Q. Find the Laurent series for \( \frac{\sin z}{z^3} \) in \( |z| > 0 \)

**Sol:** \( \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \) valid for any \( z \).

Therefore for \( |z| > 0 \)

\[
\frac{\sin z}{z^3} = z^{-3} \left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots \right) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} \frac{z^{2j+1}}{z^3}
\]

\[
= \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} \frac{z^{2j-2}}{z^3}
\]
Q: Find the Laurent series for \( \frac{x+1}{z(z-4)^2} \) in 
\[ 0 < |z-4| < 4 \]

Sol: \[
\frac{x+1}{z(z-4)^2} = \frac{1}{(z-4)^2} + \frac{1}{(4+z-4)(z-4)^2}
\]

Note that \( 0 < \frac{|z-4|}{4} < 1 \), and we have
\[
\frac{1}{1 + \frac{z-4}{4}} = 1 - \left( \frac{z-4}{4} \right) + \left( \frac{z-4}{4} \right)^2 + \ldots
\]
\[
= \sum_{j=0}^{\infty} \left( -\frac{1}{4} \right)^j \left( \frac{z-4}{4} \right)^j
\]

Therefore
\[
\frac{x+1}{z(z-4)^2} = (z-4)^{-2} + \frac{1}{4} \left( z-4 \right)^{-1} \left( \sum_{j=0}^{\infty} \left( -\frac{1}{4} \right)^j \left( z-4 \right)^j \right)
\]
\[
= (z-4)^{-2} + \sum_{j=0}^{\infty} \left( -\frac{1}{4} \right)^j \cdot \frac{1}{4} \left( z-4 \right)^{j-2}
\]
\[
= \frac{5}{4} (z-4)^{-2} - \frac{1}{16} (z-4)^{-1} + \frac{1}{64}
\]
\[
+ \sum_{j=3}^{\infty} \left( -\frac{1}{4} \right)^j \cdot \frac{1}{4} \left( z-4 \right)^{j-2}
\]

Valid for \( 0 < |z-4| < 4 \).
Q. Find and classify the isolated singularities of each of the following functions

(a) \( \frac{1}{e^z - 1} \)  
(b) \( \cot \left( \frac{1}{z} \right) \)
(c) \( \frac{\sin(z)}{z^2} - \frac{1}{z} \)

Sol

(a) Since \( e^z = 1 \) for \( z = 2\pi ni, \ n = 0, \pm 1, \pm 2, \ldots \)
and we have for \( z = 2\pi ni \),
\[
\lim_{z \to 2\pi ni} \left( \frac{z - 2\pi ni}{e^z - 1} \right) = \lim_{z \to 2\pi ni} \frac{1}{e^z} = 1
\]
(here by L'Hopital rule).

We get that \( z = 2\pi ni, \ n = 0, \pm 1, \pm 2, \ldots \) are simple poles

(b) \( \cot \left( \frac{1}{z} \right) = \frac{\cos \left( \frac{1}{z} \right)}{\sin \left( \frac{1}{z} \right)} \), the isolated singularities are \( z = 0, \frac{1}{n\pi}, \ (n = \pm 1, \pm 2, \pm 3, \pm 4) \)

The point \( z = 0 \) is non-isolated singularity.
The point \( z = \frac{1}{n\pi} \), for \( n = \pm 1, \pm 2, \pm 3, \ldots \) are simple poles because

\[
\lim_{z \to \frac{1}{n\pi}} \left( \frac{(z - \frac{1}{n\pi}) \cos(\frac{1}{z})}{\sin(\frac{1}{z})} \right)
\]

\[= \lim_{z \to \frac{1}{n\pi}} \left[ \frac{\cos(\frac{1}{z}) + (z - \frac{1}{n\pi}) \left( -\sin(\frac{1}{z}) \right) \frac{-1}{z^2}}{\cos(\frac{1}{z}) \cdot \left( -\frac{1}{z^2} \right)} \right] = \frac{\cos(n\pi)}{\cos(n\pi)} \cdot n^2 \pi^2 = -\frac{1}{n^2 \pi^2} \neq 0.\]

(C) The isolated singularity for \( \frac{\sin z}{z^2} - \frac{1}{z} \) is \( z = 0 \) which is a removable singularity.

