Last time

Consequence of the M.V.T. (assuming \( f' \) is continuous on some \([A,B]\) and \( f' \) is differentiable on \((A,B)\))

- If \( f'(x) > 0 \) for all \( A < x < B \) \( \Rightarrow \) \( f \) is increasing on \((A,B)\)
- If \( f'(x) < 0 \) for all \( A < x < B \) \( \Rightarrow \) \( f \) is decreasing on \((A,B)\)

**Rule:** if the inequality is replaced by \( \geq 0 \) or \( \leq 0 \)

the function is monotonous (increasing or decreasing)

but not strictly, e.g., \( f \) is increasing but not strictly increasing

Notice that in order for the derivative to change sign, it must • either pass through zero \( f'(a) = 0 \) or (jump) • have a singular point.

We indicate local maxima and minima.
(Quick example): Determine the monotonicity of \( f(x) = \frac{1}{3} x^3 - 4x \)

Since \( f \) is differentiable everywhere (being a polynomial),

\[
f'(x) = x^2 - 4 \quad f'(x) = 0 \iff x = \pm \sqrt{4} = \pm 2
\]

Make a table:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( -\infty )</th>
<th>(-2)</th>
<th>( 2 )</th>
<th>( +\infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f'(x) )</td>
<td>+</td>
<td>0</td>
<td>-</td>
<td>0</td>
</tr>
<tr>
<td>( f(x) )</td>
<td>↑</td>
<td>↘</td>
<td>↓</td>
<td>↑</td>
</tr>
</tbody>
</table>

For \( x \in (-\infty, -2) \), to determine the sign of \( f'(x) \),

it suffices to plug in any number. Notice

\[
f'(x) = (x-2)(x+2)
\]

so any number smaller than \(-2\) makes both factors negative \( \implies f'(x) > 0 \).

\( x \in (2, +\infty) \): \( \ldots \implies f'(x) > 0 \)

\( x \in (-2, 2) \): one is positive, the other negative

\( \implies f'(x) \leq 0 \)

\(-2\)
"any": we know the derivative only become zero at 
-2. If we want to find its sign on, say (-\infty,-2): 
suffice to plug in any point, check its sign and 
we know it would be the same sign on the whole 
interval. Because IF NOT, there would be points 

\[ x_1, x_2 \in (-\infty, -2) \] 

with \[ f'(x_1) > 0 \text{ and } f'(x_2) < 0 \] 
\( \text{if it were to change sign} \) 

But \( f'(x) = \text{continuous} \) \( \Rightarrow \exists x_0 \in (x_1, x_2) \subset (-\infty, -2) \text{ or } (x_2, x_1) \) 

s.t. \( f'(x_0) = 0 \) which is impossible since the 
only roots of \( f'(x) = 0 \) are \( \pm 2 \) (and \( x_0 < -2 \))

we can plug in any point knowing that 
the derivative won't change its sign before the 
next critical or singular point.
Putting a +1 for \( f'(x) \) when \( f''(x) > 0 \) and a -1 for \( f'(x) \) when \( f''(x) < 0 \). We complete the table to determine:

\[
f(x) > 0 \text{ on } (-\infty, -2) \cup (2, \infty) \\
\]

and \( f(x) < 0 \text{ on } (-2, 2) \).

For more information about the shape of \( f(x) \), we can consider the second derivative, \( f''(x) \).

In particular, what the sign of \( f''(x) \) says:

1. \( f(x) = x^2 + 1 \) \( \rightarrow \) \( f'(x) = 2x \) \( \rightarrow \) \( f''(x) = 2 > 0 \)

Consider:

\[
0 
\]

2. \( f(x) = -x^2 - 1 \) \( \rightarrow \) \( f'(x) = -2x \) \( \rightarrow \) \( f''(x) = -2 < 0 \)

\[
-4
\]
In the first case: $f''(x) > 0 \implies f'$ increasing

slopes of the tangent lines increase.

\[ \text{slope is positive} \]
\[ \text{slope on zero} \]
\[ \text{slope is negative} \]

And similarly $f''(x) < 0 \implies$ slopes decrease.

Looking at the figure, we see that when the case

$f''(x) > 0$ : the graph always lies above the tangent lines; in case like this we call $f(x)$ concave up (or convex).

$f''(x) < 0$ : the graph lies below the tangent lines; we call it concave (or concave down).
If \( f''(c) = 0 \) for some \( a < c < b \) and the concavity changes across \( x = c \), then we call \( x = c \) an inflection point.

Ex: Determine the concavity of \( f(x) = x^3 + 3x \).

\[
f'(x) = 3x^2 + 3, \quad f''(x) = 6x
\]

For \( x \in (-\infty, 0) \): \( f''(x) < 0 \) → concave down

For \( x \in (0, \infty) \): \( f''(x) > 0 \) → concave up

\( x = 0 \): \( f''(0) = 0 \) and the concavity changes across

\( \text{Inflection Point} \)

\(-6-\)
Dealing with common misconceptions

1. It is possible for a concave down function to be positive.

\[ a \quad b \]

concave up \( \rightarrow \) negative

2. The function may still be monotone of the same type after a change in concavity.

But function always increasing!
3. \( f''(x_0) = 0 \neq \quad x_0 = \text{inflection pt} \)

Eng: \( f''(x) = x^4 \rightarrow f'(x) = 3x^3 \rightarrow f''(x) = 9x^2 \)

\[ x = 0 \rightarrow f''(0) = 0 \]

but \( f''(x) > 0 \) for \( x \in (-\infty, 0) \cup (0, +\infty) \)

concavity does not change at \( x = 0 \)

We are now in a position to start talking about how to sketch a graph of a function.

Let's summarize some tools and give an outline of the approach before we apply what we know.
Tools

1st derivative test: sign of $f'(x) \rightarrow$ monotonically

2nd: sign of $f''(x) \rightarrow$ Concavity

Useful Theorem: (Tests for minima and maxima)

Let $x_0 \in (a,b)$ be \begin{itemize}
  \item[a] a critical point of $f$
  \item[b] a singular number
\end{itemize}

and suppose $f$ is \begin{itemize}
  \item[a] continuous at $x_0$
  \item[b] differentiable near it
\end{itemize}

1. Either of the following is sufficient to show that $f$ has a local minimum at $x_0$

   \begin{itemize}
     \item[(a)] $f''(x_0) > 0$
     \item[(b)] $f'(x)$ negative to the left, positive to the right
   \end{itemize}

2. Either of the following is sufficient to show that $f$ has a local maximum at $x_0$

   \begin{itemize}
     \item[(a)] $f''(x_0) < 0$
     \item[(b)] $f'(x)$ positive to the left, negative to its right
   \end{itemize}
Curve sketching protocol:

0th derivative stuff:
- domain and range
- domain of continuity
- x and y intercepts
- horizontal: \( \lim_{x \to \pm \infty} f(x) = L \) asymptote
  \[ y = L \]
  \( \uparrow \) check both!
- vertical: \( \lim_{x \to a^{\pm}} f(x) = \pm \infty \) asymptote
  \[ x = a \]
  \( \uparrow \) check both!

1st derivative stuff:
- using \( f'(x) \) determine
  - intervals where \( f'(x) > 0, f'(x) < 0 \)
  - critical and singular number

2nd derivative stuff:
- using \( f''(x) \) determine
  - intervals where \( f''(x) > 0, f''(x) < 0 \)
    (and concavity)
  - point where \( f''(x) = 0 \) and inflection points