1. Let $u$, $v$, $w$ be vectors in $\mathbb{R}^3$. Let $0 = (0, 0, 0)$ be the zero vector in $\mathbb{R}^3$.

(a) What geometric conclusions (involving points, lines, plane, angles, etc.) can you draw from each of the following? Your answer should not involve sine, cosine, area, volume, equations, or inequalities.

(i) [2 pts] $u \cdot v < 0$  \[ \Rightarrow \cos \theta < 0 \]

The angle between $u$ and $v$ is obtuse.

(ii) [2 pts] $u \times (v \times w) = 0$ while $v \times w \neq 0$.

$u$ is orthogonal to the plane formed by $v \times w$

(or $u$ is a normal vector of any plane that is parallel to both $v \times w$)

(b) [3 pts] Assume that $u$ and $v$ have the same length, find the angle between $u + v$ and $u - v$ using dot product.

\[
\begin{align*}
\text{1 pt} & \quad (u + v) \cdot (u - v) \\
\text{1 pt} & \quad = u \cdot u - u \cdot v + v \cdot u - v \cdot v \\
\text{1 pt} & \quad = |u|^2 - |v|^2 \\
\text{1 pt} & \quad = 0 \quad \Rightarrow \quad \text{the angle between } u + v \text{ and } u - v \\
\text{1 pt} & \quad \text{is } \frac{\pi}{2}
\end{align*}
\]
2. Given 3 points $A(-1, 2, 1)$, $B(0, 1, 2)$, and $C(-1, 4, 2)$. This problem has 4 questions.

(a) [3 pts] Find the area of the triangle $ABC$.

\[
\vec{AB} = \langle 1, -1, 1 \rangle, \quad \vec{AC} = \langle 0, 2, 1 \rangle
\]

\[
\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ 0 & 2 & 1 \end{vmatrix} = \langle -3, -1, 2 \rangle
\]

\[
\text{answer: } \frac{1}{2} |\vec{AB} \times \vec{AC}| = \frac{1}{2} \sqrt{14}
\]

(b) [3 pts] Find the coordinates of the point $D$ on $BC$ such that $AD$ is perpendicular to $BC$.

\[
\vec{BA} = \langle -1, 1, -1 \rangle, \quad \vec{BC} = \langle -1, 3, 0 \rangle
\]

\[
\vec{BD} = \text{proj}_{\vec{BC}} \vec{BA} = \frac{\vec{BC} \cdot \vec{BA}}{||\vec{BC}||^2} \vec{BC} = \frac{4}{10} \langle -1, 3, 0 \rangle
\]

\[
= \langle -\frac{2}{5}, \frac{6}{5}, 0 \rangle
\]

Coordinates of $D$

\[
x_D = x_B + \frac{2}{5} = \frac{2}{5}; \quad y_D = y_B + \frac{6}{5} = \frac{11}{5}; \quad z_D = z_B + 0 = 2
\]
(c) [2 pts] Find a scalar equation for the plane \( S \) containing \( A, B, \) and \( C \).

\[
\text{normal} = \vec{AB} \times \vec{AC} = \langle -3, -1, 2 \rangle
\]

from part (a) (1pt)

answer: 
\[-3(x+1) - (y-2) + 2(z-1) = 0 \quad (1pt)\]

(or \(- 3x - y + 2z - 3 = 0\))

(d) [4 pts] The intersection between the plane \( S \) in part (c) and the \( xy \)-plane is a line. Find a parametric equation of this line.

There are other solutions, here's a solution using the method in the textbook: \( S \) has normal \( \vec{n}_1 = \langle -3, -1, 2 \rangle \)

\( z = 0 \) \(\implies\) normal \( \vec{n}_2 = \langle 0, 0, 1 \rangle \)

The desired line is parallel to \( \vec{n}_1 \times \vec{n}_2 \)

\[
\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
-3 & -1 & 2 \\
0 & 0 & 1 \\
\end{vmatrix} = \langle -1, 3, 0 \rangle
\]

Pick a point on the intersection: \( \begin{cases} -3x - y + 2z - 3 = 0 \\ z = 0 \end{cases} \)

Say take \( y = 0 \), get \( x = -1, z = 0 \) \(\implies\) \((-1, 0, 0)\)

answer: \( \begin{cases} x = -1 - t \\ y = 3t \\ z = 0 \end{cases} \)  (rough scheme: \( * \) perfect: 4 pts  
\( * -1 \) for each mistake)
3. Given the points $A(-1, 1, 2), B(1, 0, 1)$. Let $T$ be the plane $x + 2y = 3z$.