Indeed, for \( z \neq 0 \)

\[
\frac{\sin z}{z^2} = \frac{1}{z^2} \left( z - \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots \right)
\]

\[= \frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} + \ldots\]

Therefore for \( z \neq 0 \),

\[
\frac{\sin z}{z^2} - \frac{1}{z} = g(z), \quad \text{where} \quad g(z) = -\frac{z}{3!} + \frac{z^3}{5!} + \ldots
\]

is an analytic function on \( \mathbb{C} \). Hence the singularity \( z = 0 \) is removable.
Q. Define the function

\[ h(z) = \frac{1}{\sin z} - \frac{1}{z} + \frac{2z}{z^2 - \pi^2}. \]

(a) Show \( h(z) \) is analytic in the disk \( |z| < 2\pi \), except for the removable singularities at \( z = 0, \pm \pi \).

(b) Find the first four terms of the Taylor series about \( z = 0 \) for \( h(z) \). What is the radius of convergence of this series?

(c) Use the result of (b) to obtain the first few coefficients (with positive and negative indices) in the Laurent series expansion for \( \csc z = \frac{1}{\sin z} \), valid in the annulus \( \pi < |z| < 2\pi \).

Sol: (a) The singularities of \( h(z) \) in \( \{ |z| < 2\pi \} \) are \( z = 0, \pm \pi \). We now analyze them separately.
For \( z = 0 \), we have

\[
\lim_{z \to 0} \left( \frac{h(z)}{z^2} \right) = \lim_{z \to 0} \left( \frac{1}{\sin z} - \frac{1}{z} \right) + \lim_{k \to 2^+} \left[ \frac{2z}{z - k} \right]
\]

\[
= \lim_{z \to 0} \left[ \frac{z}{\sin z} \cdot \frac{1 - \frac{\sin^2 z}{z^2}}{z} \right] + 0
\]

\[
= \lim_{z \to 0} \left( \frac{z}{\sin z} \right) \cdot \lim_{z \to 0} \left( \frac{1 - \frac{\sin^2 z}{z^2}}{z} \right)
\]

By L'Hospital rule, we have

\[
\lim_{z \to 0} \frac{z}{\sin z} = \lim_{z \to 0} \frac{1}{\cos z} = 1
\]

On the other hand,

\[
\frac{\sin^2 z}{z^2} = \frac{z^2 - \frac{z^4}{3!} + \frac{z^6}{5!} - \ldots}{z^2}
\]

\[
= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \ldots
\]

Clearly then,

\[
\lim_{z \to 0} \left( \frac{1 - \frac{\sin^2 z}{z^2}}{z} \right) = 0
\]

Hence the point \( z = 0 \) is removable.
Next consider \( z = \pi \).

Recall that in the case \( z = 0 \), we have proved

\[
\lim_{z \to 0} \left( \frac{1}{\sin z} - \frac{1}{z} \right) = 0
\]

It follows that (observe \( \sin (z \pm \pi) = -\sin z \)),

\[
\lim_{z \to \pi} \left( \frac{1}{\sin z} + \frac{1}{z-\pi} \right) = 0 \tag{*}
\]

Now by (*), we have

\[
\lim_{z \to \pi} h(z) = \lim_{z \to \pi} \left[ \frac{1}{\sin z} - \frac{1}{z} + \frac{2 \pi}{z^2 - \pi^2} \right]
\]

\[
= \lim_{z \to \pi} \left( \frac{1}{\sin z} + \frac{1}{z-\pi} \right) + \lim_{z \to \pi} \left( -\frac{1}{z} \right) + \lim_{z \to \pi} \left[ \frac{2 \pi}{z-\pi} - \frac{1}{z-\pi} \right]
\]

\[
= -\frac{1}{\pi} + \lim_{z \to \pi} \frac{z^2 - 2\pi z + \pi^2}{z^2 - \pi^2}
\]

\[
= -\frac{1}{\pi} + \lim_{z \to \pi} \frac{z-\pi}{z+\pi}
\]

\[
= -\frac{1}{\pi}.
\]
Therefore the singularity $z = \pi$ is removable.

In a similar way and by using

\[
\lim_{z \to -\pi} \left( \frac{1}{\sin z} + \frac{1}{z + \pi} \right) = 0,
\]

we can show the singularity $z = -\pi$ is also removable.

(b) To compute the coefficients near $z = 0$, we can consider $|z| < \delta$, and $\delta > 0$ is sufficiently small.

