Midterm Exam #2—Math 200, Section 104
November 13, 2015
Duration: 45 minutes

Surname (Last Name)    Given Name    Student Number

Do not open this test until instructed to do so! No textbooks, notes, calculators, or other aids are allowed; phones, pencil cases, and other extraneous items cannot be on your desk. Turn off cell phones and anything that could make noise during the exam.

Complete arguments and explanations are required. No credit is given to solutions having only final answers without justifications.

UBC rules governing examinations:

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.

2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.

3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.

4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.

5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
   (a) speaking or communicating with other examination candidates, unless otherwise authorized;
   (b) purposely exposing written papers to the view of other examination candidates or imaging devices;
   (c) purposely viewing the written papers of other examination candidates;
   (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
   (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)—(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).

6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.

7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.

8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

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1. (a) [2 pts] Find the domain of $f(x, y) = \frac{1}{\sqrt{x^2 + y^2} - 1}$.

\[ D: \quad x^2 + y^2 > 1 \]  \hspace{1cm} (2 pt)

(b) [2 pts] Let $f(x, y)$ be as in part (a). Given $k > 0$, describe precisely the level curve of $f(x, y)$ at level $k$.

Set \[ \frac{1}{\sqrt{x^2 + y^2} - 1} = k \quad \text{so} \quad \frac{2}{k^2} + 1 \]

\[ \Rightarrow \quad \text{Circle of radius } \sqrt{\frac{2}{k^2} + 1} \quad \text{center } (0, 0) \]  \hspace{1cm} (1 pt)

(c) [2 pts] Let $g(x, y)$ be differentiable and let $h(r, t) = g(r \cos t, r \sin t)$. Find $\frac{\partial h}{\partial t}$ in terms of $r$, $t$, and the partial derivatives of $g$.

\[ x = r \cos t, \quad y = r \sin t \]

\[ \frac{\partial h}{\partial t} = \frac{\partial g}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial g}{\partial y} \frac{\partial y}{\partial t} \]  \hspace{1cm} (1 pt)

\[ = g_x(r \cos t, r \sin t)(-r \sin t) + g_y(r \cos t, r \sin t)(r \cos t) \]  \hspace{1cm} (1 pt)
2. For this problem (especially part (c)), you don't need to simplify your final answer.

(a) [5 pts] Find an equation of the tangent plane to the surface \( yz + x \ln y = z^2 + 2 \) at the point \( P(2, e, e) \).

\[
\nabla F = \langle \ln y, \; z + \frac{x}{y}, \; y - 2z \rangle
\]

At \( P \):

\[
\nabla F = \langle 1, \; e + \frac{2}{e}, \; -e \rangle
\]

Tangent plane

\[
x - 2 + \left( e + \frac{2}{e} \right)(y - e) - e(z - e) = 0
\]

(b) [2 pts] Let \( z \) be defined implicitly by \( yz + x \ln y = z^2 + 2 \). Find \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \) at the point \( P(2, e, e) \).

\[
\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{1}{e}
\]

\[
\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{e + \frac{2}{e}}{e} \quad \left( = \frac{e^2 + 2}{e^2} \right)
\]

(c) [3 pts] Let \( z \) be the same implicit function as in part (b). Approximate the value of \( z \) at \( x = 1.99 \) and \( y = e + 0.01 \) using linear approximation.

\[
Z \approx Z_0 + \frac{\partial z}{\partial x}(x_o, y_0)(x - x_0) + \frac{\partial z}{\partial y}(x_o, y_0)(y - y_0)
\]

\[
Z \approx e + \frac{1}{e}(e - 2) + \frac{e^2 + 2}{e^2}(y - e)
\]

Answer: \( e + \frac{-0.01}{e} + \frac{e^2 + 2}{e^2} \cdot 0.01 \)
3. Let the temperature $T$ at the point $(x, y, z)$ be $T(x, y, z) = 4 + xy - \frac{z^2}{2}$.

