Biol 301 Winter 2001
Assignment 2: Matrix algebra
Due Wednesday, January 17, 2001

Reading assignment: Hastings, Population Biology, pages 24-29, 96-101.

1. Compute the following:

$$\begin{pmatrix} 6 & 10 \\ 3 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 5 \end{pmatrix} \quad \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

What is special about the last calculation?

2. Consider two 2-vectors

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ and } y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Then we can define the vector addition

$$w = x + y$$

by saying that the two components of the new vector w are the sum of the respective components of x and y:

$$w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}.$$

Thus the two components of $w = x + y$ are $w_1 = x_1 + y_1$ and $w_2 = x_2 + y_2$.

Similarly, for any number r we can define a new vector rx whose components are the components of x multiplied by r: $rx = \begin{pmatrix} rx_1 \\ rx_2 \end{pmatrix}$. The new vector rx is simply obtained by stretching the vector x by a factor r.

Now let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a 2x2-matrix, and consider two vectors x and y, as well as their sum $w = x + y$ defined above, and the stretched vector rx. Then we can consider the new vectors $A \cdot w$, $A \cdot x$ and $A \cdot y$, and $A \cdot rx$which are obtained by applying the matrix A to the vectors w, x, y, and rx. Show that for these new vectors we have:

$$A \cdot w = A \cdot x + A \cdot y$$

and

$$A \cdot rx = r(A \cdot x).$$

Conclude that for any two vectors $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$and $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ and any two numbers r and x we have:

$$A \cdot (rx + sy) = r(A \cdot x) + s(A \cdot y).$$

3. Using the matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \quad C = \begin{pmatrix} i & j \\ k & l \end{pmatrix} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and the rules for matrix multiplication, show that:

(a) AB does not equal BA (matrix multiplication is not commutative, unlike multiplication of real numbers!)

(b) $A(B+C) = AB+AC$ (matrix multiplication is distributive, just like multiplication of real numbers)

(c) $(AB)C = A(BC)$ (matrix multiplication is associative, just like multiplication of real numbers)

(d) $AI = IA = A$ (multiplication of a matrix by the *identity matrix* I from either side leaves the matrix unchanged, just like multiplication by 1 leaves any real number unchanged)

4. (Do this problem by hand.) Find the eigenvalues and the corresponding eigenvectors for the following matrices:

$$\begin{pmatrix} 3 & 7 \\ 2 & 8 \end{pmatrix} \quad \begin{pmatrix} a & 1-b \\ 1-b & a \end{pmatrix}$$

5. (Do this problem by hand.) Show that the following matrices have an eigenvalue that is 0:

$$\begin{pmatrix} 6 & 10 \\ 3 & 5 \end{pmatrix} \quad \begin{pmatrix} a & b \\ 3a & 3b \end{pmatrix}$$

Matrices for which one or more eigenvalues are zero are known as *singular* matrices. Such matrices have rows that are functions of the other rows in the matrix (in the first example the second row is half the first row; in the second example the second row is three times the first row). Singular matrices are special and generally rare.

6. (Do this problem using Mathematica.) Find all eigenvalues and eigenvectors of the following matrices:

$$\begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{pmatrix} \quad \begin{pmatrix} 3 & 7 & 0 & 0 \\ 2 & 8 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 4 \end{pmatrix} \quad \begin{pmatrix} a & e & e & e \\ 0 & b & e & e \\ 0 & 0 & c & e \\ 0 & 0 & -0 & d \end{pmatrix}$$

7. Let A be a 2x2 matrix that has the two eigenvalues r and s with corresponding eigenvectors x and y (so that $A \cdot x = rx$ and $A \cdot y = sy$). Consider vectors w of the form

$$w = ax + by,$$

where a and b are arbitrary real numbers (cf. Problem 1 above).

(a) Show that $A \cdot w = arx + bsy$. (Hint: Use Problem 2 and the fact that x and y are eigenvectors of A).

(b) Since the $A \cdot w = arx + bsy$ is again a vector, we can apply the matrix A to this vector, i.e. we can calculate $A \cdot (A \cdot w) = A \cdot (arx + bsy)$, which we denote by $A^2 \cdot w$. (Thus, $A^2 \cdot w$is simply the vector that is obtained by applying the matrix A twice.) Show that

$$A^2 \cdot w = ar^2 x + bs^2 y.$$

(c) Similarly, we can reapply the matrix A to the vector $A^2 \cdot w$ to get a vector $A^3 \cdot w$, to which we can apply A again to get a vector $A^4 \cdot w$, and so on. In this way, we can calculate the vector $A^t \cdot w$ by applying the matrix A t times. Show that

$$A^t \cdot w = ar^t x + bs^t y.$$

(d) What happens with $A^t \cdot w$ for large t when both r and s have an absolute value that is smaller than 1?

8. In a population of an imaginary organism that lives 2 years the average number of births for 1-year-olds is 1/3, the average number of births for 2-year-olds is 4, and the survival probability from 1 to 2 is 2/3. Death is certain after 3 years.

(a) Set up a Leslie matrix model for this population.

(b) Find the long term growth rate (largest eigenvalue) and the stable age distribution (corresponding eigenvector; the ratio of 1-year-olds to 2-year-olds) for this population.