1. Find the derivative of:

\[ f(x) = e^x \sin x \log(\cos x) \]

**Solution:** (If \( f = g \cdot h \cdot l \), then)

\[ f'(x) = g'(x)h(x)l(x) + g(x)h'(x)l(x) + g(x)h(x)l'(x) \]

So,

\[ f'(x) = e^x \sin x \log(\cos x) + e^x \cos x \log(\cos x) + \]

\[ + e^x \sin x \frac{1}{\cos x} \]  \( \text{chain rule} \)

2. \( f(x) = 2^x \)

**Solution:** Alter: Remember the derivative: \( f'(x) = \ln 2 \cdot 2^x \)

**Solution:** Logarithmic differentiation:

\[ \log f(x) = x \ln 2 \quad \Rightarrow \quad \frac{f'(x)}{f(x)} = \ln 2 \]

\[ \Rightarrow f'(x) = 2^x \ln 2 \]

3. \( f(x) = (\sin x)^x \)

**Solution:** We know how to differentiate \( [f(x)] \) or \( \sqrt{f(x)} \)
\[ f(x) = (\sin x)^{x^2} \Rightarrow \log f(x) = x^2 \log x \]

\[ \Rightarrow \frac{f'(x)}{f(x)} = 2x \log x + x^2 \frac{1}{x} \]

\[ = 2x \log x + \frac{1}{x} \]

\[ \Rightarrow f'(x) = (\sin x)^{x^2} \left( 2x \log x + \frac{1}{x} \right) \]

(2) \( g(x) = 3 + x + e^x \), find \( g^{-1}(41) \)

\[ g(3 + x + e^x) = 41 \]

\[ \Rightarrow x : g(x) = 4 \]

Notice \( g(x) = 7 \Rightarrow 1 - 1 \)

\[ \text{if } x_1 < x_2 \]

\[ e^{x_1} < e^{x_2} \]

\[ x_1 + e^{x_1} < x_2 + e^{x_2} \]

\[ 3 + e^{x_1} + x_1 < 3 + e^{x_2} + x_2 \]

\[ g(x_1) < g(x_2) \Rightarrow g^{-1} \]

\[ \Rightarrow g^{-1} \]
\[ \Rightarrow g = 1 - 1 \Rightarrow \text{there is a unique } x \]
\[ \text{s.t. } g(x) = 4 \text{ :} \]
\[ 3 + e^x + x = 4 \Rightarrow e^x + x = 1 \]
\[ \text{by inspection} \quad \Rightarrow x = 0 \]

2. Find the inverse of \( f(x) = e^{x^3} \)

Solution:

Need to solve for \( x \) \( \Rightarrow \) take \( \ln \) (\( \cdots \) )

\[ y = f(x) = e^{x^3} \Rightarrow \ln y = x^3 \ln e = x^3 \]

\[ \Rightarrow x = \sqrt[3]{\ln y} \]

3. Solve:

\[ g(x) = 3 \Rightarrow (3^x)^{x-5} \]

\[ \Rightarrow 3^{ex-10} = 3^4 \]

\[ \Rightarrow \log_3 (\text{ex - 10}) = 4 \]

\[ \Rightarrow \frac{\text{ex} - 10}{\log_3 (\text{ex})} = 4 \]

\[ \Rightarrow \frac{\text{ex} - 10}{\log_3 (\text{ex})} = 4 \Rightarrow \frac{\text{ex} - 10}{\text{ex} - 10} = 4 \Rightarrow x = \frac{11}{2} \]
$\ln x + \ln (x-1) = 1.$

$\ln x + \ln (x-1) 
\iff \ln(x(x-1)) = 1 = \ln e$

$\ln(\text{arb}) \implies x(x-1) = e$

$\implies x^2 - x - e = 0$

Quadratic formula: $x_{1,2} = \frac{1 \pm \sqrt{1 + 4e}}{2}$

4. (Implicit differentiation)

Find the $x$-coordinate(s) of the points where the tangent line has slope $-1$.

Curve: $x^2y^2 + xy = 2$

Solution: $(xy)^2 + xy = 2$

Define $z = z(x, y) \implies z(x)$:

$z^2 + z = 2 \iff 2z \frac{dz}{dx} + \frac{dz}{dx} = 0$

$\iff \frac{dz}{dx} (2z + 1) = 0$
\[
\frac{dz}{dx} = \frac{d}{dx}\left( x y(x) \right) = y + x \frac{dy}{dx} = y + xy'
\]

For \( y \), we have:
\[
(1 + 2(xy)) \left( y + xy' \right) = 0
\]

Either \( 1 + 2xy = 0 \) (*) or \( y + xy' = 0 \) (**).

Looking for \( x \) where \( y' = -1 \) \( \Rightarrow \) (**), \( y + x(-1) = 0 \)
\[\Rightarrow y = x \] (**).

