Optimization (§ 7.5, second help)

We will start with a very artificial example.

Ex 1] Suppose a company has calculated that the cost for the production of their product (per month) is modelled by the equation \( C(X) = X^2 - 6X + 10 \) (in thousand dollars) where \( X \) is the # units produced without taking any other factors into consideration.

What is the optimal # unit in the sense that it minimizes the production cost?

Sln] The question is asking for the \( X \) that minimizes the function \( C(X) \), where \( C(X) \) attains its minimum. Check \( C'(X) = 2X - 6 = 2(X-3) \) \( \Rightarrow X = 3 \) (\( f' \), no sp)

The monotonically analysis suggests that \( X = 3 \) is a global min.
Ex 2: Find the rectangle that maximizes the area among all rectangles of perimeter 10.

Want to maximize area. First, labelling:

| generic rectangle | A = x \cdot y | (A depends on two variables) |

Perimeter constraint: 2x + 2y = 10 \Rightarrow x + y = 5
Want to maximize $A(x, y) = x \cdot y$ but know how to find max/min if I have a function of a single variable. => replace one of them using the perimeter constraint: $y = 5 - x$.

$A(x) = x(5-x) = 5x - x^2$

Want to maximize $A(x)$, we know:

**Theorem (Extreme Value Theorem):** A function $f(x)$ continuous on $[a, b]$.

- Attains its max/min on $[a, b]$.
- The max/min is among $\{ f(a), f(b), \text{the CPs} \}$.

1. Take derivative: $A'(x) = 5 - 2x \Rightarrow x = \frac{5}{2} = CP$, no SP
2. $0 \leq x \leq \ell = \text{length}$
\[ y \geq 0 \text{ but } y = 5 - x \Rightarrow 5 - x \geq 0 \]
\[ \Rightarrow x \leq 5 \]
\[ \Rightarrow 0 \leq x \leq 5 \]

\[ A \left( \frac{5}{2} \right) = \frac{25}{4} \]
\[ A(0) = 0 \]
\[ A(5) = 0 \]

but before we say \( x = \frac{5}{2} \) are the side lengths
\[ y = \frac{5}{2} \]

give the

\[ \text{that maximal area we need to check } x = \frac{5}{2} \]

\[ \text{is a maximum.} \]

\[ A'(x) = 5 - 2x \]
\[ A(x) \]

and by the comparison above \( \rightarrow \) the square with

\[ \text{side } x = \frac{5}{2} \text{ maximize the one among rectangles of fixed perimeter} \]
As with related rates problem, the hard part is to translate the problem from words to math, i.e., find the function that describes the quantity we want to optimize.

Once this is done, we follow the standard procedure to deduce where the max/min are. (1) take derivative \( f'(x) \) and solve \( f'(x) = 0 \) or \( f'(x) \) is undefined.
(2) check endpoints (if on a closed interval).
General procedure

1. Read the question carefully! Re-read the information (maybe use a data table).

2. Draw a diagram if applicable.

3. Label diagram and assign variables. Don't be afraid of unknowns.

4. Find relations between the variables (in the previous example we needed to eliminate one of them in favour of the other) → 5 Reduce to a function of a single variable.

6. Maximize/Minimize or use known...
EX 3

A closed rectangular container with a square base is to be made from two different materials. Material for base \( \rightarrow \) cost $5/m^2

other sides \( \rightarrow \) cost $2/m^2.

Find the dimensions of the container which have the largest possible volume if the total cost of materials is $72.

height: \( h \), base: \( b \)

\( A_b \): area of base, \( A_s \): area of \( 5 \) of the other \( 5 \) sides.

\( C: \) Cost (in dollars); \( V: \) Volume (m³)
\[ A_b = b^2 \]
\[ A_s = 4b^2 + b^2 \]

\[ V = b^2h \]

\[ C = 5 \cdot A_b + 4 \cdot A_s = 5b^2 + 4bh + b^2 \]
\[ = 6b^2 + 4bh \]

(C depends on both \( b \) and \( h \); want one)

\[ 72 = 6b^2 + 4bh \implies h = \frac{72 - 6b^2}{4b} \]

\[ V = b^2h = b^2 \left( \frac{72 - 6b^2}{4b} \right) = \frac{3b}{2} (12 - b^2) \]
\[ = 18b - \frac{3}{2}b^3 \]

Since \( V > 0 \implies 12 - b^2 > 0 \implies b \leq \sqrt{12} \)
\[
V'(b) = 18 - 9b^2 \quad \Rightarrow \quad b = \pm 2
\]

Check:

<table>
<thead>
<tr>
<th></th>
<th>V'</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>\sqrt{12}</td>
<td>-</td>
<td></td>
</tr>
</tbody>
</table>

Max

Compute

\[V(2) = 24\]
\[V(0) = 0 = V(\sqrt{12})\]

\[\Rightarrow \text{Maximum value when } b = 2 \text{ m} \quad (V(2) = 24)\]

My answer: \[2 \times 2 \times 6 \text{ is the largest possible volume or area.}\]
**Ex 4** Find the point on the line $y = 6 - 3x$ that is closest to the point $(7, 5)$.

A point on the line has coordinates $(x, y) = (x, 6 - 3x)$.

Let $l(x)$ be the distance from $(7, 5)$:

$$l(x) = \sqrt{(x-7)^2 + (y-5)^2}$$
We want to minimize $l(x)$. Notice that it suffice (think about it) to minimize

$$l^2(x) = (x - 1)^2 + (y - 5)^2$$

$$y = (x - 1)x$$

$$l(x) = 10(x^2 - 2x + 5)$$

$$= 10\sqrt{x^2 - 2x + 5}$$

$$\frac{dl}{dx} = \frac{2x - 2}{2\sqrt{x^2 - 2x + 5}}$$

$$CP: x = 1$$

$$SP:$$

$$x^2 - 2x + 5 = x^2 - 2x + 4 + 1$$

$$= (x - 2)^2 + 1 > 0$$

$$\Rightarrow$$ no SPs

No endpoints ... so what do I do now?
Monotonicity:

\[ l(x) = \sqrt{10 \sqrt{x^2 - 2x + 4}} \]

Notice \( \lim_{x \to \infty} l(x) = \infty \)

\[
\begin{array}{c|c|c}
 l'(x) & 0 & + \\
 l(x) & \downarrow & \uparrow \\
\end{array}
\]

\( x = 1 \) local minimum

global minimum

Thus:

Let \( f/\phi \) on \( \mathbb{R} \)

1. if \( \lim_{x \to \infty} f(x) = \infty \) and \( f \) has a global minimum \( c \)
   - \( c = C_F \) or \( S_F \)

2. if \( \lim_{x \to \infty} f(x) = -\infty \) and \( f \) is max
   - \( c = C_F \) or \( S_F \)