Solutions to Homework #2, Biology 301 (Spring 2001)

1. (a) $\begin{pmatrix} 6 & 10 \\ 3 & 5 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6-10 \\ 3-5 \end{pmatrix} = -2\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

   (b) $\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 5 \end{pmatrix} = 2\begin{pmatrix} 6 \\ 7 \end{pmatrix}$

   (c) $\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -a \end{pmatrix}$

   (d) In the last calculation, $\begin{pmatrix} 0 \\ -a \end{pmatrix}$ is a multiple of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ namely $-a\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.In other words, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$is an eigenvector of $\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$ with corresponding eigenvalue $-a$.

   (a) Show: $Aw = Ax + Ay$

   $Aw = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix} = \begin{pmatrix} a(x_1+y_1) + b(x_2+y_2) \\ c(x_1+y_1) + d(x_2+y_2) \end{pmatrix}$ by rules for the multiplication of matrices

   $= \begin{pmatrix} ax_1 + ay_1 + bx_2 + by_2 \\ cx_1 + cy_1 + dx_2 + dy_2 \end{pmatrix} = \begin{pmatrix} ax_1 + bx_2 + ay_1 + by_2 \\ cx_1 + dx_2 + cy_1 + dy_2 \end{pmatrix}$ by rearrangement of terms

   $= \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix} + \begin{pmatrix} ay_1 + by_2 \\ cy_1 + dy_2 \end{pmatrix}$ by rules for the addition of matrices

   $= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

   $= Ax + Ay$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

   (b) Show: $A \cdot rx = r(A \cdot x)$

   $A \cdot rx = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} rx_1 \\ rx_2 \end{pmatrix} = \begin{pmatrix} arx_1 + brx_2 \\ crx_1 + drx_2 \end{pmatrix} = \begin{pmatrix} rax_1 + rbx_2 \\ rcx_1 + rdx_2 \end{pmatrix}$ by a rearrangement of terms

   $= r\begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix} = r\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = r(A \cdot x)$

   (c) Show: $A \cdot (rx + sy) = r(A \cdot x) + s(A \cdot y)$

   $A \cdot (rx + sy) = A \cdot rx + A \cdot sy$ by (a)

   $= r(A \cdot x) + s(A \cdot y)$ by (b)

   (a) Show: $AB \neq BA$

   $AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$ by multiplication of matrices

   $BA = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ea + fc & eb + fd \\ ga + hc & gb + hd \end{pmatrix}$ by multiplication of matrices

$$\begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix} = AB \neq BA = \begin{pmatrix} ea+fc & eb+fd \\ ga+hc & gb+hd \end{pmatrix}$$

(b) Show: $A(B+C) = AB + AC$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} e & f \\ g & h \end{pmatrix} + \begin{pmatrix} i & j \\ k & l \end{pmatrix} \Bigg] = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} e+i & f+j \\ g+k & h+l \end{pmatrix} \text{ by addition}$$

$$= \begin{pmatrix} a(e+i)+b(g+k) & a(f+j)+b(h+l) \\ c(e+i)+d(g+k) & c(f+j)+d(h+l) \end{pmatrix} \text{ by multiplication}$$

$$= \begin{pmatrix} ae+ai+bg+bk & af+aj+bh+bl \\ ce+ci+dg+dk & cf+cj+dh+dl \end{pmatrix} \text{by expansion}$$

$$= \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix} + \begin{pmatrix} ai+bk & aj+bl \\ ci+dk & cj+dl \end{pmatrix} \text{by rearrangement and factoring}$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} e & f \\ g & h \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} i & j \\ k & l \end{pmatrix} \text{ by reverse multiplication}$$

$$= AB + AC$$

(c) Show: $(AB)C = A(BC)$

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix} \text{ by multiplication of matrices}$$

$$(AB)C = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix} \cdot \begin{pmatrix} i & j \\ k & l \end{pmatrix}$$

$$= \begin{pmatrix} i(ae+bg)+k(af+bh) & j(ae+bg)+l(af+bh) \\ i(ce+dg)+k(cf+dh) & j(ce+dg)+l(cf+dh) \end{pmatrix} \text{ by multiplication of matrices}$$

$$BC = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \cdot \begin{pmatrix} i & j \\ k & l \end{pmatrix} = \begin{pmatrix} ei+fk & ej+fl \\ gi+hk & gj+hl \end{pmatrix} \text{ by multiplication of matrices}$$