There should be points on the curve:
\[(**') \quad x^2 \left( x \right)^2 + x \cdot (x) = 2 \]
\[\Rightarrow x^4 + x^2 = 2 \]

Define \( W = x^2 \geq 0 \) \( \text{A1} \)
\[W^2 + W = 2 = 0 \Rightarrow W = -1 \pm \sqrt{1 + 8} = -1 \pm \frac{3}{2} \]
\[\Rightarrow W = x^2 = 1 \Rightarrow x = \pm 1 \]
\[ f(x) : \quad 1 + 2xy = 0 \]

\[ \Rightarrow 2xy = -1 \]

\[ \Rightarrow y = -\frac{1}{2x} \]

\[ x^2 \left( -\frac{1}{2x} \right)^2 + x \left( -\frac{1}{2x} \right) = 2 \]

\[ \Rightarrow x^2 \frac{1}{4x^2} - \frac{1}{2} = 2 \quad \text{impossible} \]

Only possible \( x \)-coordinates: \( x = \pm \frac{1}{\sqrt{2}} \)

5. Differentiate: 0

\[ y = \sqrt{\tan^{-1}(x)} \]

\[ \frac{dy}{dx} = \frac{1}{2} \cdot \left( \tan^{-1}(x) \right)' \cdot 2 \sqrt{\tan^{-1}(x)} \]

\[ = \frac{1}{2 \sqrt{\tan^{-1}(x)}} \cdot \frac{1}{1 + x^2} \]
\[ y = \sin^{-1}(\frac{x}{2}) \]

\[
\frac{dy}{dx} = \frac{1}{\sqrt{1-(\frac{x}{2})^2}} \\
= \frac{1}{\frac{1}{2} \sqrt{1-(\frac{x}{2})^2}}
\]

6. Simplify \( \tan(\cos^{-1}(x)) \):

Set \( \cos^{-1}(x) = y \), \( \cos y = x = \frac{1}{2} \)

\[ \tan y = \frac{\sqrt{1-x^2}}{x} \]

7. A person was found dead in a park. The police arrived on the scene at 3am and measured the body temperature to be 33°C. One hour later it was 23°C. The outside temperature was constantly 13°C. Normal body temperature is 37°C. Assume the body cools after death according to Newton's Law of Cooling.
(a) How much time before the police or med did the homeless person die?

\[ T(t) = [T(0) - A] e^{ut} + A \]

\[ T(0) = 33^\circ C \quad T(1) = 23^\circ C \quad A = 13^\circ C \]

\[ T(0) - A = 33 - 13 = 20 \]

\[ \Rightarrow T(t) = 20 e^{ut} + 13 \]

To find \( u \):

\[ 23 = T(1) = 20 e^{u} + 13 \]

\[ \Rightarrow 10 = 20 e^{u} \Rightarrow e^{u} = \frac{1}{2} \Rightarrow u = -\ln 2 \]

\[ \Rightarrow T(t) = 20 e^{-\ln 2} t + 13 \]

Now set \( T(t) = 37 \) and solve for \( t \):

\[ 37 = 20 e^{-\ln 2} t + 13 \]

\[ \frac{24}{20} = e^{\ln 2} t \Rightarrow e^{\ln 2} t = \frac{6}{5} \]

\[ \Rightarrow t = \frac{\ln (6/5)}{\ln 2} \]

\[ \text{Note: negative time is not possible.} \]
(b) Did the death occur before or after 2 am?

\[ \text{Need to compare } |t| = \frac{\ln(6/5)}{-\ln 2} = \frac{\ln(6/5)}{\ln 2} \]

\[ \leq 1 \]

\[ \frac{6}{5} = 2^2 \]

\[ \ln \left( \frac{6}{5} \right) < \ln 2 \]

\[ \leq \frac{\ln(6/5)}{\ln 2} < 1 \]

\[ |t| < 1 \text{ and the police came at 3 am} \]

\[ \Rightarrow \text{death occurred after 2 am.} \]

8. A radioactive sample of \(^{3}H\) decays according to

\[ \frac{dq}{dt} = kq \]

(a) Suppose that \(1/3\) of the sample remains after 8 hours. What is the half-life?}

\[ \frac{\text{no.} \text{ general solution} \text{ of the form } q(t) = q(0) e^{-kt}}{-t} - 9 - \]
Call $q_0 = q(0)$; first need to find $U$.

\[ q(8) = \frac{1}{3} q_0 \]

\[ q_0 e^{8U} = \frac{1}{3} q_0 \]

\[ 8U = -\ln 3 \Rightarrow U = -\frac{\ln 3}{8} \]

