Let's demonstrate the use of the I.V.T. with a particular example:

**Ex 1** (a) Show that \( f(x) = 2x^3 - 5x + 1 \) has a root in \( 0 \leq x \leq 1 \) (i.e., the equation \( 2x^3 - 5x + 1 = 0 \) has a solution)

**Sln:** We are looking for a \( c \) in \([0,1]\) s.t. \( f(c) = 0 \)

Hence, I want to use I.V.T. with \( L = 0 \).

It suffices to show I have a configuration like one of these:

\[
\text{[Diagram: Points labeled 0, L, and another point with indication of positive and negative values of } f(\text{point})]}
\]

(i.e., there is a point on this curve that corresponds to a positive value of \( f \) and another that corresponds to a negative one.)

By continuity:

\[
\text{[Diagram: Curve with zero crossing at 0, L = 0]}\]

I have to hit zero
In our case observe:
\[ f(0) = 1, \quad f(1) = 2 - 5 + 1 = -2 < 0 \]
and \( f \in C[0,1] \) being a polynomial.

→ By the I.V.T. (1st configuration) we affirm the existence of a \( c : 0 < c < 1 \) such that:
\[ f(c) = 0 \]

**Rule 1:** a and b are not necessarily the endpoints we can try them but if we get, say, + or - we need to try again with different point(s).

**Rule 2:** I.V.T. can be also applied to showing there is a solution to an equation of the form \( f(x) = g(x) \). (E.g. continuous)

How? Set \( h(x) = f(x) - g(x) \) (continuous by \( \text{Theorem} \))

\[ f(x) = g(x) \implies h(x) = 0 \]
(b) Show \( x = \cos x \) has a solution.

**Proof:** I suspect I need to use the I. V. T. but they don't give me \([a, b]\). :-(

**Idea 1:** Plugging in 0, 1, -1 (or multiples of \( \pi \approx 3.14 \) for try functions) is usually a good idea.

**Idea 2:** Sketch? Remember \(-1 \leq \cos x \leq 1\) so for \( f(x) = x \) and \( g(x) = \cos x \) to intersect, \( x \) has to be in \([-1, 1]\).

[Sketch of graphs]

Define \( h(x) = x - \cos x \) continuous, \( h(0) = 0 - \cos 0 = -1 < 0 \)

**I. V. T.** There is a \( c \in [0, \pi/3] \) s.t. \( h(c) = 0 \)

\[
\begin{align*}
\frac{\pi}{3} & \approx 1.05 & \frac{\pi}{2} & \approx 1.6 > 1 \\
\frac{\pi}{2} & = 3.14 & & > 1 \\
\end{align*}
\]
Limits at Infinity (§15.5 on the Notes)

So far we have only worked with limits as \( x \) approaches some finite value \( a \). For a lot of applications (sketching, analysis of a population model for long times), we need to understand also how it helps to measure.

\[
\lim_{x \to \infty} [\ldots]\]

\[
\lim_{x \to -\infty}
\]

become very large and positive
very large and negative

Definition: \( \lim_{x \to \infty} f(x) = L \) when the value of \( f(x) \) gets closer and closer to \( L \) as we make \( x \) larger and larger and positive

\( \lim_{x \to -\infty} f(x) = L \) \( L \) negative

The numbers \( L \) are horizontal asymptotes for graph of \( f(x) \).
Theorem (Basic Limits) // (Standard arithmetic)

\[ \lim_{x \to \pm \infty} c = c \]
\[ \lim_{x \to +\infty} \frac{1}{x} = 0, \quad \lim_{x \to -\infty} \frac{1}{x} = 0 \]

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Let \( f, g \) be two functions and \( F, G \) sit,
\[ \lim_{x \to \infty} f(x) = F \text{ and } \lim_{x \to \infty} g(x) = G \text{ exist} \]

Then:
1. \[ \lim_{x \to \infty} f(x) \pm g(x) = F \pm G \]
2. \[ \lim_{x \to \infty} f(x) \cdot g(x) = FG \]
3. \[ \lim_{x \to \infty} \frac{f(x)}{g(x)} = \frac{F}{G} \text{ (provided } G \neq 0) \]

and for rational numbers \( r \):
\[ \lim_{x \to \infty} (f(x))^r = F^r \text{ (provided } f(x) \text{ is defined for all } x) \]

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Hints: 1. Analogous results for the limits to \(-\infty\)
2. Typo in the "book" if there should be no "\( c \)" or "neighbourhood of \( a \)."
e.g. \( \lim_{x \to +\infty} \frac{1}{x^r} = 0 \)

but when \( \lim_{x \to a} \frac{1}{x^r} \) we can only say this is 0 if \( r (= \frac{a}{q}) \) has a denominator that is not even

\[ \lim_{x \to 0} \frac{1}{x^{1/2}} = 0 \]

\[ \lim_{x \to -\infty} \frac{1}{x^{4/3}} = 0 \]

Examples (Evaluate the following limits)

(a) \( \lim_{x \to -\infty} \frac{x^2 + 1}{x - 3} = 1 + 0 \) 
\[ \lim_{x \to -\infty} \frac{x^2 + 1}{x - 3} = \lim_{x \to -\infty} \frac{x^2 (1 + \frac{1}{x^2})}{x - 3} = \lim_{x \to -\infty} \frac{x^2 (1 - 3/x)}{x - 3} = 1 - 3/x \to 1 - 0 \]
\[ \lim_{x \to -\infty} \frac{x^2 + 1}{x - 3} = +\infty \]

(b) \( \lim_{x \to +\infty} \frac{x^2 + 8}{2x^3 - 1} = 0 \)
\[ \lim_{x \to +\infty} \frac{x^2 + 8}{2x^3 - 1} = \lim_{x \to +\infty} \frac{x^2 (1 + \frac{8}{x^2})}{x^3 (2 - \frac{1}{x^3})} = \lim_{x \to +\infty} \frac{1}{x^3 (2 - \frac{1}{x^3})} = 0 \]
(c) \lim_{x \to 2^+} \frac{\sqrt{x^4 + \sin x}}{x^2 - \cos x} = \lim_{x \to 2^+} \frac{x^2 \sqrt{1 + \frac{\sin x}{x^4}}}{x^2 \cdot (1 - \cos x) / x^2}

\theta \leq \frac{\sin x}{x^4} \leq \frac{1}{x^4}

\text{So, } \lim_{x \to 2^+} \frac{\sin x}{x^4} = 0 \quad \text{(Squeeze Theorem)}

\theta \leq \frac{\cos x}{x^2} \leq \frac{1}{x^2}

\text{So, } \lim_{x \to 2^+} \frac{\cos x}{x^2} = 0

Combining all the above we see that the limit is 1.

(d) \lim_{x \to \infty} \left( \sqrt{x^2 + 2x} - \sqrt{x^2 - 1} \right)

\text{Problematic...}

Some trick at last time: multiply & divide by conjugate expression:

\lim_{x \to \infty} \left( \frac{\sqrt{x^2 + 2x} - \sqrt{x^2 - 1}}{1 - \sqrt{x^2} + 1} \right) = \lim_{x \to \infty} \frac{2x + 1}{x^2 \cdot (1 + \frac{2}{x^2}) + \sqrt{x^2(1 - \frac{1}{x^2})}} = -47