Example (Curve Sketching): Sketch the graph of

\[ f(x) = \frac{x^2 - 9}{x^2 + 3} \]

(a) Domain: \( \mathbb{R} \) and continuous everywhere (since rational with “good” denominators)

\[ f(x) = \frac{(x-3)(x+3)}{x^2+3} \]

- **x-intercept**: \( f(x) = 0 \Rightarrow x = \pm 3 \)

- **y-intercept**: \( f(0) = \frac{-9}{3} = -3 \)

- Positive on \( (-\infty, -3) \cup (3, \infty) \)

- Negative on \( (-3, 3) \)

- No vertical asymptotes (candidates are points where continuity breaks down because of infinite jumps)

- Horizontal asymptotes:

\[ \lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{x^2 (1 - \frac{9}{x^2})}{x^2 (1 + \frac{3}{x^2})} = 1 \]

Similarly \( \lim_{x \to -\infty} f(x) = 1 \) (so \( y = 1 \) is a H.A., in with direction, \( -11 \))
(b) \( f'(x) = \frac{2x \cdot (x^2 + 3) - 2x \cdot (x^2 - 9)}{(x^2 + 3)^2} \)

\[ = 2x \left[ \frac{x^2 + 3 - x^2 + 9}{(x^2 + 3)^2} \right] = \frac{24x}{(x^2 + 3)^2} \]

- Defined everywhere \( \Rightarrow \) No 5.8.

- \( f'(x) = 0 \iff x = 0 \) \( \Rightarrow \) C.P.

\( f''(x) > 0 \iff x > 0 \) \( \Rightarrow \) f is concave up on \((0, \infty)\)

\( f''(x) < 0 \iff x < 0 \) \( \Rightarrow \) f is concave down on \((-\infty, 0)\)

(c) \( f''(x) = \frac{24 \left( x^2 + 3 \right)^2 - 24x \cdot 2 \left( x^2 + 3 \right) \cdot 2x}{(x^2 + 3)^4} \)

\[ = \frac{72}{(x^2 + 3)^3}(1 - x^2) \]

\( (x^2 + 3)^3 \to > 0 \) for all x.

\( f''(x) = 0 \iff x = \pm 1 \),

\( f''(x) > 0 \) on \((-1, 1)\)

\( f''(x) < 0 \) on \((-\infty, -1) U (1, \infty)\)

\( \Rightarrow \) \( x = \pm 1 \) are inflection points.
Interesting Points: -3, -1, 0, 1, 3

<table>
<thead>
<tr>
<th>x</th>
<th>(-3, -3)</th>
<th>-3</th>
<th>(-1, -1)</th>
<th>-1</th>
<th>(0, 0)</th>
<th>0</th>
<th>(1, 1)</th>
<th>1</th>
<th>(3, 5)</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>f''</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0</td>
<td>+</td>
<td>0</td>
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<tr>
<td>f'</td>
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</table>

Also notice:
\[
f(-x) = \frac{(-x)^2 - 9}{(-x)^2 + 3} = \frac{x^2 - 9}{x^2 + 3} = f(x)
\]

f is even (symmetric around the y = ax1s)
Look at the table and do again

or notice

\[ f(-x) = f(x) \Rightarrow \text{graph} \]

symmetric w.r.t. the y-axes
\[ f(x) = x^{2/3} (x - 1) \]

- Domain: \( \mathbb{R} \) (since \( x^{2/3} \) and \( x^{1/3} \) are defined everywhere)

- \( f' = \frac{5}{3} x^{2/3} (x - 1) + x^{2/3} = \frac{2(x - 1) + 3x}{3 x^{1/3}} = \frac{5x - 2}{3 x^{1/3}} \)

- \( f(0) = 0 \) (y-intercept)
- \( f'(x) = \frac{5}{3} x^{2/3} (x - 1) + x^{2/3} \)

- Continues everywhere and no V, A.

