Math 100 - Homework Set 2 (Differentiation)

due date (strict): Thursday March the 10th at 8am (in class or by e-mail)

**Basic Skills** required to work through the problems:

- determining the domain of a function;
- applying the concept of derivative at a point as the instantaneous rate of change of a function to find the slope of the tangent line to the graph at that point;
- applying the concept of derivative as the slope of the tangent line to a curve at a point;
- relating the coordinates of a point that lies on a curve of given equation.
- identify a composite function and break it down into simpler functions using products, quotients, function composition;
- manipulating expressions and performing computations involving logarithms and exponentials

**Learning Goals:** After completing this set, you should be able to:

- master the basic skills listed above;
- find inverse functions of commonly-used functions, explain their properties (such as domain and range), compute their derivatives;
- compute derivatives using implicit differentiation.
- compute the derivative of exponential and logarithmic functions;
- apply logarithmic differentiation, that is, compute the derivative of a complicated function by taking the logarithms and differentiating the resulting equation implicitly;

1. **(Warm up)** Find the missing entries in the table:

<table>
<thead>
<tr>
<th></th>
<th>f(x)</th>
<th>g(x)</th>
<th>f(g(x))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>x - 4</td>
<td>x + 4</td>
<td></td>
</tr>
<tr>
<td>(ii)</td>
<td>1/(x+1)^2</td>
<td>x - 1</td>
<td>1/x^2</td>
</tr>
<tr>
<td>(iii)</td>
<td>x^{1/3}</td>
<td>x^3</td>
<td>2x + 3</td>
</tr>
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**Solution:**

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2. There is more than one way to find the derivative of the function

\[ f(x) = \frac{1}{2x^5 + 40x^2}. \]

(a) Use the quotient rule to compute \( f'(x) \), then check your result by computing the same derivative using the chain rule.

**Solution:**

- Use quotient rule to compute \( f'(x) \):

\[
\frac{d}{dx} \frac{1}{2x^5 + 40x^2} = \frac{0 \times (2x^5 + 40x^2) - 1 \times (10x^4 + 80x)}{(2x^5 + 40x^2)^2} = -\frac{10x^4 + 80x}{(2x^5 + 40x^2)^2}.
\]

- Use chain rule to compute \( f'(x) \): We imagine \( f(x) = g(h(x)) \), where \( g(x) = \frac{1}{x} \) and \( h(x) = 2x^5 + 40x^2 \). Thus

\[
\frac{d}{dx} \frac{1}{2x^5 + 40x^2} = \frac{-1}{(2x^5 + 40x^2)^2} (10x^4 + 80x) = -\frac{10x^4 + 80x}{(2x^5 + 40x^2)^2}.
\]

In either case we could factor the expression and cancel a factor of \( x \) but we need to remember that the function \( f(x) \) does not have \( x = 0 \) in the domain.

(b) When is \( f'(x) = 0 \)? What is this telling you about the graph of \( f(x) \)?

**Solution:** We have \( f'(x) \) potentially zero when the numerator is 0 and so when \( 10x^4 + 80x = 0 \) which is either for \( x = 0 \) or \( x = -2 \). Now \( x = 0 \) is not in the domain of \( f(x) \), so we discard it. Then we have only the one case at \( x = -2 \). This is telling you that the slope of the tangent line to \( f \) at \( x = -2 \) is zero, thus you can deduce that the tangent line is horizontal and parallel to the \( x \)-axis. This also means that the function is neither increasing or decreasing at \( x = -2 \).

3. Find the domain of each of the following functions. Then calculate the derivative of each. Use whichever methods you like.

(a) \( f(x) = e^{\sqrt{\tan x}} \)

**Solution:** The domain is \( x \) such that \( \tan(x) \geq 0 \) and so \( 0 \leq x < \pi/2 \) but also choices shifted by \( \pi \) namely \( n\pi \leq x < \pi/2 + n\pi \) for any integer \( n \).
Now the derivative is
\[
\frac{d}{dx} e^{\sqrt{\tan x}} = e^{\sqrt{\tan x}} \frac{1}{2\sqrt{\tan x}} \sec^2(x).
\]

(b) \( f(x) = e^x (\sin x)(\cos x) \)

**Solution:** The domain is all \( x \). The derivative is
\[
\frac{d}{dx} (e^x (\sin x)(\cos x)) = e^x \sin(x) \cos(x) + e^x (\cos^2(x) - \sin^2(x)).
\]

(c) \( f(x) = \frac{e^x}{x^2+4} \)

**Solution:** The domain is all \( x \). The derivative is
\[
\frac{d}{dx} e^{\frac{e^x}{x^2+4}} = e^{e^x}(x^2+4) - e^{e^x}(2x) = 2xe^{e^x}(x^2 + 3) - e^{e^x}(2x).
\]