Observe that for $0 < |z| < \delta$,

\[
\frac{z}{\sin z} = \frac{1}{\sin z} = \frac{1}{z} - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} + \cdots,
\]

Now using the formula

\[
\frac{1}{1 - x} = 1 + x + x^2 + x^3 + x^4 + \cdots, \quad \text{valid in } |x| < 1,
\]

we get for $0 < |z| < \delta$,

\[
(\delta) = 1 + \left[ (-\frac{z^3}{3!} + \frac{1}{5!} z^5 - \frac{1}{7!} z^7 + \frac{1}{9!} z^9 + \cdots) \right] \\
+ \left[ (-\frac{z^5}{5!} + \frac{1}{7!} z^7 - \frac{1}{9!} z^9 + \cdots) \right] \\
+ \left[ (-\frac{z^7}{7!} + \frac{1}{9!} z^9 + \cdots) \right] + O(z^8)
\]
= 1 + \frac{2}{6} + \frac{7z^4}{360} + \frac{31z^6}{15120} + O(z^8) \quad - -(a')

Therefore

\frac{1}{\sin z} - \frac{1}{z} = \frac{1}{z} \left( \frac{z}{\sin z} - 1 \right)

= \frac{z}{6} + \frac{7z^3}{360} + \frac{31z^5}{15120} + O(z^7) \quad - -(b)

On the other hand, for |z| < \pi, we have

\frac{2z}{z^2 - \pi^2} = \frac{2z}{\pi^2} \cdot (1 - \left( \frac{z}{\pi} \right)^2)^{-1}

= -\frac{2z}{\pi^2} \left( 1 + \left( \frac{z}{\pi} \right)^2 + \left( \frac{z}{\pi} \right)^4 + \left( \frac{z}{\pi} \right)^6 + \ldots \right)

= -\frac{2z}{\pi^2} - \frac{2z^3}{\pi^4} - \frac{2z^5}{\pi^6} + O(z^7)

Therefore by (a) and (b), we get

h(z) = 0 + \left( \frac{1}{6} - \frac{2}{\pi^2} \right) z + (o) z^2

+ \left( \frac{7}{360} - \frac{2}{\pi^4} \right) z^3

+ (o) z^4 + \ldots

Since h(z) can be re-defined as an analytic function inside \{ |z| < 2\pi \}, we conclude that
the Taylor series converges for |z| < 2\pi.
The radius of convergence is precisely $2\pi$ since at $z = 2\pi$, $h(z)$ has a simple pole there.

(C) For $\pi < |z| < 2\pi$, we have $\frac{\pi}{|z|} < 1$, and therefore

$$\frac{2z}{z^2 - \pi^2} = \frac{2}{z} \left(1 - \left(\frac{\pi}{z}\right)^2\right)^{-1}$$

$$= \frac{2}{z} \left[1 + \left(\frac{\pi}{z}\right)^2 + \left(\frac{\pi}{z}\right)^4 + \left(\frac{\pi}{z}\right)^6 + \cdots\right].$$

Hence by using part (b), we get the Laurent series for $\csc z = \frac{1}{\sin z}$ in $\pi < |z| < 2\pi$. is given by (recall $h(z) = \frac{1}{\sin z} - \frac{1}{z} + \frac{2z}{z^2 - \pi^2}$)

$$\frac{1}{\sin z} = h(z) + \frac{1}{z} - \frac{2z}{z^2 - \pi^2}$$

by (c)

$$= 0 + \left(\frac{1}{6} - \frac{2}{\pi^2}\right)z + (0)z^3 + \left(\frac{7}{360} - \frac{2}{\pi^4}\right)z^5$$

$$+ (0)z^7 + \cdots$$

$$+ \left(\frac{1}{2}\right) + \left(-\frac{2}{z} - \frac{2\pi^2}{z^3} - \frac{2\pi^4}{z^5} - \frac{2\pi^6}{z^7} + \cdots\right)$$

$$= \cdots - \frac{2\pi^6}{z^7} - \frac{2\pi^4}{z^5} - (0)z^7 - \frac{2\pi^2}{z^3}$$

$$- (0)z^5 - \frac{1}{z} + \left(\frac{1}{6} - \frac{2}{\pi^2}\right)z + (0)z^2$$

$$+ \left(\frac{7}{360} - \frac{2}{\pi^4}\right)z^3 + (0)z^4 + \cdots.$$