Math 100 - Homework Set 4 (Curve Sketching-Optimization)

due date: Friday, April the 15th

1. Sketch the curve for \( f(x) \) defined for all \( x \), where we are given the following information: \( f(x) > 0 \) for all \( x \) and \( f(1) = 1, f(2) = 2, f(3) = 3, \lim_{x \to \infty} f(x) = 5, \lim_{x \to -\infty} f(x) = 0 \). Also

\[
\begin{align*}
    f'(x) & \begin{cases} > 0 & \text{for all } x \neq 2 \\
    = 0 & \text{for } x = 2
\end{cases}, & f''(x) & \begin{cases} > 0 & \text{for all } x \in (-\infty, 1) \text{ and } x \in (2, 3) \\
    = 0 & \text{for } x = 1, 2, 3 \\
    < 0 & \text{for all } x \in (1, 2) \text{ and } x \in (3, \infty)
\end{cases}
\end{align*}
\]

2. Let \( f(x) = \cos(2x) + 2x \) be a function defined on the interval \([0, 2\pi]\). Determine the critical numbers, and the intervals on which \( y = f(x) \) is increasing or decreasing. Use the First Derivative Test to classify those critical numbers as local maxima, local minima, or neither.

3. For the function \( y = (x + 1)^2e^{-2x} \), identify the intervals on which \( f \) is increasing or decreasing, all local and absolute maxima and minima, the intervals of concavity and any inflection points. Then, sketch the graph.

4. For the function \( f(x) = \sqrt{(x^2 - 1)^2} \), identify the domain, intervals on which \( f \) is increasing or decreasing, all local and absolute maxima and minima, the intervals of concavity and any inflection points. Then, sketch the graph. You may use, without verifying it, the formula \( f''(x) = \frac{4(x^2 - 3)}{9(x^2 - 1)^{4/3}} \).

5. Sketch the graph of the function

\[ f(x) = \frac{x}{2x - 3} \]

over its natural domain. Make sure you find the intervals of increasing/decreasing, the intervals of concave up/concave down, and any asymptotes. Be sure to label local maxima and minima as well as any inflection points.

6. Sketch the graph of the function

\[ f(x) = \frac{x^2}{2x - 3} \]

over its natural domain. Indicate any symmetry, asymptotes, local or global extreme points, and intercepts.
Problem-Solving Strategy: When you approach an optimization problem, try to follow these steps:

(1) **Understand the problem**: Read the problem and make sure you understand it clearly.

   Ask yourself: • *What do I know?*
   → What information is given in the problem?
   • *What do I need to find?*
   → What is the problem asking?
   → Which quantity is to be optimized, and under what conditions?
   → Do I need to draw a sketch?
   → What are the relevant variables in the problem? (Use suitable notation!)

(2) **Think of a plan**: Find the connection between the given information and the unknown.

   Ask yourself: • *What calculus concept(s) do I need to apply to optimize the quantity of interest?*
   → How can I express the quantity of interest as a certain function \( Q \)?
   → How can I express in a mathematical form the conditions given in the problem?
   → How can I simplify the function \( Q \) to be a function of only one variable?
   → What calculus technique do I need to apply?

(3) **Carry out your plan**: Do your computations!

(4) **Reflect on what you are doing**: Does your answer make sense?

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7. Find the maximum area of a rectangle that can be drawn inside a circle of radius 2.

8. Find the dimensions of the rectangle of maximum area that can be inscribed in a right triangle with sides of length 5 and 12 and hypothenuse of length 13, if two sides of the rectangle lie along the two sides of the triangle. Make sure you justify that your answer is a maximum.

9. A shoebox, complete with overlapping lid, has a base with length \( \frac{5}{3} \) times its width. The rim of the lid overlaps the side of the main box by \( \frac{1}{5} \) of the height of the box. Assuming the thickness of the box is negligible and of uniform density, and the total volume of the box is 3000 cubic centimetres, what dimensions will result in the minimum weight of the empty box? Make sure you follow the steps outlined in the problem-solving strategy above. Draw a picture of the box and remember that if your computations lead to large numbers you can always express such numbers as a product of their factors: this will help you to carry out algebraic calculations by hand.
10. You are in a dune buggy at point P in the desert, 12 km due south of the nearest point A on a straight east-west road. You want to get to a town B on the road 18 km east of A. If your dune buggy can travel at a speed of 15 km/h through the desert, and 39 km/h along the road, toward what point on the road should you head to minimize your travel time from P to B?

11. A piece of wire 10 cm long is cut into two pieces. One piece is bent into a circle while the other is bent into an equilateral triangle. How should the wire be cut so that the total area enclosed is minimal? Maximal? (You can use a calculator at the very end of this problem to help you apply the extreme value theorem).

12. Find the length of the shortest ladder than can extend from a vertical wall, over a fence 2m high located 1m away from the wall, to a point on the ground beyond the fence.