Midterm 1  Duration: 50 minutes

This test has 6 questions on 9 pages, for a total of 40 points.

• Read all the questions carefully before starting to work.
• Q1 and Q2 are short-answer questions; put your answer in the boxes provided.
• All other questions are long-answer; you should give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
• Continue on the back of the previous page if you run out of space.
• Attempt to answer all questions for partial credit.
• This is a closed-book examination. None of the following are allowed: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

First Name: ___________________________ Last Name: ___________________________

Student-No: ___________________________ Section: 201

Signature: _____________________________

<table>
<thead>
<tr>
<th>Question</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Points</td>
<td>9</td>
<td>12</td>
<td>4</td>
<td>4</td>
<td>6</td>
<td>5</td>
<td>40</td>
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<td>Score</td>
<td></td>
<td></td>
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Student Conduct during Examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.

2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.

3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.

4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.

5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
   (i) speaking or communicating with other examination candidates, unless otherwise authorized;
   (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;
   (iii) purposely viewing the written papers of other examination candidates;
   (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and
   (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).

6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.

7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.

8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).
Short-Answer Questions. Questions 1 and 2 are short-answer questions. Put your answer in the box provided. Full marks will be given for a correct answer placed in the box. Show your work also, for part marks. Each part is worth 3 marks, but not all parts are of equal difficulty. Simplify your answers as much as possible in Questions 1 and 2.

1. Determine whether each of the following limits exists, and find the value if they do. If a limit below does not exist, determine whether it “equals” $\infty$, $-\infty$, or neither. You cannot use L’Hospital’s Rule (if you know what it is) for any of the limits.

(a) \[ \lim_{t \to 1} \frac{t^2 - 6t + 5}{t^2 - 1} \]

Answer: -2

Solution: Try direct substitution — get 0/0 so simplify.

\[
\frac{t^2 - 6t + 5}{t^2 - 1} = \frac{(t - 5)(t - 1)}{(t - 1)(t + 1)} = \frac{t - 5}{t + 1} \quad \text{provided } t \neq 1
\]

Hence

\[ \lim_{t \to 1} \frac{t^2 - 6t + 5}{t^2 - 1} = \lim_{t \to 1} \frac{t - 5}{t + 1} = -2 \]

(b) \[ \lim_{t \to 2} \left( \frac{1}{t - 2} - \frac{1}{|t - 2|} \right) \]

Answer: Does not exist

Solution: The left and right limits are different:

\[
\lim_{t \to 2^+} \frac{1}{t - 2} - \frac{1}{|t - 2|} = \lim_{t \to 2^+} \frac{1}{t - 2} - \frac{1}{t - 2} = 0 \quad \text{since } t - 2 > 0
\]

\[
\lim_{t \to 2^-} \frac{1}{t - 2} - \frac{1}{|t - 2|} = \lim_{t \to 2^-} \frac{1}{t - 2} - \frac{1}{-(t - 2)} = \lim_{t \to 2^-} \frac{2}{t - 2} = -\infty \quad \text{since } t - 2 < 0
\]

Thus the limit does not exist.

(c) \[ \lim_{x \to +\infty} \frac{2 \sin x + 4}{x^3} \]
Solution: $\sin x$ doesn’t have a limit as $x \to \infty$ but it is at least bounded and the denominator goes to infinity. We will use the Squeeze Theorem.

\[-1 \leq \sin x \leq 1 \Rightarrow -2 \leq 2 \sin x \leq 2\]
\[\Rightarrow -2 + 4 \leq 2 \sin x + 4 \leq 2 + 4\]
\[\Rightarrow \frac{2}{x^3} \leq \frac{2 \sin x + 4}{x^3} \leq \frac{6}{x^3}\]

Taking limits and employing the Squeeze theorem

\[\lim_{x \to +\infty} \frac{2 \sin x + 4}{x^3} = 0\]
2. (a) If \( f(x) = \frac{x^2 + 9x - 10}{\sqrt{x}}, x \neq 0 \), find \( f'(x) \).

\[ f(x) = \frac{x^2}{\sqrt{x}} + \frac{9x}{\sqrt{x}} - \frac{10}{\sqrt{x}} \]
\[ = x^{3/2} + 9x^{1/2} - 10x^{-1/2} \]
\[ f'(x) = \frac{3}{2}x^{1/2} + \frac{9}{2}x^{-1/2} - (10 \cdot -1 x^{-3/2}) \]
\[ = \frac{3}{2}x^{1/2} + \frac{9}{2}x^{-1/2} + 5x^{-3/2} \]

Alternatively, you can use the quotient rule.