(d) \( f(x) = \sin(\cos(e^x)) \)

**Solution:** The domain is all \( x \). The derivative is
\[
\frac{d}{dx} \sin(\cos(e^x)) = \cos(\cos(e^x))(-\sin(e^x))e^x.
\]

4. Consider the function
\[
F(x) = f(\sqrt{x} - 2x)g(h(x)^2)
\]
obtained from some functions \( f, g \) and \( h \). Given the following values,
\[
\begin{align*}
&f(-6) = 2, \quad g(9) = 3, \quad h(4) = -3, \\
&f'(-6) = 4, \quad g'(9) = 0, \quad h'(4) = 6;
\end{align*}
\]
determine the slope of the tangent to the curve \( F(x) \) when \( x = 4 \).

**Solution:** We compute using the product rule for the outer product \( f(\sqrt{x} - 2x) \)
times \( g(h(x)^2) \), where the derivative for \( f(\sqrt{x} - 2x) \) is computed by the chain
rule (the ‘inner’ function is \( \sqrt{x} - 2x \)) and the derivative for \( g(h(x)^2) \) is computed
by the chain rule. We have
\[
\frac{d}{dx} f(\sqrt{x} - 2x) = f'(\sqrt{x} - 2x) \times \frac{d}{dx} (\sqrt{x} - 2x) = (f'(\sqrt{x} - 2x)) \cdot \left( \frac{1}{2\sqrt{x}} - 2 \right)
\]
\[
\frac{d}{dx}g(h(x)^2) = g'(h(x)^2) \times \frac{d}{dx}(h(x)^2) = g'(h(x)^2) \cdot 2h(x)h'(x)
\]

Then
\[
\frac{d}{dx}F(x) = \frac{d}{dx}f(\sqrt{x} - 2x) \times g(h(x)^2) + f(\sqrt{x} - 2x) \times \frac{d}{dx}g(h(x)^2) =
\]
\[
= f'(\sqrt{x} - 2x) \cdot \left(\frac{1}{2\sqrt{x}} - 2\right) \cdot g(h(x)^2) + f(\sqrt{x} - 2x) \cdot g'(h(x)^2) \cdot 2h(x)h'(x).
\]

Thus
\[
F'(4) = f'(-6) \cdot \left(-\frac{7}{4}\right) \cdot g(h(4)^2) + f(-6) \cdot g'(h(4)^2) \cdot 2 \cdot h(4)h'(4)
\]

Now \(h(4) = -3\), so \(g(h(4)^2) = g(9) = 3\) and \(g'(h(4)^2) = g'(9) = 0\). So \(F'(4) = 4 \cdot (-\frac{7}{4}) \cdot 3 + (2) \cdot 0 \cdot 2 \cdot (-3) \cdot 6 = -21\). Thus \(-21\) is the slope of the tangent line to the curve \(y = F(x)\) at \(x = 4\). We could also compute the tangent line using the point \((4, F(4))\), where \(F(4) = f(-6)g(h(4)^2) = 6\), but that wasn’t asked for.

5. Consider the curve given by the equation \(y = 2 \sin^2(\pi x/4)\).

(a) Find a point on the curve at which the tangent line has slope \(\pi/2\).

**Solution:** We have \(y' = 2 \cdot 2 \sin(\pi x/4) \cos(\pi x/4) \cdot (\pi/4)\) (using the Chain Rule), which simplifies to \(y' = \pi \sin(\pi x/4) \cos(\pi x/4)\). Using the identity \(\sin(2x) = 2 \sin x \cos x\), we observe \(y' = (\pi/2) \sin(2\pi x/4)\) or \(y' = (\pi/2) \sin(\pi x/2)\). The required point is such that its x-coordinate must satisfy the equation \((\pi/2) \sin(\pi x/2) = \pi/2\). One possible solution to such equation is \(x = 1\).

(b) Is there a point on the curve at which the tangent line has slope \(\pi\)?

**Solution:** No. From above, \(y' = (\pi/2) \sin(\pi x/2)\), which can never equal \(\pi\) since the sine function has values between \(-1\) and 1.

6. Find the derivative of \(f(x) = \frac{\cos \sqrt{x}}{2 + \sin \sqrt{x}}\). Simplify your answer.

**Solution:**

Using the Quotient Rule and Chain Rule gives
\[
f'(x) = \frac{(2 + \sin \sqrt{x})(-\sin \sqrt{x})(1/2\sqrt{x}) - (\cos \sqrt{x})(\cos \sqrt{x})(1/2\sqrt{x})}{(2 + \sin \sqrt{x})^2}
\]
Expanding the numerator and using \( \sin^2 \sqrt{x} + \cos^2 \sqrt{x} = 1 \) yields a simplified answer

\[
\frac{1 + 2 \sin \sqrt{x}}{2 \sqrt{x}(2 + \sin \sqrt{x})^2}
\]

7. (Warm up) Find an inverse function for each of the following functions. In each case, describe the inverse function as best as you can using the information you have. Whenever possible determine the domain of the inverse function.