- \( \lim_{x \to -\infty} \left( x^{2/3} (x - 1) \right) = -\infty \)

- \( \lim_{x \to 0} \left( x^{2/3} (x - 1) \right) = 0 \)

- \( \lim_{x \to +\infty} \left( x^{2/3} (x - 1) \right) = +\infty \)

- No horizontal asymptotes.
\[ x = 0 \quad \text{is a local maximum} \]
\[ x = \frac{2}{3} \quad \text{is a local minimum} \]

\[ f''(x) = \frac{1}{3} \left( \frac{5x-2}{x^{1/3}} \right)' = \frac{5}{3x^{1/3}} - \frac{5x-2}{9x^{4/3}} = \frac{15x - 5x + 2}{9x^{4/3}} \]
\[ = \frac{10x + 2}{9x^{4/3}} \quad x^{4/3} = (\frac{1}{3})^4 > 0, \quad x \neq 0 \]

Notice: \[ f''(x) \quad \text{is undefined at} \quad x = 0 \] (expected: \( f'(0) \) undefined)

\[ f'''(x) = 0 \iff x = -\frac{1}{5} \quad \text{not an inflection point} \]

\[ f''(-\frac{1}{5}) > 0 \quad \text{and} \quad f''(\infty) < 0 \quad \text{too} \]
positive.
increasing (with no bound)
concave up

local max

domain

local min

-1

\( f(\frac{2}{5}) \)

Concave down

Increasing

Concave up

Concave down
Hospital's Rule

In general, we want to calculate some limit but it is of the form \[ \frac{0}{0} \] or \[ \frac{\infty}{\infty} \]

\[
\lim_{{x \to a}} f(x) = 0 = \lim_{{x \to a}} g(x) \quad \text{OK}
\]

\[
\lim_{{x \to a \pm \infty}} f(x) = \pm \infty = \lim_{{x \to a \pm \infty}} g(x)
\]

\[
\text{if RH-S. exists}
\]

Hospital's Rule: \[
\lim_{{x \to a}} \frac{f(x)}{g(x)} = \lim_{{x \to a}} \frac{f'(x)}{g'(x)}
\]

Also works for \[
\lim_{{x \to a^+}} \quad \text{not quotient rule...}
\]

Summary: For limits: 0 = 0, \( \frac{\text{non-zero}}{0} \) = \( \infty \), \( \frac{\text{non-zero}}{\infty} \) = 0, 0 - ONE
Examples (look at the files on the website for many more)

\[ \lim_{x \to 1} \frac{\ln x}{x-1} = \lim_{x \to 1} \frac{1}{x} = \frac{1}{1} = 1 \]

Ex 2: \[ \lim_{x \to 0} \frac{\cos x - 1}{x^2} = \lim_{x \to 0} \frac{-\sin x}{2x} = \frac{0}{2} \cdot \frac{0}{0} = \frac{-1}{2} \]

Ex 3: \[ \lim_{x \to 0^+} \frac{e^x}{x} \text{ is } \frac{0}{0} \text{ at } x = 0 \]

Ex 4: \[ \lim_{x \to 0} x^2 e^{-x} = \lim_{x \to 0} \frac{x^2}{1/e^{1/x}} = \lim_{x \to 0} \frac{2x}{e^{1/x}} = \frac{0}{1} = 0 \]
\[ \lim_{x \to 0^+} \frac{\ln x}{x} = \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x^2}} = \lim_{x \to 0^+} x^2 \ln x \]

\[ \lim_{x \to 0^+} x^2 \ln x = 0 \]

\[ \lim_{x \to 0^+} (2x+1)^{4/\sin x} \]

\text{Indeterminate form } 1^\infty:

\text{We'll calculate an easier limit first:}

\[ \lim_{x \to 0} \ln \left( \frac{(2x+1)^{4/\sin x}}{x} \right) = \lim_{x \to 0} \frac{\ln(2x+1)}{\sin x} \]

\[ = \lim_{x \to 0} \frac{2}{2x+1} \cdot \cos x = 2 \]
To go back:
\[ \frac{1}{x} \times \frac{\ln(x) - \ln(x)}{x - 0} = \lim_{x \to 0} \frac{\ln(2x+1)}{\sin(x)} \]

Notice
\[ \lim_{x \to 0} \frac{1}{\sin(x)} = \lim_{x \to 0} \frac{\ln(2x+1)}{\sin(x)} \]

\[ e^x \text{ (constant)} = e^{x \lim_{x \to 0} \frac{\ln(2x+1)}{\sin(x)}} = e^{\frac{1}{2}} \]

\[ \lim_{x \to 0} \frac{x^2 + 1}{x^2 + x} = \lim_{x \to 0} \frac{1}{1 + \frac{1}{x}} = 1 \]

\[ \lim_{x \to 0} \frac{1}{x^2 + x} = \lim_{x \to 0} \frac{1}{x^2 + x} = 0 \]

\[ \frac{2}{1 + x} = \lim_{x \to 0} \frac{1}{1 + x} = 1 \]