Solutions to homework #3 (Biol 301, Spring 2001)

1. Now we have a model with two age classes, juveniles and adults, in which an adult can become a member of the adult age class in the next generation by surviving. Therefore, the equation for the adult age class is $n_1(t+1) = S_0 n_0(t) + S_1 n_1(t)$, where $S_0 n_0(t)$ are the surviving juveniles that become adults and $S_1 n_1(t)$ are the surviving adults. Therefore, the Leslie matrix is now

$$\begin{pmatrix} M_0 & M_1 \\ S_0 & S_1 \end{pmatrix}$$

and the equations describing the dynamics of the population can be summarized in a Leslie matrix model of the form

$$\begin{pmatrix} n_0(t+1) \\ n_1(t+1) \end{pmatrix} = \begin{pmatrix} M_0 & M_1 \\ S_0 & S_1 \end{pmatrix} \begin{pmatrix} n_0(t) \\ n_1(t) \end{pmatrix}.$$

2. **i)** Mathematical explanation (1): The absolute values of the two eigenvalues of the matrix, $\begin{pmatrix} 0 & m_1 \\ S_0 & 0 \end{pmatrix}$ are the same. Therefore, the population will not converge to either one of the eigenvectors.

ii) Mathematical explanation (2): The Leslie matrix for a semelparous organism is $\begin{pmatrix} 0 & m_1 \\ S_0 & 0 \end{pmatrix}$. Thus, the dynamics of a population with semelparous reproduction is $\begin{pmatrix} n_0(t+1) \\ n_1(t+1) \end{pmatrix} = \begin{pmatrix} 0 & m_1 \\ S_0 & 0 \end{pmatrix} \begin{pmatrix} n_0(t) \\ n_1(t) \end{pmatrix}$. Starting with an initial population in which the sizes of each age class are represented by the vector $\begin{pmatrix} n_0(0) \\ n_1(0) \end{pmatrix}$, we find that

$$\begin{pmatrix} n_0(1) \\ n_1(1) \end{pmatrix} = \begin{pmatrix} 0 & m_1 \\ S_0 & 0 \end{pmatrix} \begin{pmatrix} n_0(0) \\ n_1(0) \end{pmatrix} = \begin{pmatrix} m_1 n_1(0) \\ S_0 n_0(0) \end{pmatrix}.$$

Likewise,

$$\begin{aligned} \begin{pmatrix} n_0(2) \\ n_1(2) \end{pmatrix} &= \begin{pmatrix} 0 & m_1 \\ S_0 & 0 \end{pmatrix} \begin{pmatrix} n_0(1) \\ n_1(1) \end{pmatrix} \\ &= \begin{pmatrix} 0 & m_1 \\ S_0 & 0 \end{pmatrix} \begin{pmatrix} m_1 n_1(0) \\ S_0 n_0(0) \end{pmatrix} \\ &= \begin{pmatrix} m_1 S_0 n_0(0) \\ S_0 m_1 n_1(0) \end{pmatrix}. \end{aligned}$$

Notice that in generation 2, $m_1 n_1(0)$ and $S_0 n_0(0)$ have switched locations in the vector compared with generation 1. To see if this behaviour continues in generation 3, calculate

$$\begin{aligned}\begin{pmatrix} n_0(3) \\ n_1(3) \end{pmatrix} &= \begin{pmatrix} 0 & m_1 \\ S_0 & 0 \end{pmatrix}\begin{pmatrix} n_0(2) \\ n_1(2) \end{pmatrix} \\ &= \begin{pmatrix} 0 & m_1 \\ S_0 & 0 \end{pmatrix}\begin{pmatrix} m_1 S_0 n_0(0) \\ S_0 m_1 n_1(0) \end{pmatrix} \\ &= \begin{pmatrix} m_1 S_0 m_1 n_1(0) \\ S_0 m_1 S_0 n_0(0) \end{pmatrix}.\end{aligned}$$

