Problem Set #7: Practice problems for population genetics (and some continuous time ecology)
(Not to be handed in).

(1) Let genotypes AA, Aa and aa have the corresponding frequencies D, H, and R. Let p be the frequency of allele A and q be the frequency of allele a.

(a) Prove: $p = D + \frac{1}{2}H$ and $q = R + \frac{1}{2}H$

(b) $p^2 + 2pq + q^2 = 1$

(c) Did you use any biological assumptions to prove (a) and (b)?

(2) Prove that if $D_t = p_0^2$, $H_t = 2p_0q_0$, and $R_t = q_0^2$ for all $t > 0$, then $p_t = p_0$ for all $t \geq 0$.

(3) Suppose that $w_{AA} = \frac{3}{4}$, $w_{Aa} = 1$, and $w_{aa} = \frac{1}{2}$ what is $\hat{p}$, where p is the frequency of allele A?

(4) Derive equation (3.34 on p.46, see handout). The selective values are $w_{AA} = 1 - s$, $w_{Aa} = 1 - s$, and $w_{aa} = 1$. The mutation rate of $a- > A$ is υ. The equation for Δp under both mutation and selection is $\Delta p = \Delta p_{sel} + \Delta p_{mut}$, where Δp_{sel} is the change in p due to selection alone and Δp_{mut} is the change in p due to mutation alone.

(a) Show that

$$\Delta p_{sel} = \frac{pw_{AA} + (1-p)w_{Aa}}{p^2 w_{AA} + 2p(1-p)w_{Aa} + (1-p)^2 w_{aa}} p - p$$

(b) Convince yourself that if p is sufficiently small, then the quantity, $p^2 w_{AA} + 2p(1-p)w_{Aa} + (1-p)^2 w_{aa}$, is effectively equal to one.

(c) Show that the assumptions in (b) leads to

$$\begin{aligned} \Delta p_{sel} &= (1-s)p - p \\ &= -sp \end{aligned}$$

(d) Show that

$$\Delta p_{mut} = \upsilon(1-p)$$

(e) Convince yourself that p is sufficiently small, then $\upsilon(1-p)$ is effectively equal to υ.

(f) Then the equation for the change in p becomes $\Delta p = -sp + \upsilon$.

(g) Solve for the equilibrium. What was the main assumption?

(5) Derive equation (3.35 on p. 46, see handout). The selective values are $w_{AA} = 1 - s$, $w_{Aa} = 1$, and $w_{aa} = 1$. The rate of $a- > A$ is again υ.

(a) Show that is p is sufficiently small, then

$$\Delta p_{sel} = (1 - sp)p - p$$
$$= -sp^2$$

(b) Show that Δp is then given by

$$\Delta p = -sp^2 + \upsilon.$$

(c) Solve for the equilibrium. What was the main assumption?

(6) You are studying competition between red and black desert scorpions. For the red scorpion, $K_1 = 100$ and $\alpha_{12} = 2$. For the black scorpion, $K_2 = 150$ and $\alpha_{21} = 3$.

Suppose the initial population size are 25 red scorpions and 50 black scorpions. Graph the state space and isoclines for each species, and plot these initial population sizes. Predict the short-term dynamics of each population and the final outcome of interspecific competition.

(7) Suppose that for two competing species, $\alpha_{12} = 1.5$, $\alpha_{21} = 0.5$ and $K_2 = 100$. What is the minimum carrying capacity for species 1 that is necessary for coexistence? How large is the carrying capacity needed for species 1 to win in competition?

(8) Diagram the isoclines for two competing species in which there is a stable equilibrium with coexistence. Show how intraguild predation could shift this to exclusion by the predatory species. (See the handout, pgs. 115-117 on intraguild predation).

(9) We have at several points in the term considered the effects of harvesting on the dynamics of populations. Analyze a model of the form

$$\frac{dN}{dt} = rN(1 - \frac{N}{K}) - \frac{cN}{d + N},$$

where N is population size, r is the average number of offspring born per individual, and K is the carrying capacity. The constants c and d are new terms in which c is the maximum harvest rate that occurs when N is large, and d is a term that sets the point where the harvest rate is one-half its maximum.

(10) At a particular locus in humans, there are two alleles that confer an individual's blood type (M and N). Evidence suggests that the alleles M and N are selectively neutral relative to one-another, and that they are in Hardy-Weinberg proportions in most populations. In this and the following two problems assume that for the given problem the alleles M and N are in Hardy-Weinberg proportions. In one particular population there are 10 times as many MN genotypes as there are NN genotypes. What is the frequency of the N allele?

(11) In a population 42 percent of the individuals are MN.

(a) What is the frequency of the M allele?

(b) If this question cannot be answered, what is the frequency if you are told that M is more common than N?

(12) From population data we could have estimated p_M by taking the square root of the frequency of MM genotypes instead of by counting genes. Why is the latter method preferred?

(13) In some varieties of sheep the presence of horns is determined by an allele that is dominant in males but recessive in females. If 96 percent of males are horned, what proportion of females have horns?

(14) Two equal-sized population, 1 and 2, have frequencies q_1 and q_2 of the recessive allele a. The populations are fused into a single, randomly mating unit.

(a) What is the proportion of aa homozygotes in the mixed population?

(b) What is the answer to (a) if population 1 is four times as large as population 2?

(15) What is the equilibrium allele frequency at an overdominant locus if the three fitnesses are $1, 1 + hs$, and $1 - s$ (where h and s are positive)?

(16) Cystic fibrosis is a recessive disease that until very recently was almost invariably lethal. Assume that $s = 1$. The incidence is 0.0004.

(a) What mutation rate would be required to maintain such a frequency if the disease is completely recessive?

(b) If this value seems too high for a typical mutation rate, perhaps the mutation is maintained by heterozygote advantage. Compute hs using the parameterization of Problem 15.

(17) Invent a pattern of viabilities so that the three genotypes remain in Hardy-Weinberg ratios after differential mortality.

(18) If the fitness of AA, $AA^{'}$, and $A^{'}A^{'}$ are 1, $1 - hs$, and $1 - s$, what range of values of h lead to

(a) the ultimate loss of $A^{'}$?

(b) an unstable polymorphic equilibrium?

(c) a stable polymorphic equilibrium?