Midterm 1 Duration: 50 minutes
This test has 5 questions on 10 pages, for a total of 40 points.

- Read all the questions carefully before starting to work.
- Q1 and Q2 are short-answer questions; put your answer in the boxes provided.
- All other questions are long-answer; you should give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. None of the following are allowed: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

First Name: ___________________________ Last Name: ___________________________

Student-No: ____________________________ Section: 201

Signature: ______________________________

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Student Conduct during Examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBC card for identification.

2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.

3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.

4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.

5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
   (i) speaking or communicating with other examination candidates, unless otherwise authorized;
   (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;
   (iii) purposely viewing the written papers of other examination candidates;
   (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
   (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).

6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.

7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.

8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).
**Short-Answer Questions.** Questions 1 and 2 are short-answer questions. Put your answer in the box provided. Each part is worth 2 marks, but not all parts are of equal difficulty. **Simplify your answers as much as possible in Questions 1 and 2.**

1. (a) Let \( y = \tan(\sin(x^3)) \). Find \( \frac{dy}{dx} \).

**Answer:** \( \frac{dy}{dx} = 3x^2 \cos(x^3) \cdot \sec^2(\sin(x^3)) \)

**Solution:** Use the chain rule twice

\[
\frac{dy}{dx} = \sec^2(\sin(x^3)) \cdot \frac{d}{dx} \sin(x^3)
= \sec(\sin(x^3)) \cdot \cos(x^3) \cdot \frac{d}{dx} (x^3)
= 3x^2 \cos(x^3) \cdot \sec^2(\sin(x^3))
\]

(b) Let \( f(x) = \arctan(\log(x/2)) \). Find \( f'(x) \). Note: \( \arctan \) is inverse tangent and \( \log \) is the natural logarithm.

**Answer:** \( f' = \frac{1}{x} \cdot \frac{1}{1+(\log(x/2))^2} \)

**Solution:** Use the chain rule

\[
f'(x) = \frac{1}{1+(\log(x/2))^2} \cdot \frac{d}{dx} \log(x/2)
= \frac{1}{x} \cdot \frac{1}{1+(\log(x/2))^2}
\]

(c) If \( f(x) = \frac{3x-1}{2x+3} \), find the inverse function \( g(x) = f^{-1}(x) \).

**Answer:** \( g(x) = \frac{3x+1}{3-2y} \)

**Solution:**

\[
y = \frac{3x-1}{2x+3}
(2x+3)y = 3x - 1
x(2y-3) = -3y - 1
x = \frac{3y+1}{2y-3}
\]

(d) If \( \log(x) - \log(x-1) = 1 \), find \( x \). Again note: \( \log \) is the natural logarithm.

**Answer:** \( x = \frac{e}{e-1} \)
Solution:

\[
\log(x) - \log(x - 1) = 1
\]

\[
\log x = 1 + \log(x - 1) = \log(e(x - 1))
\]

\[
x = e(x - 1) = ex - e
\]

\[
e = x(e - 1)
\]

\[
\frac{e}{e - 1} = x
\]
2 marks 2. (a) If \( f(x) = e^x + \cos(\pi)\sin(\pi) - \sin(x) \), find \( f'(x) \).

**Answer:** \( f'(x) = e^x - \cos(x) \)

**Solution:** Straightforward, except we need to note that \( \cos(\pi)\sin(\pi) \) is a constant so its derivative is zero:

\[ f'(x) = e^x + 0 - \cos(x) \]

2 marks (b) If \( g(x) = e^{x^2 - 2x + 1} \), find \( g'(x) \).

**Answer:** \( e^{x^2 - 2x + 1}(2x - 2) \)

2 marks (c) Compute \( \arcsin\left( \cos\left(\frac{25\pi}{22}\right) \right) \).

**Answer:** \(-\frac{4\pi}{11}\)

**Solution:** Notice that \( \frac{25\pi}{22} > \pi \). Going \( \pi \) back (draw on the unit circle to see; alternatively, draw a sketch of \( \cos x \) in one period):

\[
\cos \frac{25\pi}{22} = -\cos \frac{3\pi}{22} = -\sin \left( \frac{\pi}{2} - \frac{3\pi}{22} \right) = -\sin \left( \frac{8\pi}{22} \right) = \sin \left( \frac{-8\pi}{22} \right) = \sin \left( -\frac{4\pi}{11} \right) \]

Notice \(-\frac{4\pi}{11} \in \left[ -\frac{\pi}{2}, -\frac{\pi}{2} \right]\) hence \( \arcsin\left( \cos\left(\frac{25\pi}{22}\right) \right) = -\frac{4\pi}{11} \)

2 marks (d) If \( y = \left( \frac{1}{x} \right)^{1/x} \) find \( \frac{dy}{dx} \). Your answer must be expressed in terms of \( x \) only.

**Answer:** \( \frac{dy}{dx} = x^{-1/x} \cdot \frac{\log(x) - 1}{x^2} \)

**Solution:** Take log of both sides:

\[
y = x^{-1/x} \\
\log y = -\frac{\log x}{x}
\]
now differentiate

\[
\frac{1}{y} \frac{dy}{dx} = \frac{-x/x - 1 \cdot \log x}{x^2} \\
= \frac{\log(x) - 1}{x^2}
\]

\[
\frac{dy}{dx} = x^{-1/x} \cdot \frac{\log(x) - 1}{x^2}
\]
Full-Solution Problems. In questions 3–5, justify your answers and show all your work. If a box is provided, write your final answer there. Unless otherwise indicated, simplification of answers is not required in these questions.