It does, suggesting that for even numbered generations the vector of population sizes is $\begin{pmatrix} n_0(2t) \\ n_1(2t) \end{pmatrix} = \begin{pmatrix} m_1^t S_0^t n_0(0) \\ S_0^t m_1^t n_1(0) \end{pmatrix}$ and for odd numbered generations the vector is $\begin{pmatrix} n_0(2t-1) \\ n_1(2t-1) \end{pmatrix} = \begin{pmatrix} m_1^t S_0^{t-1} n_1(0) \\ S_0^t m_1^{t-1} n_0(0) \end{pmatrix}$. It is now easy to see that the ratio of the number of 0-year-olds to 1-year-olds in generation $t+1$ will in general never the same as in generation t, so that there is no stable age distribution.

iii) Biological explanation: Starting with $n_0(0)$ and $n_1(0)$, the fate of $n_0(0)$ is independent of the fate of $n_1(0)$ for all future times. Therefore, there are two independent populations, and there is no a priori reason why the relative proportions should remains the same over time.

3.

$$L = \begin{pmatrix} 1/3 & 4 & 2 \\ 2/3 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}$$

To determine the long term growth rate and stable age distribution, find the largest eigenvalue of L and its eigenvector.

4. (a) In Figure 1 the upper set of three curves is for parameter values $\lambda = 2, K = 100, 500, 1000$ and the bottom set is $\lambda = 1, K = 100, 500, 1000$. We see that the function $f(x)$ takes on larger values for larger λ and K.

   (b) K is a measure for the intensity of competition: the higher K, the less intense is competition. λ is an intrinsic rate of increase.

   (c) Using the function for the per capita number of offspring, we get $f(x)$, $x(t+1) = f(x(t)) \cdot x(t) = F(x(t))$.

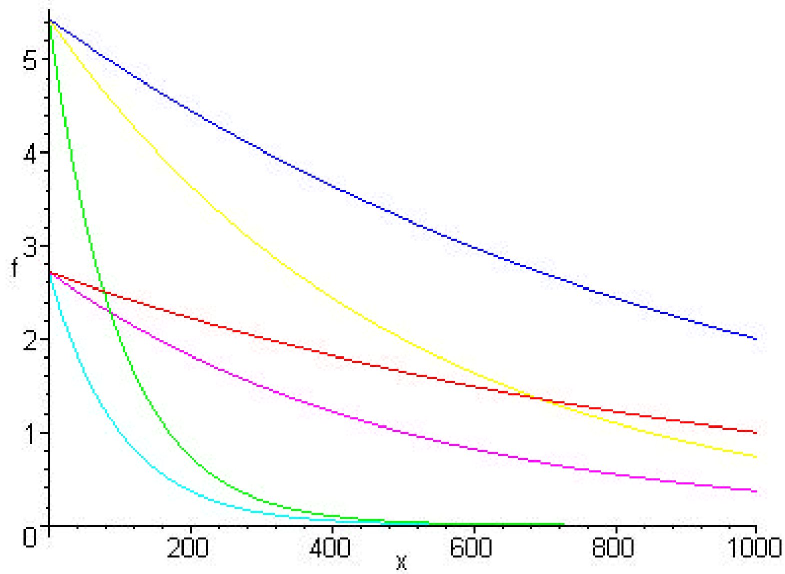

Figure 1:

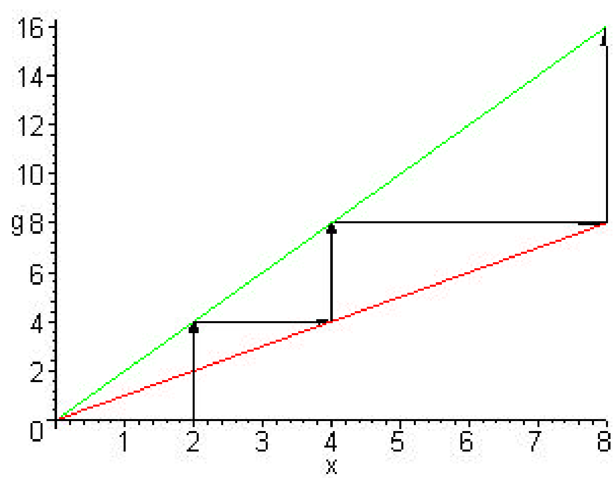

Figure 2:

NOTE: IN PROBLEMS 5-7 $x(t+1) = g$ in the figures.

