To find the inverse for \( y = f(x) \):

1. "Solve for \( x \)". Get \( x = g(y) \).
2. "Exchange \( x, y \)". Get \( y = g(x) \).

Not always easy:

Cannot solve for \( x \) \( \Rightarrow \) since no unique \( x \) for each \( y \) \( \Rightarrow \) no inverse function.
Examples

1) Find the function inverse to \( y = x^2 + 3 \)

\[ \text{Solving for } x: \quad y - 3 = x^2 \implies x = \sqrt[2]{y - 3} \]

We write the inverse function as \( g(x) = \sqrt[2]{x - 3} \)

2) Consider the function \( y = \sqrt{x - 1} \) (domain \( x \geq 1 \))

Find the inverse function in the form \( x = g(y) \)

\[ y = \sqrt{x - 1} \implies y^2 = x - 1 \]
\[ \implies x = y^2 + 1, \text{ for } y \geq 0 \]

\[ \text{Domain of inverse is the range of the original function.} \]
For the inverse function to exist (and pass the vertical line test) we need our original function \( f(x) \) to pass the horizontal line test, or equivalently to be 1-1 ("one-to-one").

If \( x_1 \neq x_2 \), then \( f(x_1) \neq f(x_2) \).

Or, no horizontal line \( y = c \) intersects the graph of \( y = f(x) \) more than once.
Let \( f \) be 1-1 with domain \( A \) and range \( B \). Then its inverse function is denoted by \( f^{-1} \) and has domain \( B \) and range \( A \).

It is defined by \( f^{-1}(y) = x \) whenever \( f(x) = y \) for any \( y \in B \).

\[ f^{-1} \text{ maps } y \text{ back to } x \]

\( f^{-1} \) undoes \( f \)

Because of this we have

\[ f^{-1}(f(x)) = x \text{ for any } x \in A \]

\[ f(f^{-1}(y)) = y \text{ for any } y \in B \]

\[ f^{-1}(x) \neq \frac{1}{f(x)} \]
Ex 1. **Inverse Trigonometric Functions** (§ 2.11)

Let \( y = f(x) = \sin x \). Find the inverse function, that is, the function which takes \( y \) and returns a unique \( x \)-value so that \( \sin x = y \).

For each real number \( y \), the number of \( x \)-values that obey \( \sin x = y \) is exactly the number of times the horizontal straight line \( y = y \) intersects the graph of \( \sin x \).

- When \(-1 \leq y \leq 1\): infinitely many intersection points
- \( y < -1 \) or \( y > 1 \): \( y = \) no intersection

The horizontal line test shows that \( \sin(x) \) is not 1-1.

Now consider the function \( y = \sin x \) with domain \( -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \).
Same formula but the domain has been restricted so that the horizontal line test is satisfied.

\[ -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \]

So for each \( -1 \leq y \leq 1 \), there is exactly one \( x \), call it \( x \) that obeys both:

\[ \sin x = y \]

\[ -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \]

→ this unique value \( x \) is typically denoted \( \arcsin(y) \).

That is:

\[ \sin(\arcsin(y)) = y \]

\[ -\frac{\pi}{2} \leq \arcsin(y) \leq \frac{\pi}{2} \]

Renaming \( y \rightarrow x \), the inverse function \( \arcsin \) is defined for all \(-1 \leq x \leq 1\) and determined by the equation.
\[ \sin^{-1}(\sin(x)) = x \quad \text{and} \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \]

(we will sometime see \( \sin^{-1}(x) \) for \( \arcsin x \),

Remember! \( \sin^{-1}(x) \neq \frac{1}{x} \))

Examples:

1. What is \( \arcsin 1 = ? \)

\[ \sin: \text{ we are looking for } z = \arcsin 1 \]

\[ \Rightarrow \sin z = 1, \quad \text{for} \quad -\frac{\pi}{2} \leq z \leq \frac{\pi}{2} \]

\[ \sin \frac{\pi}{2} = 1 \quad \Rightarrow \quad z = \arcsin 1 = \frac{\pi}{2} \]

2. \( \arcsin \frac{1}{2} = ? \)

\[ \text{we are looking for } z = \arcsin \frac{1}{2} \]

\[ \Rightarrow \sin z = \frac{1}{2}, \quad \text{for} \quad -\frac{\pi}{2} \leq z \leq \frac{\pi}{2} \]

\[ \sin \frac{\pi}{6} = \frac{1}{2} \quad \text{and} \quad -\frac{\pi}{2} \leq \frac{\pi}{6} \leq \frac{\pi}{2} \]

\[ \Rightarrow \arcsin \frac{1}{2} = \frac{\pi}{6} \]
3. Is $\arcsin 0 = 2\pi$ or $0$?

\[ \sin(2\pi) = \sin(0) = 0 \]

but only $-\frac{\pi}{2} \leq 0 \leq \frac{\pi}{2}$

Generally

$\arcsin (\sin x) = $ the unique angle $\theta$

$= x$

between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$

obeying $\sin \theta = \sin x$

4. What is $\arcsin (\sin \left( \frac{11\pi}{16} \right))$?

Since we would be tempted to say it is

\[ \frac{11\pi}{16} \]

but $\frac{11\pi}{16} > \frac{\pi}{2}$ (check!)

so how do we find the correct answer?
We start by sketching the graph of $\sin x$.

Notice that the lobe is symmetric about $x = \frac{\pi}{2}$.

