Biol 301, Spring 2001, Solutions to Homework #1

1.a.

$$\ln(a^t) = b$$
$$t\ln(a) = b$$
$$t = \frac{b}{\ln(a)}$$

1.b.

$$a^t = b$$
$$t\ln(a) = \ln(b)$$
$$t = \frac{\ln b}{\ln a}$$

1.c.

$$\ln(ax) + \ln(bx) - \ln(c) = d$$
$$\ln\left(\frac{abx^2}{c}\right) = d$$
$$\frac{abx^2}{c} = e^d$$
$$x^2 = \frac{c}{ab}e^d$$
$$x = \pm\sqrt{\frac{c}{ab}e^d}$$

1.d.

$$f(x) = \ln(x)$$
$$\frac{df(x)}{dx} = \frac{1}{x}$$
$$f(x) = \ln(x^a)$$
$$\frac{df(x)}{dx} = \frac{1}{x^a}ax^{a-1}$$
$$= \frac{a}{x}$$

$$f(x) = e^{ax} = g(h(x))$$
$$\frac{df(x)}{dx} = \frac{dg(h(x))}{dh(x)}\frac{dh(x)}{dx} \text{ (Chain Rule)}$$
$$= ae^{ax}$$

1.e.

$$\int e^{ax}dx$$
$$let$$
$$\{t = ax$$
$$dt/dx = a$$
$$dx = dt/a\}$$
$$\int \frac{1}{a} e^t dt$$
$$= \frac{1}{a} e^t$$
$$= \frac{1}{a} e^{ax} + C$$

$$\int \frac{1}{x} dx$$
$$= \ln(x) + C$$

$$\int \ln(x)dx$$
$$\{\frac{d}{dx} f(x)g(x) = f'(x)g(x) + f(x)g'(x)$$
$$\int \frac{d}{dx} f(x)g(x)dx = \int f'(x)g(x)dx + \int g'(x)f(x)dx$$
$$f(x)g(x) = \int f'(x)g(x)dx + \int g'(x)f(x)dx$$
$$\int g'(x)f(x)dx = f(x)g(x) - \int f'(x)g(x)dx\}$$
$$let$$
$$\{f(x) = \ln(x) \qquad g'(x) = 1$$
$$f'(x) = 1/x \qquad g(x) = x\}$$
$$\int g'(x)f(x)dx = x\ln(x) - \int dx$$
$$= x\ln(x) - x + C$$

2.a.

| Hour | No. yeast |
|---|---|
| 1 | 2 |
| 2 | 4 |
| 3 | 8 |
| 5 | 32 |
| 10 | 1024 |

2.b.

$N(t + 1) = 2N(t)$

2.c.

$$
\begin{aligned}
&N(t) = 10^8 \\
&N(0) = 1 \\
&N(t+1) = 2N(t) \\
&N(t) = 2N(t-1) \\
&\Rightarrow N(t+1) = 2(2N(t-1)) \\
&N(1) = 2N(0) \\
&N(2) = 2N(1) = 2(2(N(0))) \\
&\ldots \\
&N(t) = 2^t N(0) \\
&10^8 = 2^t \\
&2^t = 10^8 \\
&t = \frac{\ln(10^8)}{\ln 2} \\
&\quad = 26.58 \\
&27 \textit{ hours}
\end{aligned}
$$

3.

$$
\begin{aligned}
&10^{-12}\,\text{g/E.coli} \\
&5.9763\times10^{24}\,\text{kg/earth}\times1000\,\text{g/kg} = 5.9763\times10^{27}\,\text{g/earth}
\end{aligned}
$$

$$
\begin{aligned}
&\text{E. coli divides once every 20 minutes :} \\
&24\text{h/day}\times60\text{min/h} = 1440\text{min/day} \\
&1440\text{min/day}\times1\text{gen/(20min)} = 72\text{gen/day}
\end{aligned}
$$

$$
\begin{aligned}
N(t) &= 2^t N(0) \\
&= 2^{72} \\
&= 4.7224\times10^{21}\,\text{E.coli}\times10^{-12}\,\text{g/E.coli} \\
&= 4.7224\times10^{9}\,g
\end{aligned}
$$

Crichton is off by 18 orders of magnitude

4.a

$$N(t+1) = \frac{2}{3}N(t)$$
$$N(t) = (2/3)^t N(0)$$
$$N(0) = 10^8$$

4.b.

$N(t)$ asymptotically approaches 0, therefore ask the question : What' s the smallest $t$ in which $N(t) = 1$?

$$t = \frac{\ln\left(\frac{1}{10^8}\right)}{\ln(2/3)}$$

$= 45.43$ intervals of 2 hours

~ 92 hours

4.c.

There is an underlying element of stochasticity. That is, the time it takes for a yeast cell to die is a random variable.Therefore, as population size becomes small there is an increasing probability that all of the yeast die off in less time than is expected. Likewise, it is possible to have a few yeast survive for a very long time, thus lengthing the time it takes for the population to die off.

5.a.

Let $J_i$ be the number of juveniles at time $i$ and $A_i$ be the number of adults at time $i$.

$$J_{t+1} = 0.9A_t$$
$$A_{t+1} = 0.5J_t$$

5.b.

| Time | Juveniles | Adults |
|---|---|---|
| 0 | 100 | 200 |
| 1 | 180 | 50 |
| 2 | 45 | 90 |
| 3 | 81 | 22 |
| 4 | 19 | 40 |
| 5 | 36 | 9 |

5.c.

From b it is evident that the population is decreasing in size to 0. The reason for this is that if we look at the average number of offspring an adult has that then survive to adulthood and reproduce, this number is less than one ($1/2 * 0.9 = 0.45$).

More formally,

$$A_{t+1} = \frac{1}{2} 0.9 A_{t-1}$$

$$A_t = \left(\frac{1}{2} 0.9\right)^{t/2} A_o$$

$$\left(\frac{1}{2} 0.9\right)^{t/2} \to 0 \ \ as \ \ t \to \infty$$

$\therefore$ As $t \to \infty$, $J_t, A_t \to 0$

A similar approach for $J_t$

5.d.

If $x$ is the number of juvenile offspring per adult and $y$ the fraction of juveniles surviving to adulthood, let

$$xy \geq 1.$$

This means that the average number of offspring that survive to adulthood is greater than or equal to 1. If this is the case, the population will stay at a constant size or increase in size. For example, if the chance of survival to adulthood were 0.5, and the number of offspring per adult were 2.1, then the population would survive in the long run.