Midterm Exam—Math 200, Section 104
October 14, 2016
Duration: 45 minutes

Surname (Last Name)       Given Name       Student Number

Do not open this test until instructed to do so! No textbooks, notes, calculators, or other aids are allowed; phones, pencil cases, and other extraneous items cannot be on your desk. Turn off cell phones and anything that could make noise during the exam.

Complete arguments and explanations are required. No credit is given to solutions having only final answers without justifications.

UBC rules governing examinations:
1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBC card for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
   (a) speaking or communicating with other examination candidates, unless otherwise authorized;
   (b) purposely exposing written papers to the view of other examination candidates or imaging devices;
   (c) purposely viewing the written papers of other examination candidates;
   (d) using or having available at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
   (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)—(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

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Remarks:

- Only a rough grading scheme, the point is that we try to be consistent during the whole process.

- Some questions are graded leniently while some are graded harshly.

- In the final, there might be no hints or not as many given formulas.
1. This problem has 2 questions.

(a) Let \( S \) be the plane \( 2x - y + 2z + 4 = 0 \). Write down a normal vector of \( S \) and find the distance from the origin \( O \) to \( S \). (2pts)

\[ \vec{n} = \langle 2, -1, 2 \rangle \quad (1pt) \]

\[
\text{distance} = \frac{|0 - 0 + 0 + 4|}{\sqrt{4 + 1 + 4}} = \frac{4}{3} \quad (1pt)
\]

(b) Let \( S_1 \) be the plane \( y + z = 1 \), find a symmetric equation of the line of intersection of \( S_1 \) and the plane \( S \) in part (a). (4pts)

\( S: \ 2x - y + 2z + 4 = 0 \)

\( S_1: \ y + z = 1 \)

Pick a point in the intersection: say \( y = 0 \Rightarrow z = 1 \)

and \( 2x + 2 + 4 = 0 \Rightarrow x = -3 \). Point \((-3, 0, 1)\)

The line is parallel to \( \vec{n} \times \vec{n}_1 \) \( (1pt) \)

\[
\begin{vmatrix}
1 & i & j & k \\
2 & -1 & 2 & 0 \\
0 & 1 & 1 & 1
\end{vmatrix} = \langle -3, -2, 2 \rangle \quad (1pt)
\]

Symmetric equation:

\[
\frac{x + 3}{-3} = \frac{y}{-2} = \frac{z - 1}{2} \quad (1pt)
\]

Comment: should give answer in the form \( \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \). Although I give full marks for something like \( \frac{2}{3}x + 2 = y = -z + 1 \), we might not do so next time.
2. Given $A(0,2,4)$ and $B(1,4,3)$ and let $O$ be the origin. This problem has 4 questions.

(a) Find the area of the triangle $OAB$. (3pts)
\[
\overrightarrow{OA} \times \overrightarrow{OB} = \begin{vmatrix} i & j & k \\ 0 & 2 & 4 \\ 1 & 4 & 3 \end{vmatrix} = (-10, -(-4), -2) = \langle -10, 4, -2 \rangle
\]
Area
\[
\text{area} = \frac{1}{2} \left\| \overrightarrow{OA} \times \overrightarrow{OB} \right\| = \frac{1}{2} \sqrt{100 + 16 + 4} = \frac{1}{2} \sqrt{120}
\]

(b) Find a parametric equation of the line $AB$, then find the intersection of this line and the $xy$-plane. (3pts)
\[
\overrightarrow{AB} = \langle 1, 2, -1 \rangle
\]
Parametric equation:
\[
\begin{cases}
  x = t \\
  y = 2 + 2t \\
  z = 4 - t
\end{cases}
\]
Intersection with $xy$-plane: set $z = 0$
\[
4 - t = 0 \\
t = 4 \Rightarrow x = 4, y = 10
\]
Answer: $(4, 10, 0)$

Comment: we intersect the line $AB$ and the $xy$-plane and get the point $(4, 10, 0)$. Several students take $\langle 1, 2, -1 \rangle \times \langle 0, 0, t \rangle$ as if we are intersecting 2 planes!!! Even worse: use $\langle 1, 1, 0 \rangle$ as normal vector of the $xy$-plane.
(c) Let \( \ell \) be the line with the parametric equation:

\[
\ell : \quad x = -1 + t, \quad y = -t, \quad z = -1 + 2t.
\]