5. In figure 2, $\lambda = 2$, and the population grows indefinitely.

   In figure 3, $\lambda = 0.5$, and the population decays to extinction.

   In figure 4, $\lambda = 1$, and the population stays at its initial size, no matter what this initial size is (which makes sense biologically, for $\lambda = 1$ means that the average number of offspring per generation is 1, so that any population simply replaces itself, regardless of its size).

6. In figure 5, $\lambda = 0.75$ which we will take as a general case for $\lambda < 1$. Here we see that the population goes extinct.

   In figure 6, $\lambda = 3$, no matter whether the initial population size is above or below 2, the population converges on an equilibrium size of 2 individuals. In fact for any $\lambda > 1$, the population will approach a size that is greater than zero.

7. a. In figure 7, $\lambda = 8$ and $h = 3$, and we see some interesting behaviour, in that there are now two positive equilibrium points. If the population starts at a density greater than 1, it will converge to a density of 2. If the population is at a density of 1, it will stay there indefinitely. If the population starts at a density less than one it will go extinct. For $\lambda < 1$, the curve of $x(t+1)$ vs. $x(t)$ is always lower than the $y = x$ line irrespective of h, which tells us that the population will eventually go extinct (if we assume negative population sizes is extinction).

   b. Compared to 6., changing h will cause the curve $x(t+1)$ vs. $x(t)$ to move up or down. As h approaches 0, the dynamics are the same as in problem #7. As h increases from 0, the range of initial population sizes increases in which the population size will go to negative values, i.e. extinct. In addition, the carrying capacity, i.e. the larger of the two equilibria, will be lower than the carrying capacity in problem 6. For sufficiently large h, the curve $x(t+1)$ vs. $x(t)$ will always be lower than the $y = x$ which means that, as with the case when $\lambda < 1$, the population will always go to negative values (extinction). Biologically, this means that high enough predation will cause extinction of the prey.

   c. The model allows for negative population sizes, which is nonsensical!

   d. To avoid negative population sizes, predation should be modeled as a factor h □ 1, and NOT by subtracting a constant h! Thus, a biologically more meaningful model of predation would be $x(t+1) = h \cdot f(x(t)) \cdot x(t)$, where h □ 1.

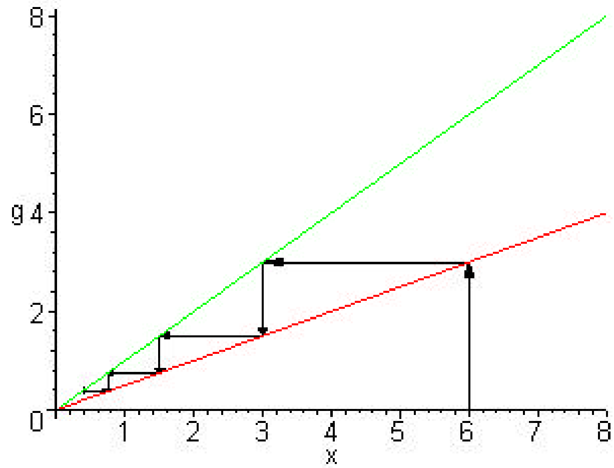

Figure 3:

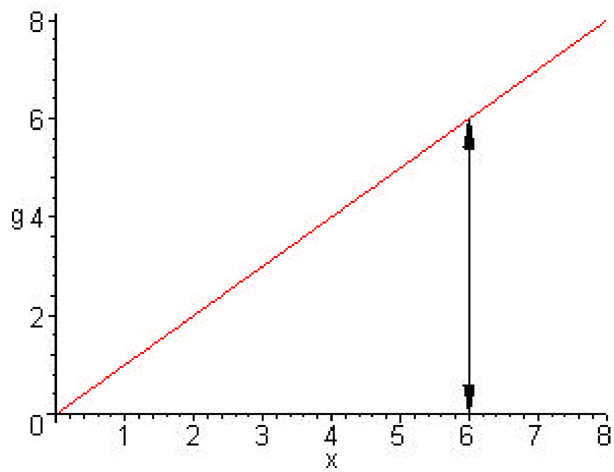

Figure 4:

