Math 100 - Homework Set 3 (Taylor Approximation-Critical Points)
due date: Thursday March the 31st in class only

Basic skills required to work through the homework problems:

- computing and evaluating derivatives;
- determining an upper bound on the absolute value of a function on an interval;
- performing computations using powers, exponentials, trigonometric functions, logarithms;
- computing and evaluating derivatives.

Learning Goals: After completing this problem set, you should be able to:

- find the Taylor polynomial of a specified degree for a function at a given point;
- use Taylor polynomials to approximate a function;
- use the remainder in Taylor’s formula to estimate the error in approximating a function with its Taylor polynomial;
- find the critical points of a function on a given interval;
- find local and global maximum and minimum values of a function;
- use the Mean Value Theorem to show that under certain conditions the derivative of a function takes a specific value at some point within a given interval, and apply the theorem to prove a given mathematical relationship.

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Taylor’s Formula with Remainder
Given the $n$–th degree Taylor polynomial of $f(x)$ about $x = a$, if $|f^{(n+1)}(t)| \leq M$ for all values of $t$ in the interval between $a$ and $x$, then the remainder $R_n(x)$ in Taylor’s formula satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x - a|^{n+1}$$

This means that when you use the Taylor polynomial of degree $n$ to approximate $f(x)$, the error in the approximation can be estimated by finding an upper bound on the $(n + 1)$-th derivative of $f$ that works for all values in the intervals between $x$ and $a$, where $a$ is the point about which you define the Taylor polynomial.

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The Mean Value Theorem Let $f$ be a function that satisfies the following hypotheses:

1. $f$ is continuous on the closed interval $[a, b]$.
2. $f$ is differentiable on the open interval $(a, b)$. 
Then there is a number $c$ in $(a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

1. (Warm-up Problems)
   (a) Let $f(x)$ be a function. What is the equation of the first degree Taylor polynomial of $f(x)$ about $x = a$? How does this compare to the equation of the tangent line to $f(x)$ at $x = a$?
   (b) Suppose you wanted to estimate $\ln 0.5$ using the first degree Taylor polynomial of $f(x) = \ln(x)$ about $x = 1$. Would your estimate be too high or too low? (Hint: sketch a graph containing both $f(x)$ and the first degree Taylor polynomial).
   (c) Find an upper bound on the absolute value of $f(x) = 3/x^4$ on the interval $[-2, -1]$.
   (d) Find an upper bound on the absolute value of $f(x) = \frac{2x-1}{x^2-5}$ on the interval $[5, 10]$.
   (e) Compute the 5th-degree Maclaurin polynomial for $e^x$ and use it to approximate $e$.

2. (a) Using a suitable linear approximation of $f(x) = \sqrt{x}$, explain why the approximation $\sqrt{1.1} \approx 1.05$ is reasonable.
   (b) Using the linear approximation of the function $f(x) = \sqrt{1-x}$ about $x = 0$ (i.e. $a = 0$), approximate the number $\sqrt{1.1}$. Compare your result with the approximation of $\sqrt{1.1}$ given in part (a) and discuss.
   (c) Can you use the linear approximation of $f(x) = \sqrt{1-x}$ about $x = 1$ to approximate $\sqrt{1.1}$? Why?

3. Consider the function $f(x) = e^x \cos(x)$.
   Calculate $T_n(x)$, the $n$-th degree Taylor polynomial to $f(x)$ about $x = 0$, for the cases:
   (i) $n = 0$ (the constant approximation)
   (ii) $n = 1$ (the linear approximation)
   (iii) $n = 2$ (the quadratic approximation)
   (iv) $n = 3$ (the cubic approximation)

4. (a) Find the 3rd-degree Taylor polynomial to $y = (1 + x)^{3/2}$ at $x = 3$.
   (b) Use your result from part a) to estimate $4.1^{3/2}$.
   (c) Give an upper bound on the magnitude of the error of your estimate from b).

5. (a) Approximate $\sqrt{101}$ by using appropriate Taylor polynomials of degree 1 and 2.
(b) Use Taylor’s formula with remainder to estimate the accuracy of the approximations of degrees 1 and 2 of \( f(x) = \sqrt{x} \), for \( 95 \leq x \leq 105 \).

6. Find the critical numbers of \( f(x) = x + \cos(2x) \) on the interval \([0, \pi/2]\).

7. Determine the \( x \)-coordinates of local maxima and minima, if any, of the following function:
   \[
   f(x) = \begin{cases} 
   e^{x-1}, & x < 1; \\
   \frac{x^2 + 3}{2(x + 1)}, & x \geq 1.
   \end{cases}
   \]

8. Use the Intermediate Value Theorem and the Mean Value Theorem to show that the equation \( 7x - 1 - \sin x = 0 \) has exactly one real root.

9. Suppose that a function \( f \) is twice differentiable on \([0, 4]\) and that \( f(1) = f(2) = 0 \) and \( f(3) = 1 \). Show that:

   (i) \( f'(a) = 1/2 \) for some point \( a \) in \((0, 4)\).

   (ii) \( f''(b) > 1/2 \) for some point \( b \) in \((0, 4)\).

10. Find the absolute maximum and absolute minimum values of the function
    \[
    f(x) = (1 - x) \cdot \sqrt{x + 3/2}
    \]
    on the interval \([-1, 1]\).

11. (For bonus points) Suppose \( f(x) \) is a function that is differentiable for all \( x \). Let \( g(x) \) be the new function defined by \( g(x) = f(x) + f(1 - x^2) \). Prove that \( g'(c) = 0 \) for some positive real number \( c \) with \( 0 < c < 1 \).

12. (For bonus points) Use the Mean Value Theorem to show that \( \sin x < x \) for all \( x > 0 \).