Some Problems of Unlikely Intersections in Diophantine Geometry and Algebraic Dynamics

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Example: solve $x^n + y^n = z^n$ with $n \in \mathbb{N}$, $n \geq 2$, and $x, y, z \in \mathbb{Z}$.

Algebraic dynamics studies the families of iterates

$$\phi, \phi^2 := \phi \circ \phi, \phi^3 := \phi \circ \phi \circ \phi, \ldots$$

where $X$ is a variety over a field $K$ and $\phi$ is a $K$-morphism from $X$ to itself.

Example: when $X = \mathbb{C}$, we can regard $\phi$ as a polynomial with complex coefficients.
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**Example:** consider the family of quadratic polynomials \( \{ f_c(t) = t^2 + c \} \) parametrized by \( c \in \mathbb{C} \). We have:

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f_c(t) = t^2 + c, \quad f_c^2(t) = t^4 + 2ct^2 + c^2 + c, ...
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The orbit of 0 is \( \{ f_c^n(0) : n \in \mathbb{N} \} = \{ c, c^2 + c, (c^2 + c)^2 + c, \ldots \} \).

\( f_c^n(0) - f_c^m(0) \) for \( m \neq n \) is a nonzero polynomial in \( c \).

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Figure: The Mandelbrot set
Highly interesting questions combining number theory and dynamics:

Perhaps this is well-known

**Conjecture:** every algebraic number in the Cantor set is a rational number.

How about

**Question:** is it true that every algebraic number \( c \) in the boundary of the Mandelbrot set must be a root of the polynomial

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Roughly, **the principle of unlikely intersections** predicts that when the *intersection* of two arithmetic objects is *larger than expected* there should be an *underlying geometric reason*.

**Example:** is it likely for a plane curve $V$ to contain infinitely many points $(\alpha, \beta)$ where both $\alpha$ and $\beta$ are roots of unity?

Other words: is it likely for a polynomial equation $P(X, Y) = 0$ to have infinitely many solutions $(\alpha, \beta)$ where both $\alpha$ and $\beta$ are roots of unity?
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Many curves do: the vertical curve \( x = \zeta \) (where \( \zeta \) is a root of unity), or the diagonal \( x = y \), or the curve \( xy = -1 \), etc. **But all these curves are “special”**.

This question was asked by Lang and answered by Ihara, Serre, and Tate. They proved that if \( V \) is such a curve then \( V \) is defined by an equation of the form \( x^m y^n = \zeta \) where \( m, n \in \mathbb{Z} \) and \( \zeta \) is a root of unity.

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The (somewhat vague) principle of unlikely intersections includes several of the most spectacular diophantine results recently.

We will discuss these examples and their dynamical analogues:

- the Manin-Mumford Conjecture (Raynaud’s Theorem),
- André’s result which is part of the more general André-Oort Conjecture,
- the Bombieri-Masser-Zannier Bounded Height Conjecture (Habegger’s Theorem),
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Let $\phi$ be a self-map of a set $S$ and let $V$ be a subset of $S$. We say that:

(i) $V$ is $\phi$-periodic (or simply periodic) if $\phi^n(V) = V$ for some $n \in \mathbb{N}$.

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How to come up with dynamical analogues?

- Many important diophantine results involve abelian varieties (or tori, or more generally semiabelian varieties) and the multiplication-by-\(d\) maps.

- Naively, replace the above data by a variety and a morphism to itself. Torsion points \(\leftrightarrow\) preperiodic points, (torsion translates of) algebraic subgroups \(\leftrightarrow\) preperiodic subvarieties, etc.

But sometimes this naive approach does not work...
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Bottom line:

- This dynamical analogy program is still a major work-in-progress. There are many interesting questions and phenomena to be discovered.

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References:

\( \mathbb{A}^n \): the affine space of dimension \( n \) and \( \mathbb{G}_m^n \): the torus of dimension \( n \). Over \( \mathbb{C} \), think of these as \( \mathbb{C}^n \) and \((\mathbb{C}^\ast)^n\).

