$\introduction - Motivation - Tangent Lines -$

$\text{The Velocity Problem - Intro to Limits}$

(§1.1 - 1.3 from the notes)

Course = Differential Calculus

→ deal with "derivatives"

to properly describe them, need some tools:

- tangent lines
- limits

$\text{Tangent Line Problem (§1.1 on the notes)}$

Need to define "tangent" - won't do formally, let's draw some pictures

$\text{[Diagram showing tangent lines and non-tangent lines]}$

It's tempting to say a line is tangent to a curve at a point if they only intersect once. Not good → not a tangent line
On the other hand: \( \text{tangent to the curve at point } p \)

Remark: tangency is local

Instead of

"Definition": If we zoom in on the point \( p \), the more the graph of the function looks like a straight line - that line is the tangent line at \( p \).

* If we zoom in at a different point, we will find a different tangent line.

Now go back and compare the previous pictures to make sure we understand what a tangent line is.

Using this idea of zooming in at a particular point, drawing a tangent line is not hard.

How about finding its equation then?
Ex: Find the equation of the tangent line to the curve $y = x^2$ at the point $\theta$ with coordinates $(x, y) = (1, 1)$

**Sln:** First of all, equation of a line:

$$y - y_1 = m(x - x_1)$$

where

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{(slope)} \quad (x_1, y_1) \quad \text{points the line goes through}$$

So we need either 2 points on the line or 1 point and the slope $\rightarrow x$ (for line)

At this point we don't really know how to find the slope (require "differentiation" - will get to that soon)

so we can only try to see if the other approach works (if we have two points, can also find the slope by $\theta$)
Problem (?): But we only have one point!

\( P(1,1) \)

What we will do is to approximate:

We will approximate the tangent line by drawing a line that passes through \( P(1,1) \) and some nearby point \( a \).

Schematically:

```
\[ \text{\includegraphics{tangent-line-diagram.png}} \]
```

(The idea is that letting a “come closer” to \( P \) this sequence of lines we draw will eventually “get close” to the tangent line)

Rather than picking actual numbers \( a \) on the curve we will write the second point \( A \) as \( A(1+h, (1+h)^2) \). Some remarks are in order:
A point on the curve $y = x^2$ has in general coordinates $(a, a^2)$ since if I tell you $X = a$ (for any $a$ in the domain) $y = x^2 \Rightarrow a^2$. Thus to $x = a$, the corresponding $y = a^2$.

I want to pick a very close to $P(1,1)$, so it makes sense to only consider $h = \text{very small}$ since $1 + h \approx 1$. Then to $x = 1 + h$ corresponds the $y = (1 + h)^2$ (since $x$ want a point $Q$ on the curve).

[It's all if you are not comfortable yet with thinking $h$ as a variable, for now think of it as very small number. $h$ as $h$ varies (getting closer to 0), the corresponding points $Q$ come closer and closer to $P$.]
But now we have two points to work with $(x_1, y_1)$ and for a fixed $h$, $Q(1+h, (1+h)^2)$, so back to our formula for a line:

Find $M = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(1+h)^2 - 1}{1+h - 1} = \frac{2h + h^2}{h} = h + 2$

Slope of this approximating line, we call it the "secant" line.
Now, every different \( h \) will give us a different slope; hence a different line (and these lines are NOT our tangent line) but notice that the idea was to kind of zoom in at \( P \), which in our process translates to taking \( Q \) to be very close to \( P \) (which can be achieved by taking \( h \) very close to 0)

If we, e.g., take \( h \) smaller and smaller

<table>
<thead>
<tr>
<th>( h )</th>
<th>( m = 2 + h )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.1</td>
</tr>
<tr>
<td>0.01</td>
<td>2.01</td>
</tr>
<tr>
<td>0.001</td>
<td>2.001</td>
</tr>
</tbody>
</table>

we see that \( m \) becomes closer and closer to 2. Mathematically we write this as

\[
\lim_{h \to 0} (2 + h) = 2
\]

limits will be the topic of another discussion
Our tangent line can be thought of as the end of this process: the slope of the secant lines (i.e., lines like our \( l_1, l_2, l_3, l_4 \)) become closer to that of the tangent line.

