The University of British Columbia
Midterm 1 - September 30, 2014
Mathematics 200

Closed book examination

Last Name: ____________________ First: ________________________

Section (check one): □ 101 (MWF 9-10, Peterson) □ 102 (MWF 11-12, Fraser)
□ 103 (MWF 11-12, Nguyen) □ 104 (MWF 1-2, Liu)
□ 105 (TuTh 9:30-11, Roe) □ 107 (TuTh 3:30-5, Roe)

Special Instructions:
- Be sure that this examination has 8 pages. Write your name on top of each page.
- No books, notes, calculators, or any other aids are allowed.
- For full credit, you must justify your answers and include all steps in your solutions, unless otherwise indicated.

Rules governing examinations
- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBC card for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
  (a) speaking or communicating with other candidates, unless otherwise authorized;
  (b) purposely exposing written papers to the view of other candidates or imaging devices;
  (c) purposely viewing the written papers of other candidates;
  (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
  (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)–(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

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<table>
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<tbody>
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<td>Total</td>
<td>50</td>
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</table>
1. Let \( u, v, \) and \( w \) be vectors in \( \mathbb{R}^3 \). What geometric conclusions (involving points, lines, planes, angles, etc.) can you draw from each of the following? Your answer should not involve sine, cosine, area, volume, equations, or inequalities.

(a) \( u \cdot v < 0 \).

**Solution:**
the angle between \( u \) and \( v \) is obtuse.

(b) \( u \times v = 0 \).

**Solution:**
\( u \) and \( v \) are co-linear.

(c) \( (u \times v) \cdot w = 0 \).

**Solution:**
\( u, v, \) and \( w \) are co-planar.

(d) \( |k \times w| = |w| \).

**Solution:**
\( w \) is horizontal, i.e. parallel to the \( xy \)-plane.
2. Consider the points $A(1, 4, 3)$, $B(3, 2, 5)$, $C(1, 6, 4)$, $D(9, 2, 0)$. Find the following:

(a) the area of the triangle $ABC$.

Solution:
\[
\vec{AB} = \langle 2, -2, 2 \rangle
\]
\[
\vec{AC} = \langle 0, 2, 1 \rangle
\]
\[
\text{area} = \frac{1}{2} |\vec{AB} \times \vec{AC}| = \frac{1}{2} |\langle -6, -2, 4 \rangle| = \sqrt{14}
\]

(b) the cosine of the angle between the sides $AB$ and $AC$ of triangle $ABC$.

Solution:
\[
\frac{\vec{AB} \cdot \vec{AC}}{|\vec{AB}| |\vec{AC}|} = \frac{-2}{\sqrt{12} \sqrt{5}} = -\frac{1}{\sqrt{15}}
\]

(c) an equation for the plane $M$ containing $A$, $B$, and $C$.

Solution:
\[-\frac{1}{2} \vec{AB} \times \vec{AC} = \langle 3, 1, -2 \rangle \text{ is a normal vector to the plane.}
\]

The equation of the plane is
\[3(x - 1) + (y - 4) - 2(z - 3) = 0\]

or
\[3x + y - 2z = 1.
\]

(d) the distance from the point $D$ to the plane $M$.

Solution:
\[
\frac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|3 \cdot 9 + 1 \cdot 2 - 2 \cdot 0 - 1|}{\sqrt{3^2 + 1^2 + (-2)^2}} = \frac{28}{\sqrt{14}} = 2\sqrt{14}
\]
3. Find parametric equations for the line in which the planes $x - y + 2z = -6$ and $3x + y - z = 2$ intersect.

**Solution:**

Let $n_1 = \langle 1, -1, 2 \rangle$ and $n_2 = \langle 3, 1, -1 \rangle$.

Then $n_1 \times n_2$ is a vector parallel to the line in which the planes $x - y + 2z = -6$ and $3x + y - z = 2$ intersect.

\[
\begin{vmatrix}
  i & j & k \\
  1 & -1 & 2 \\
  3 & 1 & -1 \\
\end{vmatrix} = (-1, 7, 4).
\]

To find a point on the line, we let $z = 0$ in

\[
\begin{align*}
x - y + 2z &= -6 \\
3x + y - z &= 2
\end{align*}
\]

to get

\[
\begin{align*}
x - y &= -6 \\
3x + y &= 2
\end{align*}
\]

which implies $x = -1$ and $y = 5$.

Hence $(-1, 5, 0)$ is on the line. Thus parametric equations of the line are:

\[
\begin{align*}
x &= -1 - t \\
y &= 5 + 7t \\
z &= 4t
\end{align*}
\]
4. Find the point on the line $x = 4 + t$, $y = 1 - t$, $z = 2t$ that is closest to the point $Q(2, 9, -1)$.

**Solution 1:**

Setting $t = 0$ in the parametric equations of the line, we find that $P(4, 1, 0)$ is a point on the line. Also, $\mathbf{v} = \langle 1, -1, 2 \rangle$ is a vector parallel to the line.

The point of the line that is closest to $Q$ is the point $R$ such that $\overrightarrow{PR} = \text{proj}_v \overrightarrow{PQ}$, or

$$\overrightarrow{PR} = \frac{\overrightarrow{PQ} \cdot \mathbf{v}}{|\mathbf{v}|^2} \mathbf{v}$$

$$= \frac{\langle -2, 8, -1 \rangle \cdot \langle 1, -1, 2 \rangle}{|\langle 1, -1, 2 \rangle|^2} \langle 1, -1, 2 \rangle$$

$$= \frac{-12}{6} \langle 1, -1, 2 \rangle$$

$$= \langle -2, 2, -4 \rangle$$

If $R = (a, b, c)$, then we have $\overrightarrow{PR} = \langle a - 4, b - 1, c \rangle$, and solving for $a, b, c$ in

$$\langle a - 4, b - 1, c \rangle = \langle -2, 2, -4 \rangle$$

we find that $R = (2, 3, -4)$.