To find $t_{1/2}$ ($t_{1/2} = q(t_{1/2}) = \frac{1}{2} q_0$)

\[ \frac{1}{2} q_0 = q(t_{1/2}) = q_0 e^{(-\frac{\ln 3}{8}) t_{1/2}} \]

\[ -\ln 3 t_{1/2} = -\ln 2 \]

\[ \Rightarrow t_{1/2} = \frac{8 \ln 2}{\ln 3} \text{ hours} \]

(2) A 50-gm sample is left unattended for three days. How much of it remains?

\[ \ln q_0 = q(0) = 50 \quad U = -\frac{\ln 3}{8} \Rightarrow q(t) = 50 e^{(-\frac{\ln 3}{8}) t} \]

\[ \Rightarrow q(3) = 50 e^{(-\frac{\ln 3}{8}) \cdot 3 \text{ gm}} \]
Two ships are travelling near an island. The first is located 20 km due west of it and is moving due north at 5 km/h. The second is located 15 km due south of it and is moving due south at 7 km/h. How fast is the distance between the ships changing?

To track the ships we need their location. It's helpful to put the island on the origin of a coordinate system. Then the first ship is at \((-20, y_1(t))\)

- The second ship is at \((0, y_2(t))\)

Let \(D(t)\) be the distance between the ships:

\[ D^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 \]

\[ = (20 - 0)^2 + (y_1 - y_2)^2 \]

\[ = 400 + (y_1 - y_2)^2 \]
\[-5 \frac{d \Phi}{d t} = 2 \left( \frac{d \Phi}{dt} \right)^2 + 0 + \frac{d}{dt} \left[ g_{1(t)}^2 - g_{2(t)}^2 \right] \]

\[= 2 (g_{1(t)} - g_{2(t)}) \frac{d}{dt} (g_{1(t)} - g_{2(t)}) \]

\[= 2 (g_{1(t)} - g_{2(t)}) \left( \frac{dg_{1(t)}}{dt} - \frac{dg_{2(t)}}{dt} \right) \]

\[= 2 \left( \frac{dg_{1(t)}}{dt} - \frac{dg_{2(t)}}{dt} \right) \]

At the given time $t_1$ we have $y_1 = 0$, $y_2 = -15$,

\[0 = \sqrt{2^2 + 15^2} = \sqrt{5^2 (4^2 + 3^2)} = 5 \cdot 5 \cdot 25 = 25 \]

\[\frac{dg_{1(t)}}{dt} = 5, \quad \frac{dg_{2(t)}}{dt} = -4 \] (moving south), so

\[\frac{d \Phi}{dt} \bigg|_{t=t_1} = \frac{0 - (-5)}{25} \left( 5 - (-4) \right) = \frac{3 \cdot 12}{25} = \frac{36}{25} \text{ m/s}^2 \]
10. \[ \cos^{-1} \left( \cos \left( \frac{4\pi}{5} \right) \right) = \frac{4\pi}{5} \]

0 \leq \frac{4\pi}{5} \leq \pi

\[ \sin \left( \sin^{-1} (-3) \right) = -3 \] cannot be evaluated

(domain of arcsin is \([-1, 1]\), \(-3 \) is outside the domain)

\[ \tan^{-1} \left( \tan \left( -\frac{3\pi}{8} \right) \right) = -\frac{3\pi}{8} \]

\[ -\frac{\pi}{2} = -4\pi < -\frac{3\pi}{8} < \frac{\pi}{2} \]

\[ \cos^{-1} \left( \cos \left( -\frac{5\pi}{3} \right) \right) \]

\[ \cos(-x) = \cos x \quad \Rightarrow \quad \cos \left( -\frac{5\pi}{3} \right) = \cos \left( \frac{5\pi}{3} \right) \]

Notice \[ \frac{5\pi}{3} > \frac{3\pi}{3} = \pi \] so the answer is not \[ \frac{5\pi}{3} \]

By symmetry,

\[ \cos \left( \frac{5\pi}{3} \right) = \cos \left( \frac{\pi}{3} \right) = \frac{1}{2} \]

\[ \cos^{-1} \left( \cos \left( -\frac{5\pi}{3} \right) \right) = \frac{\pi}{3} \]

\[ -3 \]
(c) \( \sin^{-1} \left( \sin \left( \frac{2\pi}{3} \right) \right) \):

\[ \pi = \frac{3\pi}{3} > \frac{2\pi}{3} > \frac{\pi}{3} \implies \text{the answer is } -\frac{2\pi}{3} \]

Check:

\[ = \sin \left( \frac{\pi}{3} \right) = \sin \left( \frac{\pi}{3} \right) \]

\[ \implies \sin^{-1} \left( \sin \left( \frac{-2\pi}{3} \right) \right) = \sin^{-1} \left( \sin \left( \frac{\pi}{3} \right) \right) = \frac{\pi}{3} \]