You are given that \( \ell \) and the line \( AB \) are skew lines. Find a vector that is orthogonal to both \( AB \) and \( \ell \), then find the distance between \( AB \) and \( \ell \). (3pts)

\[
\vec{C} = \vec{AB} \times (\text{vector parallel } \ell) = \begin{vmatrix}
1 & 1 & -1 \\
2 & 2 & -1 \\
1 & 1 & 2
\end{vmatrix} = \langle 3, -3, -3 \rangle \quad (1pt)
\]

Find distance: pick \( P_1 = (-1, 0, -1) \) on \( \ell \) and \( P_2 = A(0, 2, 4) \) on \( AB \Rightarrow \vec{p_1}p_2 = \langle 1, 2, 5 \rangle \quad (1pt) \)

\[
\text{length} = \frac{|\vec{p_1}p_2 \cdot \vec{C}|}{\|\vec{C}\|} = \frac{|3 - 6 - 15|}{\sqrt{27}} = \frac{18}{\sqrt{27}} \quad (1pt)
\]

(d) Find the point \( Q_1 \) on the line \( AB \) and the point \( Q_2 \) on the line \( \ell \) such that \( Q_1Q_2 \) is orthogonal to both \( AB \) and \( \ell \). (5pts)

Use parametric equations (with different parameters \( t \) & \( s \)) for \( AB \) and \( \ell \):

\[
\ell : \begin{cases}
Q_1(t, 2 + 2t, 4 - t), \\
Q_2(-1 + s, -s, -1 + 2s)
\end{cases}
\]

\( \Rightarrow \) \quad So \quad \vec{Q_1Q_2} = \langle -1 + s - t, -s - 2 - 2t, -5 + 2s + t \rangle
\]

Since \( Q_1Q_2 \perp \) both \( AB \) & \( \ell \), we have \( \vec{Q_1Q_2} \parallel \langle 3, -3, -3 \rangle \), hence \( \vec{Q_1Q_2} \parallel \langle 1, -1, -1 \rangle \quad (1pt) \)

\[
\begin{align*}
-1 + s - t &= -s - 2 - 2t \\
\Rightarrow 3t &= -3 \Rightarrow t = -1
\end{align*}
\]

System:

\[
\begin{cases}
-1 + s - t = 5 + 2 + 2t \\
-5 - 2 - 2t = -5 + 2s + t
\end{cases} \Rightarrow \begin{cases}
t = -1 \\
s = s = -5 + 2s - 1 \\
3s = 6 \Rightarrow s = 2
\end{cases}
\]

Answer: \( Q_1(-1, 0, 5) \), \( Q_2(1, -2, 3) \)
Comment: for 2(d)
  - just a rough grading scheme, most students can't solve this problem anyway. So this is graded on a case-by-case basis.

  - 1pt if you mention \( \vec{A_1A_2} \parallel \langle 3, -3, -3 \rangle \)

  - common mistake:
    1) say \( \vec{A_1A_2} = \langle 3, -3, -3 \rangle \)
    2) use the same parameter \( t \) for both \( A_1 \) & \( A_2 \)
3. This problem has 3 questions.

(a) Let \( f(x, y) = \frac{y}{x^2 + y^2 + 1} \). Find the domain of \( f \) and describe precisely the level curve at level \( k = \frac{1}{4} \). (3pts)

- **Domain:** need \( x^2 + y^2 + 1 \neq 0 \). Since \( x^2 + y^2 + 1 \geq 1 \) for every \( x, y \)
- domain: \( \mathbb{R}^2 \) (1pt)

- **level curve:** \( \frac{y}{x^2 + y^2 + 1} = \frac{1}{4} \) (1pt)

\[
\begin{align*}
x^2 + y^2 + 1 &= 4y \quad \Rightarrow \quad x^2 + y^2 - 4y + 4 &= 3 \\
&= x^2 + (y-2)^2 = 3 \\
&\text{circle center} \ (0, 2) \ \text{radius} \ 3
\end{align*}
\]

(b) With \( f(x, y) \) in part (a), find \( \frac{\partial f}{\partial x} (1, 1) \) and \( \frac{\partial f}{\partial y} (1, 1) \). (4pts)

\[
\begin{align*}
\frac{\partial f}{\partial x} &= \frac{y\cdot(-1)}{(x^2 + y^2 + 1)^2} \cdot 2x = -\frac{2xy}{(x^2 + y^2 + 1)^2} \quad (1pt) \\
\frac{\partial f}{\partial x} (1, 1) &= -\frac{2}{3} \quad (1pt)
\end{align*}
\]

\[
\begin{align*}
\frac{\partial f}{\partial y} &= \frac{x^2 + y^2 + 1 - y \cdot 2y}{(x^2 + y^2 + 1)^2} = \frac{x^2 - y^2 + 1}{(x^2 + y^2 + 1)^2} \quad (1pt) \\
\frac{\partial f}{\partial y} (1, 1) &= \frac{1}{3} \quad (1pt)
\end{align*}
\]