(a) \( f(x) = 7x + 2 \)

**Solution:** Solving the equation \( y = 7x + 2 \) for \( x \) gives that the inverse function is \( g(x) = \frac{1}{7}(x - 2) \), with domain all real numbers.

(b) \( f(x) = \cos x \), where \( 0 \leq x \leq \pi \)

**Solution:** The inverse function is just the usual inverse cosine function, \( \cos^{-1} x \). The domain of this inverse function is the interval \([-1, 1]\), which is the range of the given restricted cosine function.

(c) \( f(x) = x^2 \) (\( x < 0 \))

**Solution:** The inverse function is \( g(x) = -\sqrt{x} \), the “−” since the domain of \( f(x) \) is \( x < 0 \). The domain of the inverse function is the set of positive real numbers.

8. Evaluate, when possible, the following expressions, or explain why they cannot be evaluated. NOTE: \( \sin^{-1} x \) and \( \cos^{-1} x \) denote the inverse sine and cosine functions, not the reciprocals of \( \sin x \) and \( \cos x \).

(a) \( \cos^{-1}(\cos(7\pi)) \)  
(b) \( \sin(\sin^{-1}(\pi)) \)  
(c) \( \sin^{-1}(\sin(\pi)) \)

**Solution:** (a) \( \cos(7\pi) = -1 \) and \( \cos^{-1}(-1) = \pi \), which is the answer. 
(b) This expression cannot be evaluated, since the domain of the inverse sine function is \([-1, 1]\) and \( \pi \) is not in this domain. 
(c) \( \sin(\pi) = 0 \), so the answer is \( \sin^{-1}(0) = 0 \).

9. Sketch the graph of the function \( f(x) = |(x-1)^2 - 9| \). Determine on what intervals, if any, this function is one-to-one.

**Solution:** The function is one-to-one on the intervals \((-\infty, -2], [-2, 1], [1, 4], \) and \([4, \infty)\). A sketch makes this clear.
10. The following problem has two parts. Near the point $(0, -1)$, the equation $y^2 - 2xy + 3x^2 = 1$ defines $y$ implicitly as a function of $x$. You will use two different approaches to calculate $\frac{dy}{dx}$ at a point.

(a) Perform implicit differentiation to find $\frac{dy}{dx}$ at $(0, -1)$.

**Solution:** We take the equation $y^2 - 2xy + 3x^2 = 1$ and differentiate both sides with respect to $x$ to get

$$2y \frac{dy}{dx} - 2y - 2x \frac{dy}{dx} + 6x = 0$$

from which we can solve for $\frac{dy}{dx}$ to obtain

$$(2y - 2x) \frac{dy}{dx} = 2y - 6x, \quad \frac{dy}{dx} = \frac{2y - 6x}{2y - 2x}$$

Substituting $x = 0, y = -1$, we have $\frac{dy}{dx}(0, -1) = \frac{-2}{-2} = 1$.

(b) Solve for $y$ explicitly as a function of $x$ and then differentiate to find $\frac{dy}{dx}$ at $(0, -1)$. Compare this approach to your calculations in part (a). Hint: Use the quadratic formula.

**Solution:** Realizing we have a quadratic in $y$ when we view $x$ as a constant we find the roots

$$y = \frac{2x \pm \sqrt{4x^2 - 4(3x^2 - 1)}}{2} = x \pm \sqrt{1 - 2x^2}$$

We take the “-” since when $x = 0$ we need $y = -1$. We can then differentiate to find

$$\frac{dy}{dx} = 1 - \frac{-4x}{2\sqrt{1 - 2x^2}}$$

and so when $x = 0$ we have $\frac{dy}{dx} = 1$.

11. (a) Compute $\frac{dy}{dx}$ if $y = \sin^{-1}(2 - x^2)$. For which values of $x$ does the derivative make sense?

**Solution:** $y' = (-2x)/\sqrt{1-(2-x^2)^2}$, using the Chain Rule. This expression makes sense provided the expression under the square root is $> 0$, which means $(2-x^2)$ needs to be between $-1$ and $1$, which means $x^2$ needs to be between $1$ and $3$, which means $-\sqrt{3} < x < -1$ or $1 < x < \sqrt{3}$.

(b) A careless mathematics professor asked his calculus class on their final examination to find $\frac{dy}{dx}$ if $y = \sin^{-1}(2 + x^2)$. What’s wrong with this problem?
Solution: The problem is that the domain of $\sin^{-1}$ is the interval $[-1, 1]$ and yet there is no value for which $-1 \leq 2 + x^2 \leq 1$. Thus the original function has an empty domain, and so there is no derivative to speak of.