MacLaurin for \( \sin x \): 

We could do the same: \( u \to f(u)(x) \to f(u)(a) \to cu \)

but it's much faster to notice

\[
\sin x = \frac{1}{\cos x} \quad (\cos x)' = -\sin x = -\sin x \quad \sin x = -\cos x
\]

So if \( g(x) = \sin x, f(x) = \cos x \)

\[
g(0) = -f'(0) = 0 \\
g''(0) = -f'''(0) = 0 \\
g''(0) = -f''(0) = 1 \\
g'''(0) = -f''(0) = -1 \\
g''(0) = f(4)(0) = -1 \\
g''(0) = -f(6)(0) = 1
\]

\( \Rightarrow \) only odd terms survive, signs alternate

\[
\sin x : T_n(x) = \sum_{u=0}^{n} \frac{(-1)^u}{(2u+1)!} x^{2u+1}
\]

\[
= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots
\]

memorize!
Also useful (try yourselves!):

\[
\log(1-x) = -\sum_{k=1}^{n} \frac{x^k}{k} \quad \text{as } x \to (-1)
\]

\[
\log(1+x) = \sum_{u=1}^{\infty} \frac{(-1)^u x^u}{u}
\]

\[
\frac{1}{1-x} = \sum_{u=0}^{\infty} x^u
\]

So far; if they gave us \( N \), we find \( T_n(x) \) (as before \( n \to f^{(n)}(x) \to f^{(n)}(a) \to C_n \to T_n(x) \))

and to estimate some \( M = f(x_1) \):

\[
f(x_1) \approx T_n(x_1) \equiv \text{know how to calculate}
\]

In general

Of course \( T_n(x) \neq f(x) \) so there is some error.

We said: \( f(x) = T_n(x) + \text{error} \)

...call it \( R(x) \)
We constructed the Taylor polynomials hoping to approximate functions \( f \) using information about the given function \( f \) at exactly one point \( x = a \). How well does the Taylor polynomial of degree \( n \) approximate the function \( f \)?

One way of looking at this question is to ask for each value \( x \), what is the difference between \( f(x) \) and \( T_n(x) \)?

If we call this difference the remainder \( R_n(x) \), we can write

\[
f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)
\]

approximation of order \( n \) + correction

Can find an expression for this remainder:

**Lagrange Remainder Formula:**

\[
R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}
\]

for some \( c \) between \( a \) and \( x \).
Remarks:  

1. \( \lim_{x \to a} = |x - a| \to 0 \)  
   \( \Rightarrow \) \text{error becomes smaller}

2. \( \lim_{n \to \infty} \) \text{error becomes smaller}

3. \( n \)-th approximation has error involving \( \text{nth derivative} \)

4. \( C \) depends on both \( x \) and \( a \)

If we could actually find this number \( C \), we could
know the remainder exactly for any given value of \( x \).

However, if you look at the proof of this formula, you
would see that this number \( C \) comes from the
Mean Value Theorem (we will cover soon).

\( C \) tells you such a \( C \) exists but not \( \text{where} \)
its exact value is.

We can ask: \text{What is the worst error we}
could make in approximating \( f(x) \) using a Taylor
polynomial of degree \( n \) about \( x = a \)?
We focus our attention on \( |\ln (x)| \) (why are we about the magnitude of \( \ln \) ?)

Notice we know everything except \( f^{(n+1)}(c) \), so our goal is to find a bound on \( |f^{(n+1)}(t)| \) that works for all values of \( t \) in the interval containing \( x \) and \( a \).

I.e., we are looking for a positive number \( M \) such that

\[
|f^{(n+1)}(t)| \leq M \quad \text{for all } t \in (x, a) \quad \text{(or } (a, x) \text{)}
\]

\[
\Rightarrow |f^{(n+1)}(c)| \leq M \quad \text{(since } c \text{ is such a number)}
\]

\[
\Rightarrow \ln (x) \leq \frac{M}{|x-a|^{n+1}}
\]

Ex] Suppose we wish to estimate \( \ln x \) using a Taylor approximation of order 1 (L.A.), using \( a = 9 \)
and give an estimate on the size of the error $|r_1(10)|$.