(b) If \( g(x) = \frac{1}{2}(x^2 - 2x)e^x \), find \( g'(x) \).

\[ g'(x) = (x - 1)e^x + \frac{1}{2}(x^2 - 2x)e^x \]
\[ = e^x((x - 1) + \frac{1}{2}(x^2 - 2x)) \]

(c) Let \( f(x) \) be a function differentiable at \( x = 2 \) and let \( g = x \cdot f(x) \). The tangent line to the curve \( y = f(x) \) at \( x = 2 \) has slope 3 while the tangent line to the curve \( y = g(x) \) at \( x = 2 \) has slope 5. What is \( f(2) \)?

\[ g'(x) = xf'(x) + f(x) \]
\[ g'(2) = 2f'(2) + f(2) \]
\[ 5 = 2 \cdot 3 + f(2) \]
\[ f(2) = -1 \]

(d) Let
\[ h(x) = \frac{f(x)}{x} - \frac{x}{f(x)} \]

where \( f(2) = 1, f'(2) = 3 \). Compute \( h'(2) \).
Solution: We use the quotient rule twice

\[ h'(x) = \frac{x f'(x) - f(x)}{x^2} - \frac{f(x) - x f'(x)}{f(x)^2} \]

\[ h'(2) = \frac{2 \cdot 3 - 1}{2^2} - \frac{1 - 2 \cdot 3}{1^2} \]

\[ = \frac{5}{4} - \frac{-5}{1} = \frac{25}{4} \]

Answer: \( h'(2) = \frac{25}{4} \)
3. Show that the following equation has at least one solution:

\[ e^x - 2\cos x = 0 \]

**Solution:** We use the Intermediate Value Theorem.

- We see that the function \( f(x) = e^x - 2\cos x \) is the sum of two continuous functions so it is itself continuous (everywhere on the real line).
- At \( x = 0 \), \( f(0) = 1 - 2 \cdot 1 = -1 < 0 \).
- At \( x = \pi/2 \):
  \[
  f(\pi/2) = e^{\pi/2} - 2\cos(\pi/2) \\
  = e^{\pi/2} + 0 \\
  = e^{\pi/2} > 0, 
  \]
  since \( e^x > 0 \) always

By the IVT since the function is continuous and is negative at \( x = 0 \) and positive at \( x = \pi/2 \), there exists some \( 0 < c_1 < \pi/2 \) so that \( f(c_1) = 0 \).
4 marks 4. Determine the horizontal asymptotes of the graph \( y = f(x) \), where

\[
f(x) = \frac{5x^3 - 11x^2 - 9x}{4 - 11x - 2x^3}
\]

Answer: \( y = \frac{-5}{2} \) in both directions

**Solution:** Try direct substitution — undetermined, so need more work. Factor out the highest power across

\[
\frac{5x^3 - 11x^2 - 9x}{4 - 11x - 2x^3} = \frac{x^3(5 - \frac{11}{x} - \frac{9}{x^2})}{x^3(\frac{4}{x^3} - \frac{11}{x^2} - 2)}
\]

\[
= \frac{5 - \frac{11}{x} - \frac{9}{x^2}}{\frac{4}{x^3} - \frac{11}{x^2} - 2}
\]

Hence

\[
\lim_{x \to +\infty} \frac{5x^3 - 11x^2 - 9x}{4 - 11x - 2x^3} = \lim_{x \to +\infty} \frac{5 - \frac{11}{x} - \frac{9}{x^2}}{\frac{4}{x^3} - \frac{11}{x^2} - 2} = \frac{-5}{2}
\]

Also notice that

\[
\lim_{x \to -\infty} \frac{5x^3 - 11x^2 - 9x}{4 - 11x - 2x^3} = \lim_{x \to -\infty} \frac{5 - \frac{11}{x} - \frac{9}{x^2}}{\frac{4}{x^3} - \frac{11}{x^2} - 2} = \frac{-5}{2}
\]
5. Let $a$ be a constant real number and let

$$ h(x) = \begin{cases} 
4 + 2x^2 & x \leq a \\
3 + 4x - 2x^2 & x > a 
\end{cases} $$

For what values of $a$ is $h(x)$ continuous at $x = a$?

**Solution:**

- First $h(a) = 4 + a^2$

- Left-hand limit:

$$ \lim_{x \to a^-} h(x) = \lim_{x \to a^-} 4 + 2x^2 = 4 + 2a^2 = h(a) $$

- Right-hand limit:

$$ \lim_{x \to a^+} h(x) = \lim_{x \to a^+} 3 + 4x - 2x^2 = 3 + 4a - 2a^2 $$

- So we must have

$$ 4 + 2a^2 = 3 + 4a - 2a^2 $$
$$ 4a^2 - 4a + 1 = 0 $$
$$ (2a - 1)^2 = 0 $$

So $a = 1/2$.

6. Let

$$ f(x) = \sqrt{1 + 2x} $$

(a) What is the domain of this function?

**Solution:** We need $1 + 2x \geq 0 \Leftrightarrow x \geq -\frac{1}{2}$. So the domain is $[-1/2, +\infty)$

(b) Find the derivative of $f(x)$ at $x = 0$ using the definition. No points will be given for a different approach.
Solution: 0 is in the domain of the function which is also continuous there.

\[ f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} \]

\[ = \lim_{h \to 0} \frac{\sqrt{1 + 2h} - 1}{h} \]

\[ = \lim_{h \to 0} \frac{(\sqrt{1 + 2h} - 1)(\sqrt{1 + 2h} + 1)}{h(\sqrt{1 + 2h} + 1)} \]

\[ = \lim_{h \to 0} \frac{1 + 2h - 1}{h(\sqrt{1 + 2h} + 1)} \]

\[ = \lim_{h \to 0} \frac{2h}{h(\sqrt{1 + 2h} + 1)} \]

\[ = \lim_{h \to 0} \frac{2}{\sqrt{1 + 2h} + 1} \]

\[ = 1 \]