(c) Let \( F \) be a differentiable function in one variable and \( G(x, y) = F(x^2 + y^2) + x - y \). What can we say about \( yG_x - xG_y \) (hint: find \( G_x \) and \( G_y \), then simplify \( yG_x - xG_y \))? (3pts)

\[
\begin{align*}
G_x &= F'(x^2 + y^2) \cdot 2x + 1 \quad (1pt) \\
G_y &= F'(x^2 + y^2) \cdot 2y - 1 \quad (1pt)
\end{align*}
\]

\[
yG_x - xG_y = 2xy \ F'(x^2 + y^2) + y - (2xy \ F'(x^2 + y^2) - x) = y + x \quad (1pt)
\]
4. This problem has 3 questions.

(a) Let \( f(x, y) = x^2 + y^2 \). Write down an equation of the tangent plane of the graph of \( f \) at \((2, 1, 5)\). (3pts)

\[
\begin{align*}
1pt & \quad \frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y \quad \text{so} \quad \frac{\partial f}{\partial x}(2, 1) = 4, \quad \frac{\partial f}{\partial y}(2, 1) = 2 \\
\text{tangent:} & \quad z - 5 = 4(x - 2) + 2(y - 1) \quad \text{(1pt)}
\end{align*}
\]

(b) Find the differential \( df \) at \((2, 1)\), the linear approximation \( L(x, y) \) at \((2, 1)\), and use either \( df \) or \( L(x, y) \) to approximate \( f(2.01, 0.99) \). (3pts)

\[
\begin{align*}
\text{df} &= 2x \, dx + 2y \, dy \quad \text{so} \quad \text{at} \ (2, 1), \ \text{we have} \\
\text{df} &= 4 \, dx + 2 \, dy \quad \text{(1pt)} \\
L(x, y) &= 4(x - 2) + 2(y - 1) + 5 \\
f(2.01, 0.99) &\approx L(2.01, 0.99) = 8.04 - 0.02 + 5 = 8.02
\end{align*}
\]

(c) Let \( S \) be the plane \( x + 3y = \frac{1}{2}z + 8 \). Find the real number \( C \) such that \( S \) is the tangent plane at a point \( P(x_1, y_1, z_1) \) on the graph of \( x^2 + y^2 + C \). Hint: find \( P(x_1, y_1, z_1) \). (4pts)

Tangent plane at \( P \) has normal vector \( \langle 2x_1, 2y_1, -1 \rangle \) which is parallel to \( \langle 1, 3, -\frac{1}{2} \rangle \) \( \{2pt\}

\[
\begin{align*}
L_0 &\quad \frac{2x_1}{1} = \frac{2y_1}{3} = \frac{-1}{-1/2} = 2 \quad \Rightarrow \quad x_1 = 1, \ y_1 = 3 \\
since P \ on \ the \ plane: &\quad 1 + 9 = \frac{1}{2}z_1 + 8 \quad \Rightarrow \quad z_1 = 4 \quad \{2pt\} \\
l_1 &= 1 + 9 + C \quad \Rightarrow \quad C = -6
\end{align*}
\]
Comment:
* Common mistakes (some are more serious than others)

1) Tangent plane
   \[ z - 5 = 2x(x - 2) + 2y(y - 1) \]  (very serious mistake)

2) At (2,1):
   - df = 2x dx + 2y dy  (instead of 4 dx + 2 dy)
   - df = 4(x - 2) + 2(y - 1)  (this is a minor mistake, should write 4 dx + 2 dy)

   \[ L(x,y) = 4(2 - 2) + 2(1 - 1) + 5 = 5 \]
   \[ L(x,y) = 4(x - 2) + 2(y - 1) \]  (forget + 5)

3) Tangent plane = 4(x - 2) + 2(y - 1)

   Tangent plane:
   \[ 4(x - 2) + 2(y - 1) - (z - 5) \]

4) Linear approx using differential:
   \[ df = 4(2 - 2.01) + 2(1 - 0.99) \]
   \[ = -0.02 \]

5) Part (c), set \( 2x_1 = 1 \), \( 2y_1 = 3 \)