A Dynamical Analogue of Theorems by Bombieri-Masser-Zannier and Habegger

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Overview

We will present:

Results of Bombieri-Masser-Zannier and Habegger under the principle of “Unlikely Intersection”

Their analogue in arithmetic dynamics obtained by Ghioca and N.

References:

- K.N. *Some Arithmetic Dynamics of Diagonally Split Polynomial Maps.* IMRN **2015**, 1159–1199. Section 4.
- D. Ghioca and K.N. *Dynamical Anomalous Subvarieties: Structure and Bounded Height Theorems.* arXiv:1408.5455

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Arithmetic Dynamics

Dynamics: studies a self-map $\varphi : S \to S$ and all the iterates ϕ^n for $n \in \mathbb{N}$.

Arithmetic dynamics: S is a variety over K and φ is a K-morphism where K is “arithmetically interesting” (number fields, function fields,...).

From diophantine geometry to arithmetic dynamics: “torsion” vs “preperiodic”, ”(torsion translates of) algebraic subgroups” vs “(pre)periodic subvarieties”, “small subgroups” vs “orbits”,...

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Unlikely Intersection

Principle: when the intersection of two objects is larger than expected, there should be an underlying geometric reason.

Simplest example: Lang's question answered by Ihara, Serre, and Tate:

Theorem

Let X be a curve in $\mathbb{G}_m^2$. If X has infinitely many points (a, b) where both a and b are roots of unity then X is a torsion translate of an algebraic subgroup.

This has many vast generalizations. Example: Mordell-Lang Conjecture for semi-abelian varieties by Faltings, Vojta, and McQuillan. Another example: work of Bombieri, Masser, and Zannier.

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From now on: everything is over $\bar{\mathbb{Q}}$.

Think of torsion points (a, b) as "subgroups of codimension 2 in $\mathbb{G}_{\mathrm{m}}^2$".

Question: fix a curve X in $\mathbb{G}_{\mathrm{m}}^n$, what happens when intersect X with:

(a) the union of all subgroups of codimension 2?

(b) the union of all subgroups of codimension 1?

Bombieri, Masser, and Zannier treated both. Today we only focus on (b). Answer: when X is not contained in a translate of an algebraic subgroup, the intersection is infinite but it is *small* (i.e. bounded height).

Bombieri-Masser-Zannier: Part I

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h: absolute logarithmic Weil height on $\mathbb{P}^1(\bar{\mathbb{Q}})$. Define h_n on $(\mathbb{P}^1)^n$ by:

$$h_n(a_1, \ldots, a_n) := h(a_1) + \ldots + h(a_n).$$

A subset of $(\mathbb{P}^1)^n$ has bounded height: boundedness with respect to h_n.

Theorem (BMZ 1999)

Let X be a curve in $\mathbb{G}_m^n$ that is not contained in any translate of an algebraic subgroup then $\bigcup_V X \cap V$ has bounded height where V ranges over all algebraic subgroups of codimension 1.

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Dynamical Analogue: Part I

Let $d \geq 2$ and $C_d(X)$ be the polynomial of degree d satisfying $C_d(x + \frac{1}{x}) = x^d + \frac{1}{x^d}$.

Exceptional polynomials (of degree d) are polynomials that are linearly conjugate to x^d or $\pm C_d(x)$. Non-exceptional polynomials are also called “disintegrated” by Medvedev-Scanlon.

Let $n \geq 2$ and $f_1, \ldots, f_n \in \bar{\mathbb{Q}}[x]$ of degrees at least 2. Let $\varphi := f_1 \times \ldots \times f_n$ be the coordinate-wise self-map of $(\mathbb{P}^1)^n$:

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For the arithmetic dynamics of φ, it *suffices* to study the arithmetic of $\mathbb{G}_m^n$ and the case when $f_1, \ldots, f_n$ are non-exceptional.

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Dynamical Analogue: Part I

We have an analogue of the BMZ Theorem:

Theorem (N. 2013)

Let X be a curve in $(\mathbb{P}^1)^n$ whose projection to each factor $\mathbb{P}^1$ is non-constant. Assume that X is not contained in any φ-periodic hypersurface. Then $\bigcup_V X \cap V$ has bounded height where V ranges over all φ-periodic hypersurfaces.

Bombieri-Masser-Zannier: Part II

Bombieri, Masser, and Zannier tried to generalize their theorem in 1999 for intersection between a subvariety of dimension r with algebraic subgroups of codimension r.

This is rather subtle. After a series of work, they proved a "structure theorem" and asked a "bounded height conjecture" in 2007. Habegger proved this conjecture in 2009.

All these are inside $\mathbb{G}_{\mathrm{m}}^n$. The more general version for semi-abelian varieties is still open.

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Their approach: given X of dimension r in $\mathbb{G}_\mathrm{m}^n$, define "anomalous subvarieties" of X, then define $X^{\mathrm{oa}} := X \setminus \bigcup_Z Z$ where Z ranges over all anomalous subvarieties.

They prove the following:

Theorem

Let X be a subvariety of dimension r in $\mathbb{G}_m^n$.

(a) *(BMZ 2007) Structure Theorem: X^{oa} is Zariski open in X.*

(b) *(Habegger 2009) Bounded Height Theorem: the intersection $\bigcup_V X^{\mathrm{oa}} \cap V$ has bounded height where V ranges over all algebraic subgroups of codimension r.*

Part (b) was conjectured by Bombieri-Masser-Zannier in 2007.

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Consider $f_1(x), \ldots, f_n(x)$ and $\varphi = f_1 \times \ldots \times f_n$ as before.

Given X of dimension r in $(\mathbb{P}^1)^n$, we can define φ-anomalous subvarieties of X, then define $X^{oa}_{\varphi} := X \setminus \bigcup_Z Z$ where Z ranges over all the φ-anomalous subvarieties.

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We have the following:

Theorem (Ghioca-N. 2014)

Notation as above.

(a) *Structure Theorem:* $X_{\varphi}^{\mathrm{oa}}$ *is Zariski open in* X.

(b) *Bounded Height Theorem: the intersection* $\bigcup_V X_{\varphi}^{\mathrm{oa}} \cap V$ *has bounded height where* V *ranges over all* φ*-periodic subvarieties of codimension* r.

THANK YOU.