ERRATA TO THE PAPER "SOME ARITHMETIC DYNAMICS OF DIAGONALLY SPLIT POLYNOMIAL MAPS"

KHOA D. NGUYEN

All the theorems of the paper "Some Arithmetic Dynamics of Dynamically Split Polynomial Maps" [Ngu15] remain valid. However, there is a relatively minor mistake in Step 1.1.3 in the proof of Theorem 3.6 [Ngu15, pp.1175]. This note corrects the above mistake and some typos.

1. Step 1.1.3 in the proof of Theorem 3.6

There is a gap in Step 1.1.3 in the proof of Theorem 3.6 [Ngu15, pp.1175–1178] and we are grateful to Professor Paul Vojta for notifying us about this. The gap lies in the claim in the first paragraph of this step where we state that there is $\rho_2 \in M(f^\infty)$ such that $f = (\tilde{f} \circ \rho_2)^A$ so that we may assume $f = \tilde{f}^A$. *This latter assumption is not essential and is only used to simplify our computations with derivatives.* Assuming the notation used in [Ngu15, pp.1159–1174], we may replace Step 1.1.3 in [Ngu15, pp.1175–1178] by the following.

Step 1.1.3: we turn to the most difficult case, namely $\hat{h}_f(a) > 0$. For each $1 \leq j \leq n-1$, write $g_j = L_j \circ \tilde{f}^{m_j}$.

For almost all $\mathfrak{p}$, we have $v_\mathfrak{p}(f^n(a)) \geq 0$ for every $n \geq 0$. If for some $1 \leq j \leq n-1$, $b_j = \infty$ then for almost all $\mathfrak{p}$, the $\mathfrak{p}$-adic closure of the orbit of P lies in:

$$\{(x, y_1, \ldots, y_{j-1}, \infty, y_{j+1}, \ldots, y_{n-1}) : \ x \in K_\mathfrak{p}, \ v_\mathfrak{p}(x) \geq 0\}$$

which is disjoint from $V(K_\mathfrak{p})$. So we can assume $b_j \neq \infty$ for every $1 \leq j \leq n-1$.

By taking a finite extension of K if necessary, we choose an f-periodic point $\gamma \in K$ of exact period $N \geq 3$ such that every point of the form $L \circ \tilde{f}^k(\gamma)$, where $L \in M(f^\infty)$ and $k \geq 0$, is not a zero of the derivative $\tilde{f}'(X)$ of $\tilde{f}(X)$. Equivalently, we require that the $\tilde{f}$-orbit of γ does not contain any element of the form $L^{-1}(\delta)$ where $L \in M(f^\infty)$ and δ is a root of $\tilde{f}'(X)$. We briefly explain why this is possible. By Proposition 2.3 in [Ngu15, pp.1163], $M(f^\infty)$ is finite, and γ is $\tilde{f}$-periodic since $\tilde{f}$ and f have a common iterate. So we can simply require that the $\tilde{f}$-period of γ is sufficiently large.

By Lemma 3.11 in [Ngu15, pp.1171], there is an infinite set of primes R such that for every $\mathfrak{p} \in R$, all of the following hold:

$$a, b_1, \ldots, b_{n-1} \in \mathscr{O}_\mathfrak{p}, \text{ in other words } P \in \mathbb{A}^n(\mathscr{O}_\mathfrak{p}) \tag{1}$$

$$v_\mathfrak{p}(\tilde{f}'(\tilde{f}^l \circ f^k(\gamma))) = 0 \ \forall \ l, k \geq 0 \tag{2}$$

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(3)
$\tilde{f}, f$ and elements of $M(f^{\infty})$ are in $\mathscr{O}_{\mathfrak{p}}[X]$ with $\mathfrak{p}$-adic units leading coefficients

$$v_{\mathfrak{p}}(\tilde{f}^{\mu}(a)-\gamma)>0 \text{ for some } \mu=\mu_{\mathfrak{p}}. \tag{4}$$

Note that (2) is possible since there are only finitely many elements of the form $\tilde{f}^l\circ f^k(\gamma)$, and these elements are not a root of $\tilde{f}'$ by the choice of γ (and Proposition 2.3 in [Ngu15, pp.1163]).