The multiplication-by-\( d \) map of \( \mathbb{G}_m^n \) is:

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(x_1, \ldots, x_n) \mapsto (x_1^d, \ldots, x_n^d).
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Given diophantine properties of \( \mathbb{G}_m^n \), try to formulate analogues for the dynamics of \( \phi : \mathbb{A}^n \to \mathbb{A}^n \) given by:

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This is proved by Raynaud. Similar question for tori:

**Lang’s Question:** is it true that if a curve $V$ in $\mathbb{G}_m^2$ contains infinitely many torsion points then it must be a torsion translate of an algebraic subgroup?

Ihara, Serre, and Tate give a positive answer. Moreover, we can show that every such torsion translate is given by the equation $x^m y^n = \zeta$ where $m, n \in \mathbb{Z}$ and $\zeta$ is a root of unity.
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A variant of this analogue was asked by Shou-wu Zhang around 1995. Ghioca, Tucker, and Zhang found a counter-example and proposed a modified version in 2011. They note that even the below example is complicated:

Example: $X = \mathbb{A}^2$ and $\phi(x, y) = (f(x), g(y))$ with $f, g \in \mathbb{C}[t]$ having degree $d > 1$. Then ask for curves $V$ in $X$ having infinitely many points $(a, b)$ where $a$ is $f$-preperiodic and $b$ is $g$-preperiodic. The case $f(t) = g(t) = t^d$ is Lang’s question.
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Early this year, we are able to prove:

**Theorem (Ghioca-N.-Ye, 2016)**

Let \( f, g \in \mathbb{C}[t] \) having degree \( d > 1 \), let \( X = \mathbb{A}^2 \), and let \( \phi(x, y) = (f(x), g(y)) \). If \( V \) is an irreducible curve in \( X \) having infinitely many \( \phi \)-preperiodic points then \( V \) is \( \phi \)-preperiodic.

Let’s prove the theorem in the special case when \( C = \Delta \) is the diagonal in \( \mathbb{A}^2 \).
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For every polynomial $P$ with $\deg(P) > 1$, we can associate a canonical measure $\mu_P$ on $\mathbb{C}$ supported by the Julia set $J_P$.

**Example:** if $P(t) = t^d$ then $J_P$ is the unit circle and $\mu_P$ is its Haar measure.
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Equidistribution theorem: if $P \in \bar{\mathbb{Q}}[t]$ then the Galois orbits of preperiodic points are equidistributed with respect to $\mu_P$.

Example: consider $P(t) = t^d$. Equidistribution says that as $N \to \infty$, the set of primitive $N$-th roots of unity is equidistributed with respect to the Haar measure on the unit circle.
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Proof of our theorem when $C = \Delta$ is the diagonal in $\mathbb{A}^2$:

* By specialization, assume $f(t), g(t) \in \bar{\mathbb{Q}}[t]$.

* Since $C = \Delta$ has infinitely many $\phi$-preperiodic points, there are infinitely many $\alpha \in \mathbb{C}$ that is both $f$-preperiodic and $g$-preperiodic.

* By equidistribution: get $\mu_f = \mu_g$, and hence $J_f = J_g$. (It is an overkill to use equidistribution here when $C = \Delta$; but we need it for general $C$.)

* If both $f$ and $g$ are linearly conjugate to $X^d$, reduce to Lang’s question (Ihara-Serre-Tate result).

* Otherwise, $J_f = J_g$ implies $f$ and $g$ have a common iterate. Hence $\Delta$ is (pre)periodic under $\phi$, qed.
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Unlike all the other examples that happen inside a semiabelian variety, this result occurs in moduli spaces.

Complex multiplication: $E$ is an elliptic curve (over $\mathbb{C}$), then:

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Fix $d \geq 2$. Which rational functions inside the moduli space of rational functions of degree $d$ should play the role of CM elliptic curves inside the moduli space of elliptic curves?

