to the PT invariants of the resolution. Most of the work towards proving the crepant resolution conjecture is now done.

The conclusion is the following theorem. Let $PP(Y)$ denote the generating series for PP invariants, and $PT(Y)$ denote the generating series for PT invariants.

Theorem 1. *Let X be a singular Calabi-Yau threefold with crepant resolution Y. Suppose that $R\pi_*(\mathcal{O}_Y) = \mathcal{O}_X$, where $Y \overset{\pi}{\to} X$ is the resolution. Then there exist DT-type invariants, called PP invariants, counting stable objects in an abelian subcategory of $D(Y)$ such that*

$$PP(Y) = \frac{PT(Y)}{PT_{exc}(Y)},$$

where $PT_{exc}(Y)$ counts the PT objects on Y that are supported on the exceptional locus.

Future research

The complete proof of the crepent resolution conjecture is a work in progress, and I believe that it will proceed as follows. Let us consider the case of global quotients for simplicity, i.e., stacks of the form $\mathcal{X} = M/G$. In this case, the derived category of $\mathcal{X}$, $D^b(\mathcal{X})$, is isomorphic to the G-equivariant derived category of the master space M, $D^b_G(M)$. By the celebrated results of Bridgeland, King, and Reid [8], we know that there is an equivalence Ψ between the derived category of Y and the G-equivariant derived category of M. One is tempted to try to show that this equivalence takes perverse pairs on Y to stable pairs on $\mathcal{X}$, but it appears that it is easier to relate perverse pairs to Bridgeland's perverse ideal sheaves [4] (we will return to this point in a moment). Preliminary results suggest that the duality autoequivalence takes perverse pairs to perverse ideal sheaves. Now it remains to show that Ψ, the equivalence of Bridgeland, King, and Reid, takes perverse ideal sheaves to equivariant coherent sheaves $\mathrm{coh}_G(M)$ in $D^b(M)$. The perverse ideal sheaves have the property that one can test whether an object E of the derived category is a perverse ideal sheaf by looking at properties of $\mathrm{Hom}(P, E)$ for all projective perverse ideal sheaves P. Similarly, objects F of $\mathrm{coh}_G(M)$ be detected using $\mathrm{Hom}(V, \ )$ for all equivariant vector bundles V. We know that $\mathcal{O}_Y$ is sent to $\mathcal{O}_M$ by the equivalence Ψ, so it remains to show that Ψ maps an equivariant coherent sheaf to a projective object of the category of perverse ideal sheaves. This may be checked using the results of [20].

In Bridgeland's "Hall Algebras and Curve-Counting Invariants," the author proves that DT and PT invariants are equivalent. I would like to generalize this result to the orbifold case, and have reason to believe that it will be a very manageable generalization. This may be thought of as an extension of my PT crepant resolution conjecture research program: once I have proven the orbifold PT/DT correspondence, it may be combined with the proof of the PT crepant resolution conjecture to yield a proof of the DT crepant resolution conjecture, extending the results of [9].

A central theme in everything that I have studied is stability. Stability appears in algebraic geometry in a number of different ways [5], as does the variation of stability conditions [11], [19]. Bridgeland stability is desirable in the sense that it applies to objects in a triangulated category (instead of just an abelian category), and that the space of Bridgeland stability conditions is a complex manifold. However, at the time of writing, no one has been able to give an example of such a stability condition in the case we are most interested in: projective Calabi-Yau threefolds. I would like to get a better understanding of why it is so difficult to construct Bridgeland stability conditions. The heap of recent related results will provide me with a toe-hold into the subject. In mind is the long-term goal of constructing Bridgeland stability conditions on projective Calabi-Yau threefolds, and using the theory of complex manifolds to deduce information about them from their space of stability conditions.

Past research

Prior to working on the PT crepant resolution conjecture, I had completed a project whose goal was to use geometric invariant theory to construct the moduli space of stable pairs for DM stacks. Using Nironi [15], Le Potier [13], and Stoppa-Thomas [18] as a guide, the project split into a few easily identifiable pieces: proving the boundedness of the moduli problem, constructing the *master space,* (i.e., the space whose GIT quotient will be the desired moduli space), expressing the Hilbert-Mumford index of a stable pair (F, Φ) in terms of Hilbert polynomails of stable pairs associated to (F, Φ), and finally proving that geometric stability (given in terms a Joyce stability condition) is equivalent to GIT stability (given in terms of the now-more-approachable Hilbert-Mumford index). However, before I was able to finishing writing up my result, an identical result [1] appeared on the arXiv.

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