Now fix a prime $\mathfrak{p}$ in R and write $\mu=\mu_{\mathfrak{p}}$, we still use V to denote the model $y_j=L_j\circ\tilde{f}^{m_j}(x)$ over $\mathscr{O}_{\mathfrak{p}}$, hence it makes sense to write $V(\mathscr{O}_{\mathfrak{p}})$ and $V(k_{\mathfrak{p}})$. We also use P, and φ to denote the corresponding models over $\mathscr{O}_{\mathfrak{p}}$. Replacing P by $\varphi^{\mu}(P)$, we can assume that $v_{\mathfrak{p}}(a-\gamma)>0$. This gives that a is f-periodic modulo $\mathfrak{p}$ and:

(5)
$$v_{\mathfrak{p}}(\tilde{f}^l\circ f^k(a)-\tilde{f}^l\circ f^k(\gamma))>0 \text{ and } v_{\mathfrak{p}}(\tilde{f}'(\tilde{f}^l\circ f^k(a))-\tilde{f}'(\tilde{f}^l\circ f^k(\gamma)))>0\ \forall l,k\geq 0$$

The second inequality in (5) together with (2) give:

$$v_{\mathfrak{p}}(\tilde{f}'(\tilde{f}^l\circ f^k(a)))=0\ \forall l,k\geq 0 \tag{6}$$

By (6) and induction, we have:

$$v_{\mathfrak{p}}((\tilde{f}^m)'(\tilde{f}^l\circ f^k(a)))=0\ \forall m,l,k\geq 0 \tag{7}$$

By Proposition 2.3 in [Ngu15, pp.1163] and condition (3), the derivative of an iterate of f has the form $u(\tilde{f}^m)'$ for some $m\geq 0$, and some $\mathfrak{p}$-adic unit u. Therefore identity (7) (with $l=0$) implies:

$$v_{\mathfrak{p}}((f^m)'(f^k(a)))=0\ \forall m,k\geq 0 \tag{8}$$

Since the φ-orbit of P lies in $\mathbb{A}^n(\mathscr{O}_{\mathfrak{p}})$ which is closed in $(\mathbb{P}^1)^n(K_{\mathfrak{p}})$, it suffices to show that $V(\mathscr{O}_{\mathfrak{p}})$ does not intersect the $\mathfrak{p}$-adic closure of the φ-orbit of P. Assume there is η such that the mod $\mathfrak{p}$ reduction $\varphi^{\eta}(\bar{P})$ lies in $V(k_{\mathfrak{p}})$, otherwise there is nothing to prove. After replacing P by $\varphi^{\eta}(P)$, we can assume $\eta=0$, or in other words $\bar{P}\in V(k_{\mathfrak{p}})$. This means

$$v_{\mathfrak{p}}(b_j-L_j\circ\tilde{f}^{m_j}(a))>0\ \forall 1\leq j\leq n-1. \tag{9}$$

Note that $L_j\circ\tilde{f}^{m_j}$ commutes with an iterate of f, therefore (9) together with the f-periodicity mod $\mathfrak{p}$ of a give that b_j is f-preperiodic mod $\mathfrak{p}$ for $1\leq j\leq n-1$. Therefore P is φ-preperiodic mod $\mathfrak{p}$.

Inequality (9) shows that:

$$v_{\mathfrak{p}}(\tilde{f}^l\circ f^k(b_j)-\tilde{f}^l\circ f^k\circ L_j\circ\tilde{f}^{m_j}(a))>0\ \forall l,k\geq 0\ \forall 1\leq j\leq n-1. \tag{10}$$

Our next step is to show:

$$v_{\mathfrak{p}}(\tilde{f}'(\tilde{f}^l\circ f^k\circ L_j\circ\tilde{f}^{m_j}(a)))=0\ \forall l,k\geq 0\ \forall 1\leq j\leq n-1 \tag{11}$$

By Proposition 2.3 in [Ngu15, pp.1163], write $\tilde{f}^l\circ f^k\circ L_j\circ\tilde{f}^{m_j}=L\circ\tilde{f}^A$ where $L\in M(f^{\infty})$ and $A\geq 0$ depending on k,l,m_j. Let c denote the leading coefficient of L, by Proposition 2.3 in [Ngu15, pp.1163] we have:

$$(\tilde{f}\circ L\circ\tilde{f}^A)'(a)=(L^D\circ\tilde{f}^{A+1})'(a)=c^D(\tilde{f}^{A+1})'(a) \tag{12}$$

and

$$(\tilde{f}\circ L\circ\tilde{f}^A)'(a)=\tilde{f}'(L\circ\tilde{f}^A(a))c(\tilde{f}^A)'(a) \tag{13}$$

Since c is a $\mathfrak{p}$-adic unit, (12) and (13) imply:

$$v_{\mathfrak{p}}((\tilde{f}^{A+1})'(a)) = v_{\mathfrak{p}}(\tilde{f}'(L \circ \tilde{f}^A(a))(\tilde{f}^A)'(a)) \tag{14}$$

Now (11) follows from (7), and (14).