A possible answer: post-critically finite (PCF) functions which mean functions for which every critical point is preperiodic. Example: $f_c(t) = t^d + c$ is PCF iff 0 is preperiodic.
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Serre’s Open Image Theorem: the image of the $\ell$-adic Galois representation associated to an elliptic curve is large unless the curve is CM.

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Let $X$ be an irreducible algebraic curve in $\mathbb{C}^2$. Assume $X$ contains infinitely many points $(c_1, c_2)$ such that both $t^d + c_1$ and $t^d + c_2$ are PCF. Then $X$ is either vertical, or horizontal, or given by the equation $y = \zeta x$ where $\zeta$ is a $(d - 1)$-th root of unity.

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**Example**: If \( \frac{a}{b} \in \mathbb{Q} \) is in lowest terms then

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Back to Lang’s question, think of torsion points on a curve $V \subset \mathbb{G}^2_m$ as:

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What if we intersect $V$ with all subgroups of codimension 1? This intersection should be \textit{infinite}, but can it \textit{remain small in some sense}?
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Answer: this intersection has **bounded height unless** \( V \) is “special”. Work over \( \bar{\mathbb{Q}} \) (so think of \( \mathbb{G}_m^2 \) as \( (\bar{\mathbb{Q}}^*)^2 \)).

**Theorem (Bombieri-Masser-Zannier, 1999)**

Let \( V \) be a curve in \( \mathbb{G}_m^2 \) defined over \( \bar{\mathbb{Q}} \). If \( V \) is not a translate of an algebraic subgroup then

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Let \( V \) be a curve in \( \mathbb{A}^2 \) defined over \( \bar{\mathbb{Q}} \). Assume that \( V \) is not \( \phi \)-periodic and the projection from \( V \) to each coordinate \( \mathbb{A}^1 \) is non-constant. Then

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That’s all about **curves intersecting subgroups of codimension** 1. How about higher dimensional subvarieties V?

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This problem turns out to be delicate. After a series of papers, Bombieri, Masser, and Zannier come up with a subset $V^{oa}$ of $V$ by removing its “anomalous” subvarieties, prove a structure theorem, and formulate a bounded height conjecture. Then Habegger proves this conjecture.

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Let $V \subseteq \mathbb{G}_m^n$ be a subvariety of dimension $r$ defined over $\bar{\mathbb{Q}}$.

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It is proved by Faltings, McQuillan, and Vojta.

It implies the famous Mordell Conjecture: if $V$ is a curve of genus $g > 1$ defined over a number field $K$ then $V(K)$ is finite. Why? Embed $V$ into its Jacobian $A$, the group $\Gamma := A(K)$ is finitely generated, hence $V(K) = V \cap \Gamma$ is a finite union of translates of algebraic subgroups. Since $\dim(V) = 1$ and $V$ is not an elliptic curve, such algebraic subgroups are points. So $V(K)$ is finite, qed.
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**Conjecture (Dynamical Mordell-Lang)**

Let $V$ be a subvariety of a variety $X$ over $\mathbb{C}$, let $P \in X$, and let $\phi$ be a self-map of $X$. Then the set

$$\{ n \in \mathbb{N} : \phi^n(P) \in V \}$$

is a finite union of arithmetic progressions.

Why dynamical analogue?

- The “small” subgroup $\Gamma$ is replaced by the orbit of $P$.
- Given, say, the arithmetic progression $\{2 + 3k\}_{k \geq 0}$, think of the Zariski closure of $\{\phi^{2+3k}(P)\}_{k \geq 0}$, which is invariant under $\phi^3$, as an analogue for a “translate of algebraic subgroup”.