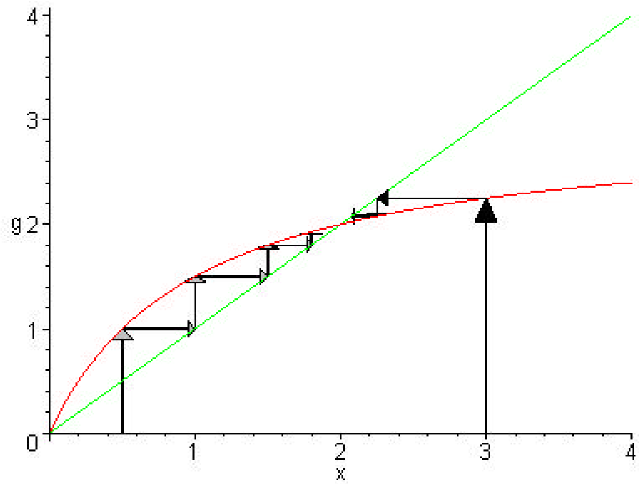

Figure 5:

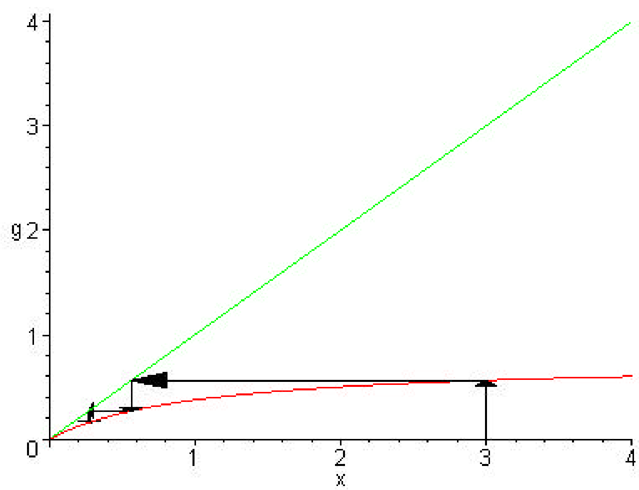

Figure 6:

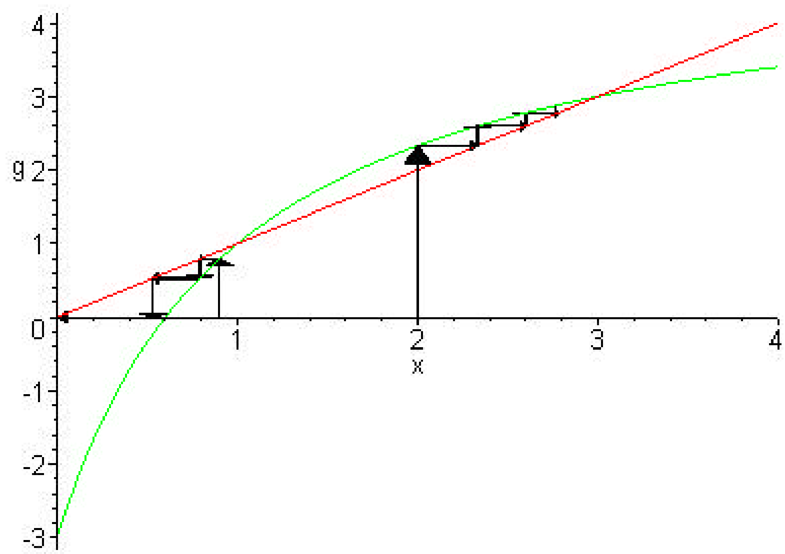

Figure 7: