A Dynamical Analogue of Theorems by Bombieri-Masser-Zannier and Habegger

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We will present:

Results of Bombieri-Masser-Zannier and Habegger under the principle of “Unlikely Intersection”
Their analogue in arithmetic dynamics obtained by Ghioca and N.

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Dynamics: studies a self-map \( \varphi : S \to S \) and all the iterates \( \varphi^n \) for \( n \in \mathbb{N} \).

Arithmetic dynamics: \( S \) is a variety over \( K \) and \( \varphi \) is a \( K \)-morphism where \( K \) is “arithmetically interesting” (number fields, function fields,...).

From diophantine geometry to arithmetic dynamics: “torsion” vs “preperiodic”, ”(torsion translates of) algebraic subgroups” vs “(pre)periodic subvarieties”, “small subgroups” vs “orbits”,
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Principle: when the intersection of two objects is larger than expected, there should be an underlying geometric reason.

Simplest example: Lang’s question answered by Ihara, Serre, and Tate:

**Theorem**

Let $X$ be a curve in $\mathbb{G}_m^2$. If $X$ has infinitely many points $(a, b)$ where both $a$ and $b$ are roots of unity then $X$ is a torsion translate of an algebraic subgroup.

This has many vast generalizations. Example: Mordell-Lang Conjecture for semi-abelian varieties by Faltings, Vojta, and McQuillan. Another example: work of Bombieri, Masser, and Zannier.
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From now on: everything is over \( \bar{\mathbb{Q}} \).

Think of torsion points \((a, b)\) as “subgroups of codimension 2 in \( \mathbb{G}_m^2 \)”.

Question: fix a curve \( X \) in \( \mathbb{G}_m^n \), what happens when intersect \( X \) with:

(a) the union of all subgroups of codimension 2?
(b) the union of all subgroups of codimension 1?

Bombieri, Masser, and Zannier treated both. Today we only focus on (b). Answer: when \( X \) is not contained in a translate of an algebraic subgroup, the intersection is infinite but it is small (i.e. bounded height).
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$h$: absolute logarithmic Weil height on $\mathbb{P}^1(\overline{\mathbb{Q}})$. Define $h_n$ on $(\mathbb{P}^1)^n$ by:

$$h_n(a_1, \ldots, a_n) := h(a_1) + \ldots + h(a_n).$$

A subset of $(\mathbb{P}^1)^n$ has bounded height: boundedness with respect to $h_n$.

**Theorem (BMZ 1999)**

Let $X$ be a curve in $\mathbb{G}_m^n$ that is not contained in any translate of an algebraic subgroup then $\bigcup_V X \cap V$ has bounded height where $V$ ranges over all algebraic subgroups of codimension 1.
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Let $d \geq 2$ and $C_d(X)$ be the polynomial of degree $d$ satisfying $C_d(x + \frac{1}{x}) = x^d + \frac{1}{x^d}$.

Exceptional polynomials (of degree $d$) are polynomials that are linearly conjugate to $x^d$ or $\pm C_d(x)$. Non-exceptional polynomials are also called “disintegrated” by Medvedev-Scanlon.

Let $n \geq 2$ and $f_1, \ldots, f_n \in \overline{\mathbb{Q}}[x]$ of degrees at least 2. Let $\varphi := f_1 \times \ldots \times f_n$ be the coordinate-wise self-map of $(\mathbb{P}^1)^n$:

$$\varphi(a_1, \ldots, a_n) = f_1(a_1) + \ldots + f_n(a_n).$$

For the arithmetic dynamics of $\varphi$, it suffices to study the arithmetic of $\mathbb{G}_m^n$ and the case when $f_1, \ldots, f_n$ are non-exceptional.
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We have an analogue of the BMZ Theorem:

**Theorem (N. 2013)**

Let $X$ be a curve in $(\mathbb{P}^1)^n$ whose projection to each factor $\mathbb{P}^1$ is non-constant. Assume that $X$ is not contained in any $\varphi$-periodic hypersurface. Then $\bigcup V X \cap V$ has bounded height where $V$ ranges over all $\varphi$-periodic hypersurfaces.
Bombieri, Masser, and Zannier tried to generalize their theorem in 1999 for intersection between a subvariety of dimension $r$ with algebraic subgroups of codimension $r$. This is rather subtle. After a series of work, they proved a “structure theorem” and asked a “bounded height conjecture” in 2007. Habegger proved this conjecture in 2009. All these are inside $\mathbb{G}_m^n$. The more general version for semi-abelian varieties is still open.
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Their approach: given \(X\) of dimension \(r\) in \(\mathbb{G}_m^n\), define "anomalous subvarieties" of \(X\), then define \(X^{oa} := X \setminus \bigcup Z\) where \(Z\) ranges over all anomalous subvarieties.

They prove the following:

**Theorem**

Let \(X\) be a subvariety of dimension \(r\) in \(\mathbb{G}_m^n\).

(a) (BMZ 2007) Structure Theorem: \(X^{oa}\) is Zariski open in \(X\).

(b) (Habegger 2009) Bounded Height Theorem: the intersection \(\bigcup V X^{oa} \cap V\) has bounded height where \(V\) ranges over all algebraic subgroups of codimension \(r\).

Part (b) was conjectured by Bombieri-Masser-Zannier in 2007.
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Let $X$ be a subvariety of dimension $r$ in $\mathbb{G}_m^n$.

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Part (b) was conjectured by Bombieri-Masser-Zannier in 2007.
Consider $f_1(x), \ldots, f_n(x)$ and $\varphi = f_1 \times \ldots \times f_n$ as before.

Given $X$ of dimension $r$ in $(\mathbb{P}^1)^n$, we can define $\varphi$-anomalous subvarieties of $X$, then define $X_{\varphi}^{oa} := X \setminus \bigcup Z$ where $Z$ ranges over all the $\varphi$-anomalous subvarieties.
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We have the following:

**Theorem (Ghioca-N. 2014)**

*Notation as above.*

(a) *Structure Theorem:* $X_{\varphi}^{oa}$ is Zariski open in $X$.

(b) *Bounded Height Theorem:* the intersection $\bigcup_{V} X_{\varphi}^{oa} \cap V$ has bounded height where $V$ ranges over all $\varphi$-periodic subvarieties of codimension $r$. 
THANK YOU.