We now know the slope of the tangent line is \( m = 2 \), and that the tangent line goes through \( P(1, 1) \):

\[
y - y_1 = m(x - x_1)
\]

\[
y - 1 = 2(x - 1) = 2x - 2 \Rightarrow y = 2x - 2 + 1
\]

\[
y = 2x - 1
\]

Alternatively, the equation of the line is given by \( y = mx + b \); \( m \) is the slope as before so to find \( b \) we use the information that the point \( (1, 1) \) lies on the line.

\( \Delta \) the graph and the tangent line share the point \( P \).
Limits and derivatives are very useful in real life applications too (especially if you are in Applied Science, Engineering, Economics or Finance).

Problem: drop a ball from the top of a very tall building. Let \( t \) be the elapsed time (measured in seconds) and \( s(t) \) the distance the ball has fallen (in meters). So \( s(0) = 0 \).

At what speed / with what velocity is the ball falling after 1 second? or better precisely 4 sec after it's dropped or velocity of the ball at the 1 second mark?

Galileo worked out that \( s(t) \) is a quadratic function: \( s(t) = 4.9 t^2 \).
Notice: something subtle is going on, what do we mean by the "velocity at \( t=1 \)?

- If an object is moving at a **constant velocity** then that velocity is just

\[
V = \frac{\text{distance travelled}}{\text{time fallen}}
\]

- However, in our case, the object is being acted on by gravity and its speed is definitely not constant.

So instead of asking for **THE** velocity, let us examine the "average velocity" over a certain window of time:

\[
\text{average velocity} = \frac{\text{distance moved}}{\text{time fallen}} = \frac{\text{difference in distance}}{\text{difference in time}} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}
\]

**e.g.** what is the Vave between 1 and 1.1 seconds:

\[
\text{Vave} = \frac{s(1.1) - s(1)}{1.1 - 1} = \frac{4.9 \cdot (1.1)^2 - 4.9 \cdot 1}{1.1 - 1} = 10.29 \text{ m/s}
\]
Notice that the line I've drawn has $\text{slope} = \frac{\text{change in } y}{\text{change in } x} = \frac{\text{difference in } s}{\text{difference in } t} = v_{ave}!$

For demonstration purposes, let's look at the average velocity over shorter and shorter time-windows:

<table>
<thead>
<tr>
<th>Time Window</th>
<th>Average Velocity ($= \text{slope of second line}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 \leq t \leq 1.1$</td>
<td>10.29</td>
</tr>
<tr>
<td>$1 \leq t \leq 1.01$</td>
<td>9.849</td>
</tr>
<tr>
<td>$1 \leq t \leq 1.001$</td>
<td>9.8049</td>
</tr>
<tr>
<td>$1 \leq t \leq 1.0001$</td>
<td>9.80049</td>
</tr>
</tbody>
</table>

→ average velocity gets closer and closer to 9.8.
More precisely, \( V_{ave} = \frac{5(1+h) - 5(1)}{(1+h) - 1} \) expand
\[
= \frac{4.9(1+h)^2 - 4.9}{h} = \frac{9.8 + 4.9h}{h}
\]
\[
= \frac{h}{h}(9.8 + 4.9h)
\]

As \( h \to 0 \) (i.e., make the time-window "infinitesimally small") the average velocity becomes the \textbf{instantaneous velocity} (same as with secant and tangent lines).

This is our new limit:
\[
V(t) = \lim_{h \to 0} \frac{5(1+h) - 5(1)}{h} = \lim_{h \to 0} \frac{h}{h}(9.8 + 4.9h)
\]
\[
= 9.8 \text{ m/s}
\]

More generally, the \textbf{instantaneous velocity} at time \( t = a \) is defined as
\[
V(a) = \lim_{h \to 0} \frac{5(a+h) - 5(a)}{h}
\]