8 marks 3. A flower is planted 5m from the base of a lamppost. A bee is flying in a circular path around the flower with a radius of 2 metres.

(a) Draw a (careful) picture describing the situation. Your picture should indicate all relevant lengths and angles.

Solution: Let \( \theta \) be the angle of the bee measured from the line between the lamp and the flower. Let \( z \) be the distance between the lamp and the bee.

(b) What is the distance between the bee and the lamp. Hint — the cosine law might help (ie \( c^2 = a^2 + b^2 - 2ab \cos \theta \)) but it is not necessary.

Solution: Using the picture and the cosine law, we have

\[
z^2 = 5^2 + 2^2 - 2 \cdot 5 \cdot 2 \cos \theta = 29 - 20 \cos \theta
\]

Alternatively, we can draw a perpendicular line from the bee to the line joining the lamp and flower. This creates a right-angled triangle of sides 2, 2\( \sin \theta \) and 2\( \cos \theta \). So the other right-angled triangle gives

\[
z^2 = (2 \sin \theta)^2 + (5 - 2 \cos \theta)^2 = 4 \sin^2 \theta + 5^2 + 4 \cos^2 \theta - 20 \cos \theta = 29 - 20 \cos \theta
\]

(c) Let \( z \) denote the distance between the bee and the lamp. At what values of \( z \) is \( \frac{dz}{dt} \) equal to zero? Explain your answer.

Solution: Differentiate the distance expression above

\[
2z \frac{dz}{dt} = 20 \sin \theta \frac{d\theta}{dt}
\]

\[
\frac{dz}{dt} = \frac{20 \sin \theta \, d\theta}{z \, dt}
\]
Since the distance $z$ is never zero, the rate of change is zero when $\sin \theta = 0$. This happens when $\theta = 0, \pi$. This corresponds to the bee being as close as possible to the lamp or as far as possible from the lamp. ie

\[
\begin{align*}
    z^2 &= 5^2 + 2^2 + 20 = 49 \\
    z^2 &= 5^2 + 2^2 - 20 = 9
\end{align*}
\]

so $z = 7$ and $z = 3$. 
4. Water is being pumped into a tank at a rate of 1 m³ per hour. The tank is a right circular cone whose base has radius 6 m and has height 6 m. Note that the tank is not inverted — the cone points up.

(a) Draw a (careful) picture describing the situation. Your picture should indicate all relevant lengths.

Solution:

![Diagram of a cone with water level and dimensions](image)

(b) What is the volume of water in the tank as a function of water depth? Hint — think about the air in the tank.

Solution: Volume of water in the tank is the total volume minus the volume of air. Hence

\[ V_t = \frac{\pi}{3} (\text{radius})^2 (\text{height}) \quad \text{total volume} \]

\[ = \frac{\pi}{3} \cdot 36 \cdot 6 = 72\pi \]

\[ V_a = \frac{\pi}{3} (\text{radius})^2 (\text{height}) \quad \text{vol of air} \]

\[ = \frac{\pi}{3} \cdot r^2 (6 - h) \]

\[ V = V_t - V_a \quad \text{volume of water} \]

\[ = 72\pi - \frac{\pi}{3} \cdot r^2 (6 - h) \]

Now by similar triangles, we see that \( \frac{r}{6-h} = \frac{6}{6} \), so \( r = 6 - h \). Thus

\[ V = 72\pi - \frac{\pi}{3} (6 - h)^3 \]

(c) How quickly is the depth changing when the water is 2 m deep?

Solution: Simply differentiate by \( t \) to get

\[ \frac{dV}{dt} = \pi (6 - h)^2 \frac{dh}{dt} \]
We know \( h = 2, \frac{dV}{dt} = 1 \), so

\[
1 = \pi \cdot 4^2 \cdot \frac{dh}{dt}
\]

\[
\frac{dh}{dt} = \frac{1}{16\pi} m/h
\]
5. You are sitting outside a fine cafe in Melbourne on a beautiful sunny 10°C winter’s day. The barista brings your espresso straight from the machine. The coffee is a perfect 90°C. Just as your coffee arrives, your phone goes off and it distracts you for 1 minute. By that time your coffee has cooled to 70°C. Just at that point a “friend” interrupts you and it takes another 2 minutes to get them to leave.

What temperature is your coffee when you finally get to drink it?

Assume that your drink is cooling according to Newton’s law of cooling. Your answers may be left in “calculator-ready” form. Be sure to define all variables and units used.

Solution:

• Let $T$ be the temperature of the coffee in C and $t$ be the time in minutes measured from its arrival.

• Let $y$ be the difference in temperature between the coffee and the surroundings. So $y = T - 10$.

• Assuming Newton’s law of cooling we have

$$y(t) = y(0)e^{-kt}$$

• Initially $T(0) = 90$, so $y(0) = 80$ and after 1 minute, $T(1) = 70$, so $y(1) = 60$.

$$y(1) = 60 = 80e^{-k}$$

$$e^k = \frac{60}{80} = \frac{3}{4}$$

$$k = \log\left(\frac{3}{4}\right)$$

• Thus after a total of 3 minutes

$$y(3) = 80e^{-3k} = 80e^{-3\log\left(\frac{3}{4}\right)} = 80e^{\log\left(\frac{3}{4}\right)^3}$$

$$= \frac{80 \times 3^3}{4^3} = \frac{80 \times 27}{64} = \frac{5 \times 27}{4} = \frac{135}{4}$$

and the temperature is then

$$T(3) = 10 + y(3) = 10 + \frac{135}{4} = \frac{175}{4}$$ degrees