(a) [3 pts] Find a symmetric equation for the line through $A$ and $B$, and find a parametric equation for the line segment $AB$.

\[ \overrightarrow{AB} = \langle 2, -1, -1 \rangle \]

Symmetric eq for line $AB$:
\[
\frac{x+1}{2} = \frac{y-1}{-1} = \frac{z-2}{-1}
\]

Parametric eq for line segment $AB$:
\[
\begin{cases}
  x = -1 + 2t \\
  y = 1 - t \\
  z = 2 - t
\end{cases}
\text{ for } 0 \leq t \leq 1
\]

(b) [4 pts] Find the angles between $T$ and each of the coordinate planes (i.e. the $xz$-plane, the $xy$-plane, and the $yz$-plane).

\[ n_T = \langle 1, 2, -3 \rangle, \quad |n_T| = \sqrt{14} \]

$xz$-plane $\Rightarrow y = 0$ \quad $n_1 = \langle 0, 1, 0 \rangle$ \quad $\cos^{-1}\left(\frac{n_T \cdot n_1}{|n_T||n_1|}\right) = \cos^{-1}\left(\frac{2}{\sqrt{14}}\right)$

$xy$-plane $\Rightarrow z = 0$ \quad $n_2 = \langle 0, 1, 1 \rangle$ \quad $\Rightarrow \pi - \cos^{-1}\left(\frac{3}{\sqrt{14}}\right)$ \quad since the angle between 2 planes must be acute

$yz$-plane $\Rightarrow x = 0$ \quad $n_3 = \langle 1, 0, 0 \rangle$ \quad $\cos^{-1}\left(\frac{1}{\sqrt{14}}\right)$

(c) [4 pts] Find an equation for the plane that passes through $A$ and $B$ and is perpendicular to the plane $T$.

The normal vector of the desired plane must be orthogonal to $\overrightarrow{AB}$ and $n_T$.\[ \Rightarrow \text{ take } \overrightarrow{AB} \times n_T = \begin{vmatrix} i & j & k \\ 2 & -1 & -1 \\ 1 & 2 & -3 \end{vmatrix} = \langle 5, 5, 5 \rangle \]

\[
\text{answer: } 5(x+1) + 5(y-1) + 5(z-2) = 0
\]
(here we pick $A(-1, 1, 2)$ on the plane)
4. Let $S$ be the sphere $(x - 1)^2 + (y - 1)^2 + (z - 1)^2 = 9$. Let $T$ be the plane $2x - 2y + z = 4$.

(a) [4 pts] Find the intersection between the line $x = 1 + t$, $y = 1 + 2t$, $z = 2 + t$ and $S$.

\[
t^2 + (2t)^2 + (1 + t)^2 = 9 \\
6t^2 + 2t + 1 = 9 \Rightarrow 3t^2 + t - 4 = 0 \\
t = 1 \Rightarrow (2, 3, 3) \\
t = -\frac{4}{3} \Rightarrow (-\frac{1}{3}, -\frac{5}{3}, \frac{2}{3})
\]

(b) [2 pts] Find the distance between the center $C(1, 1, 1)$ of $S$ to $T$.

\[
T: \quad 2x - 2y + z - 4 = 0, \quad \text{use the formula} \quad \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}
\]

\[
\frac{|2-2+1-4|}{\sqrt{2^2+2^2+1}} = \frac{3}{\sqrt{9}} = 1
\]

(c) [4 pts] The intersection between $S$ and $T$ is a circle. Find its radius and center.

By Pythagorean theorem

\[r = \sqrt{9 - 1} = \sqrt{8}
\]

Find center $D$:

Normal of $T$ is $(2, -2, 1)$, so $L$ has parametric eq

\[
\begin{align*}
x &= 1 + 2t \\
y &= 1 - 2t \\
z &= 1 + t
\end{align*}
\]

Center $D$ = intersection of $L$ and the plane $T$

\[
2(1+2t) - 2(1-2t) + 1 + t - 4 = 0 \\
gt + 3 \Rightarrow t = \frac{4}{3} \Rightarrow D\left(\frac{5}{3}, \frac{1}{3}, \frac{4}{3}\right)
\]