I am looking for the $x$-value before $\frac{\pi}{2}$ that corresponds to the $y$-value $\sin \left( \frac{11\pi}{16} \right)$.

Symmetric about $\frac{\pi}{2}$: $\sin \left( \frac{\pi}{2} - \theta \right) = \sin \left( \frac{\pi}{2} + \theta \right) = \cos \theta$.

Notice that $\sin \left( \frac{11\pi}{16} \right) = \sin \left( \frac{\pi}{2} + \frac{3\pi}{16} \right) = \sin \left( \frac{\pi}{2} - \frac{3\pi}{16} \right) = \sin \left( \frac{5\pi}{16} \right)$.

And $\frac{5\pi}{16}$ is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

$\arcsin \left( \sin \left( \frac{11\pi}{16} \right) \right) = \frac{5\pi}{16}$.
In general

\[ \sin^{-1}(\sin x) = x, \text{ for } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \]

\[ \sin(\sin^{-1} x) = x, \text{ for } -1 \leq x \leq 1 \]

The inverse cosine function is handled similarly

\[ \cos^{-1}(\cos x) = x, \text{ for } 0 \leq x \leq \pi \]

\[ \cos(\cos^{-1} x) = x, \text{ for } -1 \leq x \leq 1 \]

\[ (1): \cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta \]

\[ (2): \cos\left(\frac{\pi}{2} + \theta\right) = -\sin \theta \]

\[ (3): -\sin \theta \]

→ restrict on \[ 0 \leq x \leq \pi \]

→ \[ \arccos y = x \Leftrightarrow \cos x = y \]

and \[ 0 \leq x \leq \pi \]
\[ \cos^{-1}(\cos x) = x, \quad \text{for} \quad 0 \leq x \leq \pi \]

\[ \cos(\cos^{-1} x) = x, \quad \text{for} \quad -1 \leq x \leq 1 \]

Ex.

(1) \[ \cos^{-1}(\cos(7\pi)) = ? \]

\[ \cos(7\pi) = \cos(6\pi + \pi) = \cos(3 \cdot 2\pi + \pi) = \cos \pi = -1 \in \text{domain of arccos} \]

What if \( x \) is an \( \pi \) in the domain?

\[ \implies \cos z = -1 \quad \text{and} \quad 0 \leq z \leq \pi \]

\[ \cos \pi = -1 \quad \text{and} \quad \pi \quad \text{is in the domain} \]

\[ \implies \cos^{-1}(\cos(7\pi)) = \pi \]

(2) \[ \sin(\sin^{-1}(\pi)) = ? \quad \pi > 1 \implies \pi \notin \text{domain of } \arcsin \]

\([\text{the expression cannot be evaluated.}]\)
Can be made $-1$ by restricting to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$\tan^{-1} y = x \iff \tan x = y$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$

- **Derivatives of Inverse Trig Functions**
  - set $\theta(x) = \arcsin x$; want to find $\frac{d\theta}{dx}$

  $\Rightarrow$ take sine of both sides & get:
\[ \sin \left( \theta(x) \right) = x \quad \text{(assuming of course)} \]

\[ \frac{d}{dx} \cos \left( \theta(x) \right) \cdot \frac{d\theta}{dx} = 1 \quad \text{dom} \left( \arcsin(x) \right) \]

\[ \text{what I am looking for} \]

\[ \Rightarrow \quad \frac{d\theta}{dx} = \frac{1}{\cos(\theta(x))} = \frac{1}{\cos(\arcsin(x))} \]

\[ \frac{d}{dx} \arcsin(x) = \frac{1}{\cos(\arcsin(x))} \]

\[ this \ is \ a \ correct \ answer \ but \ it \ is \ rather \ ugly \ and \ also \ involves \ \arcsin(x) \ for \ which \ we \ don't \ have \ an \ explicit \ formula. \]

We can however do the following:
$$\theta(x) = \arcsin x \rightarrow \sin \theta = x = \frac{x}{\sqrt{1-x^2}} \quad \frac{1}{2} \leq \theta \leq \frac{\pi}{2}$$

Draw a right triangle with one angle being
\[ \theta \] (\( = \arcsin x \)).

Since by \((*)\) \( \sin \theta = \frac{x}{1} \) we can make

the side opposite to the angle \( \theta \) of length \( x \) and the hypotenuse of length \( 1 \)

\[ \begin{align*}
\text{hypotenuse} & = 1 \\
\text{opposite side} & = x \\
\text{adjacent side} & = \sqrt{1-x^2}
\end{align*} \]

(by Pythagoras, the adjacent has length \( \sqrt{1-x^2} \))

and so \( \cos (\arcsin x) = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2} \)

Thus
\[ \frac{d}{dx} \arcsin(x) = \frac{1}{\sqrt{1-x^2}} \]
Using similar ideas (work it out yourselves & check §2.12 in the notes)

one can get that:

\[ \frac{d}{dx} \arcsin(x) = \frac{1}{\sqrt{1-x^2}} \]

\[ \frac{d}{dx} \arccos(x) = -\frac{1}{\sqrt{1-x^2}} \]

\[ \frac{d}{dx} \arctan(x) = \frac{1}{1+x^2} \]

(and I don't care much about \( \frac{d}{dx} \arccsc(x) \) or \( \arccsc(x) \) or \( \arccot(x) \))

but they follow the same way.