**Solution 2:**

$\mathbf{v} = \langle 1, -1, 2 \rangle$ is a vector parallel to the line.

The vector from $Q$ to a general point $P = (4 + t, 1 - t, 2t)$ on the line is

$$\overrightarrow{QP} = \langle (4 + t) - 2, (1 - t) - 9, 2t - (-1) \rangle = \langle 2 + t, -8 - t, 2t + 1 \rangle$$

We want to find $t$ such that $\overrightarrow{QP} \cdot \mathbf{v} = 0$, or

$$\langle 2 + t, -8 - t, 2t + 1 \rangle \cdot \langle 1, -1, 2 \rangle = 0$$

$$2 + t - (-8 - t) + 2(2t + 1) = 0$$

$$6t = -12$$

$$t = -2$$

Therefore, the closest point has coordinates $x = 4 - 2 = 2$, $y = 1 - (-2) = 3$, $z = 2 \cdot (-2) = -4$, and is $(2, 3, -4)$.

**Solution 3:**

The vector from $Q$ to a general point $(4 + t, 1 - t, 2t)$ on the line is $\langle 2 + t, -8 - t, 2t + 1 \rangle$ and so the square of the distance from $Q$ to a general point on the line is

$$f(t) = (4 + t)^2 + (1 - t)^2 + (2t)^2 = 6t^2 + 6t + 5.$$ 

This is minimized when $f'(t) = 12t + 6 = 0 \iff t = -2$. Therefore, the closest point has coordinates $x = 4 - 2 = 2$, $y = 1 - (-2) = 3$, $z = 2 \cdot (-2) = -4$, and is $(2, 3, -4)$. 
5. Match the functions with their graphs and plots of level curves by filling in the table below. You do not need to justify your answers.

<table>
<thead>
<tr>
<th>Function</th>
<th>Graph</th>
<th>Contour</th>
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<tbody>
<tr>
<td>$\frac{-2xy}{1+x^2+y^2}$</td>
<td>(A)</td>
<td>(1)</td>
</tr>
<tr>
<td>$\frac{\pi}{4} + \cos(\pi y)$</td>
<td>(B)</td>
<td>(2)</td>
</tr>
<tr>
<td>$\frac{1}{2}y \sin(\pi x)^2$</td>
<td>(C)</td>
<td>(3)</td>
</tr>
<tr>
<td>$e^{-</td>
<td>x</td>
<td>-</td>
</tr>
<tr>
<td>$1 -</td>
<td>x</td>
<td>-</td>
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</tbody>
</table>
6. (a) Suppose $z$ is implicitly defined as a function of $x$ and $y$ such that:

$$xe^y + ye^z + ze^x = 2014.$$ 

Find $\frac{\partial z}{\partial x}$ in terms of $x$, $y$, and $z$ (hint: partial credit will be given if you can find $\frac{\partial}{\partial x}$ of $xe^y$, $ye^z$ and $ze^x$ correctly).

**Solution:**

Taking partial derivatives with respect to $x$, we have:

(i) $\frac{\partial}{\partial x}(xe^y) = e^y$

(ii) $\frac{\partial}{\partial x}(ye^z) = ye^z \frac{\partial z}{\partial x}$

(iii) $\frac{\partial}{\partial x}(ze^x) = \frac{\partial z}{\partial x} e^x + ze^x$

To get these, we used the chain rule for (ii) and the product rule for (iii).

Taking $\frac{\partial}{\partial x}$ of the original equation, we have:

$$e^y + ye^z \frac{\partial z}{\partial x} + \frac{\partial z}{\partial x} e^x + ze^x = 0.$$

Simplifying, we get:

$$\frac{\partial z}{\partial x} (ye^z + e^x) = -e^y - ze^x.$$

Hence we have:

$$\frac{\partial z}{\partial x} = \frac{-e^y - ze^x}{e^x + ye^z}.$$
(b) If $f$ is an arbitrary differentiable function of one variable, then $u(x, t) = 2f(x - 3t)$ is a solution of the wave equation $u_{tt} = cu_{xx}$ for some constant $c$. Find the constant $c$.

**Solution:**

By the chain rule from single variable calculus,

$$u_t = 2f'(x - 3t) \cdot \frac{\partial}{\partial t}(x - 3t)$$
$$= 2f'(x - 3t) \cdot (-3)$$
$$= -6f'(x - 3t)$$

$$u_{tt} = -6f''(x - 3t) \cdot \frac{\partial}{\partial t}(x - 3t)$$
$$= -6f'(x - 3t) \cdot (-3)$$
$$= 18f''(x - 3t)$$

$$u_x = 2f'(x - 3t) \cdot \frac{\partial}{\partial x}(x - 3t)$$
$$= 2f'(x - 3t)$$

$$u_{xx} = 2f''(x - 3t) \cdot \frac{\partial}{\partial x}(x - 3t)$$
$$= 2f''(x - 3t)$$

Therefore,

$$u_{tt} = 18f''(x - 3t) = 9(2f''(x - 3t)) = 9u_{xx}$$

and $c = 9$. 