First, we note that $f(x) = f(9) + f'(9)(x-9) + r_1(x)$

$$\Rightarrow \sqrt{x} = 3 + \frac{1}{6} (x-9) + r_1(x)$$

$$\Rightarrow \sqrt{10} \approx 3 \frac{1}{6}$$

Estimate $|r_1(10)|$: we first need to find $M$ s.t.

$$|f''(t)| \leq M \text{ for all } t \text{ in } [9,10]$$

Now, $f''(t) = \frac{-5}{4t^{3/2}} \Rightarrow |f''(t)| = \frac{5}{4t^{3/2}}$

and want to make this as big as possible on $[9,10]$

As $t \rightarrow \frac{1}{\sqrt{5}}$ ("fizz principle") so it's largest

of the left-hand endpoint, $t = 9$

$$\Rightarrow M = \frac{5}{4 \cdot 9^{3/2}} = \frac{5}{108}$$

(my error $M$ works, think about it)
x=10
\[ |h_x(10)| \leq \frac{1/108}{9!} \]
\[ |l_0 - 9\|^2 = \frac{1}{9!} 216 \]

We will discuss this later but noticing that...

on \([9,10]\) \(f''(t)\) is always negative \(\Rightarrow\) concave (down)

\(\Rightarrow\) tangent line always lies above the curve

\[ y = \sqrt{x} \Rightarrow \text{we are overestimating } \sqrt{10} \text{ by LA} \]

Ex. Approximate \(\sin(0.5)\) using a Maclaurin polynomial of degree 3.

\[ \text{Notice that instead of estimating } M_0 = \sin(-0.5) \]

since \(\sin(-0.5) = -\sin(0.5)\) we can estimate

\[ M_0 = \sin(0.5) \quad (\text{and then use } M_0 = -M_0) \]

\[ \sin x = x - \frac{x^3}{3!} + R_3(x) \]

\[ \sin(0.5) \approx 0.5 - \frac{0.5^3}{3!} = \]

\[ \frac{1}{2} - \frac{1}{24} = \frac{23}{48} \Rightarrow \sin(-0.5) \approx \frac{\pi}{2} \]

\[ -\frac{23}{48} \]
To estimate the error:
\[ \frac{d^4 \sin t}{dt^4} = \frac{\text{error}}{\sin t + 1} \]

mean 4th derivative of \( \sin t \)

An easy choice (one such bound) would be \( M = 1 \)

since \( |\sin x| \leq 1 \) always.

We can do a bit better here:

Notice that the tangent line of \( \sin x \) at \( x = 0 \)

is \( \ell_t(x) = f'(0)(x-0) + f(0) \)

\[ \ell_t(x) = t \]

\( q(x) : \text{tangent at 0} \)

and we can easily check (do it!) that it lies above the graph of \( f(x) \) for \( x \in [0, 0.5] \)

i.e., \( \sin t \leq \ell_t(0) = t \) (for \( t \in [0, 0.5] \))

\[ \Rightarrow \sin (0.5) \leq 0.5 \]

\[ - \frac{1}{8} \]

\[ \Rightarrow M = \frac{1}{169} \]
Another category of problems:

What degree Maclaurin polynomial do you need to approximate \( \cos(0.1) \) to 5 decimal places of accuracy? i.e., the give you a bound on the error and want you to solve for \( n \).

Think about it! Time permitting I will do an example like that later on.

§ 3.5 Finding maxima and minima

In the first example we went about maximizing the function

\[
\frac{1}{4(1)}
\]

on \([3,10]\) to find an upper bound for the error/minute.

We made use of the fact that \( f(x) \) is decreasing hence the maximal value will be attained at the left-hand endpoint. In that particular case we were very lucky.