Algebraic approximations to linear combinations of powers: an extension of results by Mahler and Corvaja-Zannier

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Overview

For $x \in \mathbb{C}$, let $\|x\| := \min\{|x - m| : \; m \in \mathbb{Z}\}$.

Fix $k \geq 1$ and $\alpha_1, \ldots, \alpha_k \in \bar{\mathbb{Q}}^*$, study:

$$\|q_1\alpha_1^n + \ldots + q_k\alpha_k^n\|$$

where $n \in \mathbb{N}$ and $q_1, \ldots, q_k \in \bar{\mathbb{Q}}^*$ having small logarithmic height compared to n. Special case: linear recurrence sequences.

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References

- K. Mahler, *On the fractional parts of the powers of a rational number (II)*, Mathematika **4** (1957), 122–124.
- P. Corvaja and U. Zannier, *On the rational approximations to the powers of an algebraic number: Solution to two problems of Mahler and Mendès France*, Acta Mathematica **193** (2004), 175–191.
- A. Kulkarni, N. M. Mavraki, and K. N., *Algebraic approximations to linear combinations of powers: an extension of results by Mahler and Corvaja-Zannier*, arXiv:1511.08525.

Lagrange proves that every natural number is the sum of four squares, then:

Problem (Waring)

Given $n \geq 2$, does there exist a (minimal) number $g(n)$ such that every natural number is the sum of $g(n)$ n-th powers of natural numbers? Eg: $g(2) = 4$.

Thanks to work of many mathematicians, we have:

Theorem

$g(n)$ always exists. In fact, if $\lceil (3/2)^n \rceil - (3/2)^n \geq (3/4)^n$ then $g(n) = 2^n + \lfloor (3/2)^n \rfloor - 2$. If $\lceil (3/2)^n \rceil - (3/2)^n < (3/4)^n$, then $g(n)$ has another explicit formula.

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Mahler's Result

Conjecture

$\lceil (3/2)^n \rceil - (3/2)^n \geq (3/4)^n$ *for every* $n \geq 2$.

Theorem (Mahler, 1957)

Let $\alpha \in \mathbb{Q} \setminus \mathbb{Z}$ *satisfying* $\alpha > 1$. *Let* $\theta \in (0,1)$. *Then there are only finitely many n such that* $\|\alpha^n\| < \theta^n$.

Corollary

Since $\lceil (3/2)^n \rceil - (3/2)^n \geq \|(3/2)^n\|$, *there are at most finitely many n such that* $\lceil (3/2)^n \rceil - (3/2)^n < (3/4)^n$.

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Observe: $\left(\frac{\sqrt{5}+1}{2}\right)^n + \left(\frac{\sqrt{5}-1}{2}\right)^n \in \mathbb{Z}$ for every $n \in \mathbb{N}$. Hence if $\frac{\sqrt{5}-1}{2} < \theta < 1$, then $\left\|\left(\frac{\sqrt{5}+1}{2}\right)^n\right\| < \theta^n$ for every n.

Question (Mahler, 1957)

Characterize all the real algebraic numbers $\alpha > 1$ such that for every $\theta \in (0, 1)$ there are at most finitely many $n \in \mathbb{N}$ satisfying $\|\alpha^n\| < \theta^n$.

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In 2004, Corvaja and Zannier prove that all counter-examples arise “in the same manner as $(\sqrt{5}+1)/2$”.

$\frac{\sqrt{5}+1}{2}$ is an example of Pisot numbers:

Definition

A Pisot number is a real algebraic integer greater than 1 while all the other conjugates have modulus less than 1.

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If α is Pisot, by taking the trace of α^n over $\mathbb{Q}$, we have that there exists $\theta \in (0, 1)$ such that $\|\alpha^n\| < \theta^n$ for all sufficiently large *n*.

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Let $\alpha > 1$ be a real algebraic number. Assume that for some $\theta \in (0, 1)$, there are infinitely many $n \in \mathbb{N}$ such that $\|\alpha^n\| < \theta^n$. Then there is $m \in \mathbb{N}$ such that α^m is a Pisot number.

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Linear Recurrence Sequences

A linear recurrence sequence a_n over $\bar{\mathbb{Q}}$ has the form:

$$a_n = Q_1(n)\alpha_1^n + \ldots + Q_k(n)\alpha_k^n$$

for some $k \geq 1$, $Q_i(x) \in \bar{\mathbb{Q}}[x]$, and $\alpha_i \in \bar{\mathbb{Q}}$ for $1 \leq i \leq k$. So $\{\alpha^n\}_{n\in\mathbb{N}}$ is linear recurrence.

Problem

Characterize the Q_i's and α_i's such that for every $\theta \in (0, 1)$, there are only finitely many n satisfying $\|a_n\| < \theta^n$.

We actually handle the more general sum

$$q_1\alpha_1^n + \ldots + q_k\alpha_k^n$$

where $(n, q_1, \ldots, q_k)$ varies with $n \in \mathbb{N}$ and the q_i's are algebraic numbers having small logarithmic height compared to n.