Khoa D. Nguyen

*Unlikely Intersections*
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- The “small” subgroup $\Gamma$ is replaced by the orbit of $P$.
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Ghioca and Tucker conjecture the following after earlier work of Denis and Bell:

**Conjecture (Dynamical Mordell-Lang)**

Let $V$ be a subvariety of a variety $X$ over $\mathbb{C}$, let $P \in X$, and let $\phi$ be a self-map of $X$. Then the set

$$\{ n \in \mathbb{N} : \phi^n(P) \in V \}$$

is a finite union of arithmetic progressions.

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Khoa D. Nguyen

Unlikely Intersections
There are various results by Bell, Ghioca, Tucker, Xie, Zieve, and others supporting this conjecture.

But very little is known about the following **General Dynamical Mordell-Lang Problem** when there are at least two maps acting on $X$:

**Question (General Dynamical Mordell-Lang)**

Let $X$, $V$, and $P$ be as before. Let $f_1, \ldots, f_r$ be $r$ commuting morphisms from $X$ to itself. When can we conclude that the set

$$\{(n_1, \ldots, n_r) \in \mathbb{N}^r : f_1^{n_1} \circ \ldots \circ f_r^{n_r}(P) \in V\}$$

is a finite union of translates of subsemigroups of $\mathbb{N}^r$?

**Scanlon-Yasufuku**: the above set of $(n_1, \ldots, n_r)$ can be very complicated even when $X$ is a torus and each $f_i$ is an endomorphism.
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Motivated by work of Ghioca, Tucker, and Zieve, we consider the special case called **Orbit Intersection Problem**:

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Let $\phi_1, \ldots, \phi_r$ be (not necessarily commuting) morphisms from $X$ to itself. Let $P_1, \ldots, P_r \in X$ such that $P_i$ is not $\phi_i$-preperiodic for every $i$. When can we conclude that the set

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The Orbit Intersection Problem is a special case of General DML since:

- Ambient variety $X^r$, subvariety $V = \Delta$: the diagonal, starting point $P = (P_1, \ldots, P_r)$.
- The $r$ commuting self-maps of $X^r$ are $f_i = (\text{id}, \ldots, \phi_i, \ldots, \text{id})$ ($f_i$ on the $i$-th factor and identity on other factors) for $1 \leq i \leq r$.

The set $\{(a_1k + b_1, \ldots, a_rk + b_r : k \in \mathbb{N}\}$

- is a singleton when $a_1 = \ldots = a_r = 0$
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We are able to solve the Orbit Intersection Problem for **linear spaces** and **semiabelian varieties**:

For linear spaces ($X = \mathbb{A}^N$, each $\phi_i$ is an affine transformation):

**Theorem (Ghioca-N., 2016)**

Let $r, N \in \mathbb{N}$ with $r \geq 2$. For $1 \leq i \leq r$, let $\phi_i : \mathbb{C}^N \to \mathbb{C}^N$ be an affine transformation which means there is an $N \times N$-matrix $A_i \in M_N(\mathbb{C})$ and a vector $v_i \in \mathbb{C}^N$ such that $\phi_i(x) = A_i x + v_i$ for every $x \in \mathbb{C}^N$. For every $i$, let $P_i \in \mathbb{C}^N$ that is not $\phi_i$-preperiodic. If none of the eigenvalues of $A_i$ is a root of unity for every $i$ then

$$\{(n_1, \ldots, n_r) \in \mathbb{N}^r : \phi_1^{n_1}(P_1) = \ldots = \phi_r^{n_r}(P_r)\}$$

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Why the condition on eigenvalues is necessary?

(i) Consider $\phi_1, \phi_2 : \mathbb{C} \rightarrow \mathbb{C}$ given by $\phi_1(x) = x + 1$ and $\phi_2(x) = 2x$.

(ii) $P_1 = 0$ and $P_2 = 1$ so that $\phi_1^n(P_1) = n$ and $\phi_2^m(P_2) = 2^m$.