$$A(BC) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} ei+fk & ej+fl \\ gi+hk & gj+hl \end{pmatrix}$$

$$= \begin{pmatrix} a(ei+fk)+b(gi+hk) & a(ej+fl)+b(gj+hl) \\ c(ei+fk)+d(gi+hk) & c(ej+fl)+d(gj+hl) \end{pmatrix} \text{ by multiplication of matrices}$$

$$= \begin{pmatrix} aei+afk+bgi+bhk & aej+afl+bgj+bhl \\ cei+cfk+dgi+dhk & cej+cfl+dgj+dhl \end{pmatrix} \text{ by expansion of terms}$$

$$= \begin{pmatrix} aei+bgi+afk+bhk & aej+bgj+afl+bhl \\ cei+dgi+cfk+dhk & cej+dgj+cfl+dhl \end{pmatrix} \text{ by rearrangement}$$

$$= \begin{pmatrix} i(ae+bg)+k(af+bh) & j(ae+bg)+l(af+bh) \\ i(ce+dg)+k(cf+dh) & j(ce+dg)+l(cf+dh) \end{pmatrix} \text{ by factoring}$$

$$= (AB)C$$

(d) Show: $AI = IA = A$

$$AI = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a+b\times 0 & a\times 0+b \\ c+d\times 0 & c\times 0+d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$IA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+0\times c & b+0\times d \\ 0\times a+c & 0\times b+d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$$

(a) $A = \begin{pmatrix} 3 & 7 \\ 2 & 8 \end{pmatrix}$

$\lambda_{1,2} = \frac{(a+d)\pm\sqrt{(a+d)-4(ad-bc)}}{2}$

$\lambda_{1,2} = \frac{11\pm\sqrt{11^2-4(24-14)}}{2} = \frac{11\pm\sqrt{11^2-40}}{2} = \frac{11\pm\sqrt{81}}{2} = \frac{11\pm 9}{2}$. Therefore, $\lambda_1 = 10$ and $\lambda_2 = 1$.

Eigenvector for $\lambda_1 = 10$:

Let $x_1 = \begin{pmatrix} x \\ y \end{pmatrix}$

$\begin{pmatrix} 3 & 7 \\ 2 & 8 \end{pmatrix} x_1 = 10x_1$

$\begin{pmatrix} 3x+7y \\ 2x+8y \end{pmatrix} = \begin{pmatrix} 10x \\ 10y \end{pmatrix}$

$\begin{pmatrix} 6x+14y \\ 6x+24y \end{pmatrix} = \begin{pmatrix} 20x \\ 30y \end{pmatrix}$

$-10y = 20x - 30y$

$20y = 20x \Rightarrow x_1 = k\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, i.e. the vector corresponding to the eigenvalue $\lambda_1 = 10$, is some multiple of a 2x1 vector in which the upper component is equal to the bottom component

Eigenvector for $\lambda_2 = 1$:

Let $x_2 = \begin{pmatrix} x \\ y \end{pmatrix}$

$\begin{pmatrix} 3 & 7 \\ 2 & 8 \end{pmatrix} x_2 = 1x_2$

$\begin{pmatrix} 3x+7y \\ 2x+8y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\begin{pmatrix} 6x+14y \\ 6x+24y \end{pmatrix} = \begin{pmatrix} 2x \\ 3y \end{pmatrix}$

$-10y = 2x - 3y$

$y = -\frac{2}{7}x \Rightarrow x_2 = k\begin{pmatrix} -3.5 \\ 1 \end{pmatrix}$

(b) $A = \begin{pmatrix} a & 1-b \\ 1-b & a \end{pmatrix}$

By the same procedure as in (a)

$\lambda_1 = a+b-1; x_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\lambda_2 = a-b+1; x_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(a) $\begin{pmatrix} 6 & 10 \\ 3 & 5 \end{pmatrix}$

$\lambda_{1,2} = \frac{(6+5) \pm \sqrt{(6+5)^2 - 4(30-30)}}{2} = \frac{11 \pm \sqrt{11^2}}{2} = \frac{11 \pm 11}{2} = 11, 0$

(b) $\begin{pmatrix} a & b \\ 3a & 3b \end{pmatrix}$

$\lambda_{1,2} = \frac{(a+3b) \pm \sqrt{(a+3b)^2 - 4(a3b - b3a)}}{2} = \frac{(a+3b) \pm \sqrt{(a+3b)^2}}{2} = \frac{(a+3b) \pm (a+3b)}{2} = a + 3b, 0$

2.