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Pseudo-Pisot Tuples

$(\beta_1, \ldots, \beta_k)$: distinct non-zero algebraic numbers, let:

$$B = \{\beta \in \bar{\mathbb{Q}} \setminus \{\beta_1, \ldots, \beta_k\} : \ \beta = \sigma(\beta_i) \text{ for some } i \text{ and } \sigma \in G_{\mathbb{Q}}\}.$$

$(\beta_1, \ldots, \beta_k)$ is *pseudo-Pisot* if $|\beta| < 1$ for every $\beta \in B$ and:

$$\sum_{i=1}^{k} \beta_i + \sum_{\beta \in B} \beta \in \mathbb{Z}.$$

Moreoever, if β_i is an algebraic integer for every i, we say that $(\beta_1, \ldots, \beta_k)$ is Pisot .

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Let $f : \mathbb{N} \to (0, \infty)$ be a *sublinear function* which means $\lim_{n\to\infty} \frac{f(n)}{n} = 0.$

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Extending Results by Mahler and Corvaja-Zannier

Theorem (Kulkarni-Mavraki-N., 2015)

Assume that for some $\theta \in (0,1)$, the set $\mathcal{M}$ of $(n, q_1, \ldots, q_K) \in \mathbb{N} \times (K^*)^n$ satisfying:

$$\|\sum_{i=1}^{k} q_i \alpha_i^n\| < \theta^n \text{ and } \max_{1 \le i \le k} h(q_i) < f(n)$$

is infinite. For all but finitely many $(n, q_1, \ldots, q_k) \in \mathcal{M}$:

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(ii) *Every α_i is an algebraic integer.*

(iii) *For $\sigma \in G_{\mathbb{Q}}$ and $1 \le i, j \le k$, $\sigma(q_i\alpha_i^n) = q_j\alpha_j^n$ iff $\frac{\sigma(\alpha_i)}{\alpha_j} \in \mu$.*

(iv) *For $\sigma \in G_{\mathbb{Q}}$ and $1 \le i \le k$, if $\frac{\sigma(\alpha_i)}{\alpha_j} \notin \mu$ for $1 \le j \le k$ then $|\sigma(\alpha_i)| < 1$.*

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What it says in down-to-earth terms

When $k = 1$ and $q_1 = 1$, (i)-(ii) recover Corvaja-Zannier result.

Plainly, the properties (i)-(iv) say that for all but finitely many $(n, q_1, \ldots, q_k) \in \mathcal{M}$, the sum $\sum_{i=1}^{k} q_i\alpha_i^n$ is a subsum of the larger sum $\sum_{i=1}^{k} q_i\alpha_i^n + \sum_{\ell=1}^{m} r_\ell\beta_\ell^n$ satisfying:

- This larger sum is an integer.
- The $r_\ell\beta_\ell^n$'s are exactly the Galois conjugates of the $q_i\alpha_i^n$'s that do not appear in $\{q_i\alpha_i^n : \ 1 \leq i \leq k\}$.
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Effectiveness and Questions (if there's time)

Mahler uses Roth's theorem while Corvaja and Zannier use the Subspace Theorem.

We follow the method in Corvaja-Zannier together with several modifications.

Hence these results are not effective (i.e.: Mahler's proof does not give an effective upper bound for solutions of the inequality $\|(3/2)^n\| < (3/4)^n$).

But there is an effective result.

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Theorem (Baker-Coates, 1974)

Let $2 \leq b < a$ be relatively prime integers, then there exist effectively computable numbers N and $\theta_0 > \frac{1}{b}$ such that

$$\|(a/b)^n\| \geq \theta_0^n$$

for every $n \geq N$.

But θ_0 is very close to $1/b$. They strongly suspect that "some fundamentally new ideas would be required to complete the solution to Waring's Problem".

There have been various improvements to Baker-Coates theorem. For Waring's Problem, need $\|(3/2)^n\| \geq (3/4)^n$. Currently, here's the best result obtained by Zudilin:

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Theorem (Baker-Coates, 1974)

Let $2 \leq b < a$ be relatively prime integers, then there exist effectively computable numbers N and $\theta_0 > \frac{1}{b}$ such that

$$\|(a/b)^n\| \geq \theta_0^n$$

for every $n \geq N$.

But θ_0 is very close to $1/b$. They strongly suspect that "some fundamentally new ideas would be required to complete the solution to Waring's Problem".

There have been various improvements to Baker-Coates theorem. For Waring's Problem, need $\|(3/2)^n\| \geq (3/4)^n$. Currently, here's the best result obtained by Zudilin:

$$\|(3/2)^n\| \geq 0.5803^n$$

In view of our results for sums involving more than one term, we ask the following:

Question

Let $k \geq 2$ and let $b, a_1, \ldots, a_k$ be integers with $2 \leq b < \min a_i$ and $\gcd(b, a_1, \ldots, a_k) = 1$. Are there effectively computable N and θ_0 with $\theta_0 > \frac{1}{b}$ such that:

$$\|(a_1^n + \ldots + a_k^n)/b^n\| \geq \theta_0^n$$

for every $n \geq N$.

THANK YOU!