(a) [3 pts] A bird is at the point $(1, 2, 3)$ and flies in the direction of the vector $\langle 2, 2, 1 \rangle$. Find the rate of change of temperature.

$$\nabla T = \langle y, x, -z \rangle$$ (1pt)

$$\nabla T(1, 2, 3) = \langle 2, 1, -3 \rangle$$ (1pt)

**Answer:** $\langle 2, 1, -3 \rangle = \frac{\langle 2, 2, 1 \rangle}{3} = 1$ (1pt)

(b) [3 pts] In which direction (as a unit vector in $\mathbb{R}^3$) should a bird at the point $(1, 2, 3)$ fly so that the temperature decreases at the fastest rate? Find the rate of change of temperature in this direction.

Decreases the fastest: opposite direction with $\nabla T = \langle 2, 1, -3 \rangle$ (1pt)

**Answer:** $\langle -2, -1, 3 \rangle$ (1pt)

$$\text{Rate} = \sqrt{14}$$ (1pt)
4. [8 pts] Find all the critical points of \( f(x, y) = x^3 - 12xy + y^3 \) and classify each of them as a local maximum, local minimum, or saddle point.

\[
\begin{align*}
\text{Solve} & \quad \begin{cases} f_x = 3x^2 - 12y = 0 & \quad (1pt) \\
               f_y = 3y^2 - 12x = 0 & \quad (1pt) 
\end{cases} \\
\text{So} & \quad \begin{cases} y = \frac{x^2}{4} \quad (1) \\
x = \frac{y^2}{4} \quad (2) 
\end{cases} \\
\text{Plug in (1) for (2)} & \quad x = \frac{x^4}{64} \Rightarrow x(64-x^3) = 0 \\
\Rightarrow & \quad \begin{cases} x = 0 \quad \text{and } y = 0 \quad (1pt) \\
                 x = 4 \quad \text{and } y = 4 \quad (1pt) 
\end{cases}
\end{align*}
\]

\[
\begin{align*}
& \quad f_{xx} = 6x, \quad f_{yy} = 6y, \quad f_{xy} = -12, \quad D = 36xy - 144 \\
& \text{At } (0,0): \quad D < 0 \quad \boxed{\text{saddle point}} \quad (1pt) \\
& \text{At } (4,4): \quad D > 0 \quad \& \quad f_{xx} > 0 \quad \boxed{\text{local min}} \quad (1pt) 
\end{align*}
\]

Comments: Wrong at the beginning \((f_x \times f_y) \): never \( > 6 \)
extra critical points: marks are taken off
lack one of the correct critical pt: \(-2\)
Wrong \(f_{xx}, f_{yy}, f_{xy}, D\): - mark depending on how bad the mistake is
Get \((0,0) \) \& \((4,4)\) without explanation: never \( > 6 \).
5. [10 pts] Let $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$. Find the absolute maximum and absolute minimum of $f(x, y) = \frac{1}{x^3 + y^3 + 8}$ on the set $D$.

Find max/min for $x^3 + y^3 + 8$ then conclude according to $\frac{1}{x^3 + y^3 + 8}$

1) Critical: \[
\begin{align*}
3x^2 &= 0 \\
3y^2 &= 0 \\
critical value &= 8
\end{align*}
\]  

\(1\text{pt}\)

(x, y) \Rightarrow (0, 0) \quad \text{(1pt)}

2) Max/min on boundary: boundary is given by \(x^2 + y^2 = 1\)

Lagrange multiplier: \[
\begin{align*}
3x^2 &= \lambda 2x \\
3y^2 &= \lambda 2y \\
x^2 + y^2 &= 1
\end{align*}
\]  

Want to divide $x$ and divide $y$ to solve for $\lambda$, so: make sure $x \neq 0, y \neq 0$

Case $x = 0$: $y^2 = 1 \Rightarrow (0, 1), (0, -1)$

Case $y = 0$: $x^2 = 1 \Rightarrow (1, 0), (-1, 0)$

Case $x \neq 0$ and $y \neq 0$: $\lambda = \frac{3}{2} x = \frac{3}{2} y \Rightarrow x = y$

\(2x^2 = 1 \Rightarrow \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\)

(0.5 pt for each of the above 6 solutions)

Conclusion for original $f = \frac{1}{x^3 + y^3 + 8}$: \[\max = \frac{1}{7} \text{ at } (0, 1), (-1, 0): \text{1 pt} \]

\[\min = \frac{1}{9} \text{ at } (0, 1), (1, 0): \text{1 pt} \]

(Note: if conclude according to $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$: at most (1pt) out of (2pt).)