Matrix	Eigenvalues	Corresponding Eigenvectors
1	6; 2; 2	$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$; $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$; $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$
2	10; 1; 2; 3	$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$; $\begin{pmatrix} -3.5 \\ 1 \\ 0 \\ 0 \end{pmatrix}$; $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$; $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix}$
3	a; b; c; d	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$; $\begin{pmatrix} 1 \\ \frac{(b-a)}{e} \\ 0 \\ 0 \end{pmatrix}$; $\begin{pmatrix} \frac{b-c-e}{a-c} \\ 1 \\ \frac{c-b}{e} \\ 0 \end{pmatrix}$; $\begin{pmatrix} \frac{2de+d^2-dc-be-bd+bc+e^2-ec}{d^2-da+ba-bd} \\ \frac{c-e-d}{b-d} \\ 1 \\ \frac{d-c}{e} \end{pmatrix}$

(a) Show: $Aw = arx + bsy$

$Aw = A(ax + by)$
$= Aax + Aby$ from problem 2a
$= aAx + bAy$ from problem 2b
$= arx + bsy$ from definitions of Ax and Ay and then substitution

(b) Show: $A^2w = ar^2x + bs^2y$

$Aw = arx + bsy$ from 7a
$A(Aw) = A(arx + bsy) = Aarx + Absy$ from 2a
$= arAx + bsAy$ because a, r, b, s are scalars and from 2b
$= ar^2x + bs^2y$ from definitions of Ax and Ay and then substitution

(c) Show: $A^tw = ar^tx + bs^ty$

We've shown that $Aw = arx + bsy$ and $A^2w = ar^2x + bs^2y$ which suggests,
$A^3w = ar^3x + bs^3y$
To check if this is in fact true we can apply A to A^2w
$A(A^2w) = A(ar^2x + bs^2y) = ar^2Ax + bs^2Ay$ (from 7b)
$= ar^3x + bs^3y$ again from the definitions of Ax and Ay and then substitution
From this we see the pattern is consistent suggesting
$A^tw = ar^tx + bs^ty$ for $t \geq 1$

(d) From 7c we showed that $A^tw = ar^tx + bs^ty$. Now both $r^t \to 0$ and $s^t \to 0$ as $t \to \infty$ if r and s are > 0 and < 1. Therefore, $ar^tx + bs^ty$ goes to 0

as t goes to ∞ if r and s are between 0 and 1, which means that A^tw also goes to 0.

(a) $L = \begin{pmatrix} \frac{1}{3} & 4 \\ \frac{2}{3} & 0 \end{pmatrix}$

(b) Using previous methods we find that the two eigenvalues for this 2x2 matrix are $\lambda_1 = \frac{1}{6} + \frac{1}{6}\sqrt{97}$; $\lambda_2 = \frac{1}{6} - \frac{1}{6}\sqrt{97}$

Remembering that the long term growth rate is determined by the largest eigenvalue because if $x(t+1) = Lx(t)$, then $x(t) = c_1\lambda_1^t v_1 + c_2\lambda_2^t v_2$ provided that λ_1 and λ_2 are the two eigenvalues corresponding to L and v_1 and v_2 are the corresponding eigenvectors of λ_1 and λ_2. In the equation, $c_1\lambda_1^t v_1 + c_2\lambda_2^t v_2$, as t becomes large the term with the largest λ_i will dominate, i.e. as t becomes large the vector $x(t)$ approaches some multiple of the vector v_i that is associated with the largest eigenvalue λ_i.

For the matrix L, the largest eigenvalue is $\lambda_1 = \frac{1}{6} + \frac{1}{6}\sqrt{97}$. The eigenvector associated with this eigenvalue is, $k\begin{pmatrix} \frac{1}{4} + \frac{1}{4}\sqrt{97} \\ 1 \end{pmatrix}$.

Thus, the long term growth rate of the population is $\frac{1}{6} + \frac{1}{6}\sqrt{97}$ and the stable age distribution is $\frac{1}{4} + \frac{1}{4}\sqrt{97}$ 1-year-olds for every 2-year-old. This is the stable age distribution because as t becomes large, the population vector $x(t)$ approaches some multiple of $\begin{pmatrix} \frac{1}{4} + \frac{1}{4}\sqrt{97} \\ 1 \end{pmatrix}$ in which the upper component of the matrix is the number of 1-year-olds and the bottom component is the number of 2-year-olds.