(iii) The set $\{(n, m) : \phi_1^n(P_1) = \phi_2^m(P_2)\}$ is exactly $\{(2^m, m) : m \in \mathbb{N}\}$.
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Now $X$ is a **semiabelian variety**. **Fact:** every morphism $\phi$ from $X$ to itself is an endomorphism followed by a translate (i.e. $\exists \phi_0 \in \text{End}(X)$ and $y \in X$ such that $\phi(x) = \phi_0(x) + y \ \forall x \in X$).

**Theorem (Ghioca-N., 2016)**

Let $r \geq 2$. For $1 \leq i \leq r$, let $\phi_i$ be a self-map of $X$. Write $\phi_i = \phi_{i,0} + y_i$ with $\phi_{i,0} \in \text{End}(X)$ and $y_i \in X$ as above and let $D\phi_{i,0}$ be the linear transformation of the tangent space at identity of $X$ induced by $\phi_{i,0}$. For each $i$, let $P_i \in X$ that is not $\phi_i$-preperiodic. If none of the eigenvalues of $D\phi_{i,0}$ is a root of unity for every $i$ then

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References:


If there’s enough time:

- A list of some proof techniques
- Some fractals
We have introduced dynamical analogues of 4 important topics in diophantine geometry. Here is a list of some techniques used in the proofs of those analogues:

- Equidistribution results for certain canonical measures associated to our dynamical systems.
- Geometric and analytic properties of certain Julia sets and generalized Mandelbrot sets.
- Complex and $p$-adic analytic functions.
- Properties of height and canonical height functions.
- Classical results on diophantine equations involving curves (Siegel’s theorem, Faltings’ theorem,...) and polynomial-exponential functions (Skolem-Mahler-Lech theorem, Laurent’s theorem,...),...
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Equip $\mathbb{P}^1(\mathbb{C})$ with the chordal metric:

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\rho([X_1 : Y_1], [X_2 : Y_2]) = \frac{|X_1 Y_2 - X_2 Y_1|}{\sqrt{X_1^2 + Y_1^2} \sqrt{X_2^2 + Y_2^2}}
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so that we can discuss normal families of maps.

**Fatou sets and Julia sets:** every rational function in $\mathbb{C}(z)$ is regarded as a holomorphic map from $\mathbb{P}^1(\mathbb{C})$ to itself. Let $f(z) \in \mathbb{C}(z)$ with $\text{deg}(f) \geq 2$:

- The Fatou set of $f$, denoted $F_f$, is the largest open subset of $\mathbb{P}^1(\mathbb{C})$ over which the family of iterates $\{f, f^2, f^3, \ldots, \}$ is a normal family.

- The Julia set of $f$, denoted $J_f$, is the complement of the Fatou set: $J_f := \mathbb{P}^1(\mathbb{C}) \setminus F_f$. Fact: $J_f \neq \emptyset$. 

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Unlikely Intersections
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Figure: $F_f$ and $J_f$ when $f(z) = z^2 + (-0.79 + 0.156i)$. 

Khoa D. Nguyen

Unlikely Intersections
Figure: $F_f$ and $J_f$ when $f(z) = z^2 + 0.75i$. 
Figure: $F_f$ and $J_f$ when $f(z) = 1 + \frac{-3.2 + 0.96i}{z^2}$.
Generalized Mandelbrot Sets: fix $d \geq 2$ and consider the family $\{P_c(z) = z^d + c\}_{c \in \mathbb{C}}$ of polynomials with parameter $c$. Define:

$$M_d := \{ c \in \mathbb{C} : \sup_{n \geq 0} |P_c^n(0)| < \infty \}.$$ 

Motivation: $P_c$ has two critical points, namely $\infty$ and $0$. Obviously: elements in the orbit of $\infty$ remain at $\infty$. Now define $M_d$ to be the set of $c \in \mathbb{C}$ such that elements in the orbit of $0$ “do not stay far away from each other”.

When $d = 2$, this gives the famous Mandelbrot set. Some people also call $M_d$ a “multibrot set”.
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Unlikely Intersections
Figure: $M_2$
Figure: $M_5$
Figure: $M_8$
THANK YOU!