Approximating functions near a specified point

Suppose we are interested in the values of some function \( f(x) \) for \( x \) near some fixed point \( a \).

E.g., suppose you want to compute by hand or at least estimate \( f(0.0002) \), for

\[
\frac{2x^2 - 3}{x^2 - 3x + 4}
\]

is a mess... If you are all with an approximation you can say (informally): \( x \approx 0 \), \( f(x) \) is continuous at 0 (check!!!)

So \( f(0.0002) \approx f(0) = -\frac{3}{4} \)

Remarks: (i) notice that if \( f \) is not continuous, we could have something like this

\[
\begin{array}{c}
-3 \\
\uparrow \\
\frac{3}{4} \\
\downarrow \\
0.0002
\end{array}
\]

E.g., a huge jump
(ii) even if \( f \) is continuous, we can still making a large error:

\[
\begin{array}{c}
\text{error} \\
\text{0.0002} \\
0
\end{array}
\]

and that's because the function grows very quickly between 0 and 0.0002.

We suspect that the smoother (i.e., slope of tangent line is not too steep) the function is close to the point we care about (here \( x = 0.0002 \)) the easier it would be to approximate it by a value of a nearby point (here \( a = 0 \)) in the sense we are not making a huge error.

In the above examples, there was nothing special about \( x = 0.0002 \). We could have tried to approximate \( f(x) \) for all \( x \)'s that are reasonably close to \( a = 0 \).
In other words we want to approximate \( f(x) \) by

\[
\lim_{x \to a} f(x) = f(a)
\]

for all \( x \) "close" to \( a = 0 \).

\[ f(x) \approx f(a), \quad x \approx a \]

Zeroth approximation (or constant approximation)

(approximating all values for \( x \) by a constant number for \( x \approx a \))

![Graph showing approximation error]

want to approximate \( f(x) \) (for all \( x \approx a \)) by \( f(a) \), we are of course introducing some error, which, in general, will increase the further we go from \( a \).

The thing to keep in mind is that the approximation should only be valid locally (close to the point \( a \)) and should improve the closer we are to \( a \). The smoother the function is.
More examples (estimate the given numbers below).

(a) \[ \sin(-0.00073) \]

Set \( f(x) = \sin x \). \( x = -0.00073 \approx 0 \)

\[ -\sin(-0.00073) \approx \sin(0) = 0 \]

(which is a pretty good approximation, use a calculator to see it!)

(b) \[ e^{0.1} \]

Set \( f(x) = e^x \). \( x = 0.1 \approx 0 \)

\[ e^{0.1} \approx e^0 = 1 \]

Using a calculator \( e^{0.1} = 1.105171... \)

\[ \text{error} = |\text{actual value} - \text{approximation}| \]

\[ = |1.105171... - 1| = 0.105171... \]

(c) \[ \sqrt{5.03} \]

Set \( f(x) = \sqrt{x} \) (or \( g(x) = \sqrt{4+x} \))

For \( x \approx 4 \) \( \sqrt{5.03} \approx f(4) = 2 \) or \( \sqrt{5.03} = \sqrt{4+1.03} = g(1.03) \approx g(0) = \sqrt{4} \)

"true value" \( \approx 2.4277... \) \( \approx \text{error} = 0.24277... \)

\[ -4 \]
First approximation - the linear approximation

We can improve on our zeroth approximation by allowing the approximating function to be a linear function of $x$, rather than just a constant function.

The idea is that for a (fairly smooth function)

the tangent line of the graph of the function at some point $a$ is what locally the graph looks like if we zoom in on the point $a$.

We see from the picture that

$$f(x) \approx l(x)$$

where $l(x)$ is the tangent line of $f$ at $a$.

There is of course some error in the approximation (which becomes smaller and smaller the closer we are to $a$).
Linear approximation

The 1st (or linear) approximation of \( f(x) \) about \( x = a \) is:

\[
 f(x) \approx f(a) + f'(a) (x-a) =: g(x)
\]

tangent line at \((a, f(a))\)

and the error is:

\[
|e| = |f(x) - g(x)|
\]

downward from approximated value

Use a linear approximation to estimate

Examples:

1. \( \sqrt{1.2} \): let \( f(x) = \sqrt{x} \); want to approximate \( f(1.2) \)
so we will use a L.A. about \( a = 1 \)
\[ f(a) = f(1) = \sqrt{1} = 1; \quad f'(x) = \frac{1}{\sqrt{x}} \quad \Rightarrow \quad f'(a) = f'(1) = \frac{1}{2} \]

2. **L.A.:**

\[
\sqrt{1.2} \approx 1 + \frac{1}{2} \left( 1.2 - 1 \right)
\]

\[
= 1 + \frac{1}{2} - 0.2 = 1.1
\]

**Comment:** Could have come up with a better approximation by using \( a = 1.21 \) (since \( \sqrt{1.21} = 1.1 \) and L.A.:

\[
\sqrt{1.2} \approx \ldots \approx 1.095
\]

\[ (15)^{1/4} \]: Let \( f(x) = x^{1/4} \). We need to approximate \( f(15) \). Since \( (16)^{1/4} = 2 \) is easy to calculate, we will use a L.A. about \( a = 16 \). We have \( f(a) = f(16) = 2 \) and since \( f'(x) = \frac{1}{4} x^{-3/4} \) we have:

\[
f'(a) = f'(16) = \frac{1}{4} \times 16^{-3/4}
\]

\[
= \frac{1}{4} \left[ (16)^{1/4} \right]^{-3} = \frac{1}{4} \times \frac{1}{8} = \frac{1}{32}
\]

2. **L.A.:**

\[
(15)^{1/4} \approx 2 + \frac{1}{32} (15 - 16) = 2 - \frac{1}{32} = \frac{63}{32}
\]

\[ 2. \quad (15)^{1/4} \approx 2 + \frac{1}{32} (15 - 16) = 2 - \frac{1}{32} = \frac{63}{32} \]
\[ \log 3 : \text{ always we a for which calculations are easy} \]

Let \( f(x) = \log x \), we need to approximate \( f(3) \)

**Solu 1**
\[
\begin{align*}
\log 4 &= \log 1 = 0 \\
\text{and } f'(x) &= \frac{1}{x} \\
\Rightarrow & f'(1) = 1
\end{align*}
\]

**Solu 2**
Try about \( a = 1 \) again but for a different function:
\[ f(x) = -\log \frac{1}{x} \]
Notice
\[
\log 3 = -\log \frac{1}{3} \approx -(0 + 1) \left( \frac{4}{3} - 1 \right) = \frac{2}{3}
\]

**Solu 3**
\[
a = e^0.1 \log e = 1, \quad (\log x)' = \frac{1}{x}
\]

Therefore
\[
\log 3 \approx \log e + \frac{1}{e} (3 - e) = 1 + \frac{3}{e} - 1 = \frac{3}{e}
\]