By (10) and (11), we have:

$$v_{\mathfrak{p}}(\tilde{f}'(\tilde{f}^l \circ f^k(b_j))) = 0 \ \forall l, k \geq 0 \ \forall 1 \leq j \leq n-1 \tag{15}$$

By (15) and induction, we have:

$$v_{\mathfrak{p}}((\tilde{f}^m)'(\tilde{f}^l \circ f^k(b_j))) = 0 \ \forall m, l, k \geq 0 \ \forall 1 \leq j \leq n-1 \tag{16}$$

By Proposition 2.3 in [Ngu15, pp.1163] and condition (3), the derivative of an iterate of f has the form $u(\tilde{f}^m)'$ for some $m \geq 0$, and some $\mathfrak{p}$-adic unit u. Identity (16) (with $l = 0$) implies:

$$v_{\mathfrak{p}}((f^m)'(f^k(b_j))) = 0 \ \forall m, k \geq 0 \ \forall 1 \leq j \leq n-1 \tag{17}$$

Now (8) and (17) show that the $\mathscr{O}_{\mathfrak{p}}$-morphism φ is étale at every $\mathscr{O}_{\mathfrak{p}}$-valued point in the orbit of P. Together with the fact that P is preperiodic mod $\mathfrak{p}$, we can apply Theorem 3.10 in [Ngu15, pp.1170] to get the desired conclusion. This finishes the case V is periodic and $I_V = \emptyset$.

2. Typos and further remarks

(i) In Line 7 of page 1186 in [Ngu15], the reference to Theorem B.5.9 of Hindry and Silverman's book [HS00] used to justify [Ngu15, Inequality (28)] should be replaced by original work of Néron [Nér65] (see also [Lan83, Proposition 5.4] and [BMZ99, pp.1127]).

(ii) **Added Note**: it is possible to prove Theorem 4.3 in [Ngu15] using the equation

$$|M\hat{h}(\alpha) - N\hat{h}(\beta)| = o\left(\hat{h}(\alpha) + \hat{h}(\beta)\right)$$

as $\hat{h}(\alpha) + \hat{h}(\beta) \to \infty$ [HS00, Proposition B.3.5] instead of the stronger equation given in Lemma 4.4 in [Ngu15].

(iii) **Added Note**: the Bounbed Height Theorem [Ngu15, Theorem 4.3] has been generalized in [GN14] to treat the case of intersection between a subvariety C of dimension r and the union of periodic subvarieties of codimension r. The proof in [GN14] when $r = 1$ provides another proof of [Ngu15, Theorem 4.3].

(iv) The last two sentences of page 1197 of [Ngu15] should be modified slightly, as follows.

If there is a *periodic* curve that does not have such forms, by Medvedev and Scanlon [MS13, Proposition 2.34] there exist *a positive integer* N and polynomials p_1, p_2, and q such that $f_1^N \circ p_1 = p_1 \circ q$ *and* $f_2^N \circ p_2 = p_2 \circ q$. In other words, we have the commutative diagram:

$$\begin{array}{ccc} (\mathbb{P}^1)^2 & \xrightarrow{(q,q)} & (\mathbb{P}^1)^2 \\ {\scriptstyle (p_1,p_2)}\downarrow & & \downarrow{\scriptstyle (p_1,p_2)} \\ (\mathbb{P}^1)^2 & \xrightarrow[\Phi^N]{} & (\mathbb{P}^1)^n \end{array}$$

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Khoa D. Nguyen, Department of Mathematics, University of British Columbia, and Pacific Institute for The Mathematical Sciences, Vancouver, BC V6T 1Z2, Canada

E-mail address: `khoanguyen2511@gmail.com`

URL: `www.math.ubc.ca/~dknguyen`